On affine designs and Hadamard designs with line spreads

V.C. Mavron [a] T.P. McDonough [a] V.D. Tonchev [b,1]

[a] *Department of Mathematics, The University of Wales Aberystwyth, SY23 3BZ, U.K.*

[b] *Department of Mathematical Sciences, Michigan Technological University, Houghton, Michigan 49931, U.S.A.*

Abstract

Rahilly [10] described a construction that relates any Hadamard design H on $4^m - 1$ points with a line spread to an affine design having the same parameters as the classical design of points and hyperplanes in $AG(m, 4)$. Here it is proved that the affine design is the classical design of points and hyperplanes in $AG(m, 4)$ if, and only if, H is the classical design of points and hyperplanes in $PG(2m-1, 2)$ and the line spread is of a special type. Computational results about line spreads in $PG(5, 2)$ are given. One of the affine designs obtained has the same 2-rank as the design of points and planes in $AG(3, 4)$, and provides a counter-example to a conjecture of Hamada [6].

Dedicated to Jennifer Seberry on her 60th birthday

1 Introduction

The connection between Hadamard matrices and symmetric or affine designs is well-known, see [12] for example. In this paper, we describe two constructions, based on one of Rahilly [10], that relates affine 2-designs of class number 4 with symmetric Hadamard 2-designs possessing spreads of lines where each line has size 3. In Section 2, we show that the affine design is the classical design of points and hyperplanes in the affine geometry $AG(m, 4)$ of dimension m over the field of 4 elements if, and only if, the Hadamard design is the classical design of points and hyperplanes in the projective geometry $PG(2m-1, 2)$

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of dimension $2m - 1$ over the field of 2 elements and the line spread is of a special type which we call *normal* and define in 2.5 below. In Section 3, we give an indication of the variety of affine designs produced by this construction using line spreads from the projective geometry $PG(5, 2)$ by summarizing computational results. In particular, we establish the falsity of Hamada's conjecture that, among the 2-designs with the same parameters as the 2-design of points and t-subspaces of a projective or affine geometry over a field of characteristic p, the designs whose incidence matrices are of minimum p-rank are isomorphic to the given design of points and t-subspaces of a projective or affine geometry, by exhibiting a non-geometric affine 2-(64,16,5) design, whose incidence matrix has 2-rank 16. Although it has not yet been established that 16 is the minimum 2-rank of the incidence matrices of 2-(64,16,5) designs, any 2-(64,16,5) designs of 2-rank less than 16 which might be discovered in the future will necessarily be non-geometric.

The basic design theory needed for this paper may be found, for example, in [1], [3], [11]. We give an outline here.

Let $\Pi = (\mathcal{P}, \mathcal{B}, I)$ be a design with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$. Where convenient, as is customary, we may identify a block with the subset of points incident with it, and regard incidence as set-theoretic inclusion. Π is a t-(v, k, λ) design if $\mathcal{P}$ and $\mathcal{B}$ are finite, $|\mathcal{P}| = v$ and $|B| = k$ for all $B \in \mathcal{B}$ and any t-subset of $\mathcal{P}$ is contained in λ blocks. A design is *symmetric* if $|\mathcal{P}| = |\mathcal{B}|$. A t-(v, k, λ) design is *resolvable* if $\mathcal{B}$ has a partition, called a *parallelism*, into *parallel classes* of blocks such that two distinct blocks in the same parallel class are always disjoint and every point belongs to exactly one block from each parallel class. If, further, any two non-parallel blocks (i.e. blocks from different parallel classes) meet in a constant number $\mu > 0$ of points, then Π is *affine resolvable* or simply, *affine*. It is easy to see that each parallel class consists of $m = v/k$ blocks, where we call m the *class number* of the affine design, and $\mu = k/m$. From the definition, it follows that the parallelism in an affine design is unique.

The dual design Π^* of a design $\Pi = (\mathcal{P}, \mathcal{B}, I)$ is defined to be the design $\Pi^* = (\mathcal{B}, \mathcal{P}, I^*)$, where $(x, y) \in I$ if and only if $(y, x) \in I^*$. The *line* joining two distinct points P and Q in a t-design is the intersection of all blocks which contain both P and Q. If $t \geq 2$, the maximum size of a line is $(v - 1)/k + 1$ and a line has this maximal size if, and only if, every block which does not contain it meets it in exactly one point. The set of blocks that contain the intersection of two distinct given blocks in Π forms a line in the dual design Π^*.

The parameters of a symmetric 2-(v, k, λ) design satisfy the equation $\lambda(v-1) = k(k - 1)$. A symmetric 2-design is said to be *Hadamard* if $v = 2k + 1$. It is well-known that a Hadamard design exists if, and only if, a Hadamard matrix of order $2k+2$ exists or, equivalently, a 3-$(2k+2, k+1, \frac{1}{2}(k-1))$ design, which is necessarily affine, exists. The size of a line in a Hadamard 2-design is at most $(v-1)/k+1 = 3$ since $v = 2k+1$. So, a line of size 3 has maximum size and any block either contains it or meets it in exactly one point.

A set $\mathcal{L}$ of non-empty point subsets of a design is a *spread* if it partitions the point set of the design. In a resolvable design, a parallel class of blocks is a spread of blocks.

2 Affine designs and spreads in symmetric designs

Rahilly [10] established a connection between affine 2-designs with class number 4 (the size of a parallel class) and Hadamard 2-designs with line spreads. This construction is generalized in Al-Kenani and Mavron [9]. Here, we present Rahilly's construction in a different but simpler and more transparent form that is suitable for the exposition of the results of this paper.

Construction 2.1 Let Γ be an affine 2-$(16\mu, 4\mu, \frac{1}{3}(4\mu-1))$ design, where $\mu \equiv 1 \pmod 3$. Define a design Π as follows.

Choose any point w of Γ. The points of Π are all the points of Γ except w. To define a general block of Π, consider any parallel class $\mathcal{C}$. Then $\mathcal{C}$ has four blocks. Let B_0 be the block of $\mathcal{C}$ on w. For any $B \in \mathcal{C}$ with $B \neq B_0$, we define $B \cup B_0 - \{w\}$ to be a block of Π.

It is not difficult to verify that Π is a symmetric 2-$(16\mu-1, 8\mu-1, 4\mu-1)$ design and that, for any parallel class $\mathcal{C}$, the three blocks $B \cup B_0 - \{w\}$, with $B \in \mathcal{C}$ and $B \neq B_0$, form a line in the dual Π^* of Π. The set of all such lines is a spread of lines, each of size 3, in Π^*.

Construction 2.2 Let $\Pi = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric 2-$(16\mu-1, 8\mu-1, 4\mu-1)$ design whose dual Π^* has a spread $\mathcal{L}$ of lines of size 3, that is, a set of lines of Π^* which partitions $\mathcal{B}$. Define an incidence structure Γ as follows.

The point set of Γ is $\mathcal{P} \cup \{w\}$ where w is a new point. The block set of Γ is $\mathcal{B} \cup \mathcal{L}$. We define the incidence relation I_Γ for Γ in two parts. Firstly, $w \, I_\Gamma \, L$ for all $L \in \mathcal{L}$. Secondly, let $P \in \mathcal{P}$ and $L \in \mathcal{L}$. If P is on exactly one (say B) of the three blocks of L in Π, then $P \, I_\Gamma \, B$. If P is on all three blocks of L in Π, then $P \, I_\Gamma \, L$.

It is routine to verify that Γ is an affine 2-$(16\mu, 4\mu, \frac{1}{3}(4\mu-1))$ design. A typical parallel class consists of L, B_1, B_2, B_3, where $L \in \mathcal{L}$ and the B_i are the three blocks of L in Π.

The verifications for both of the above constructions may be found in Al-Kenani and Mavron [9] in a more general setting.

The constructions are, in an obvious sense, inverses of one another. However, it should be noted that the choice of spread in the second construction is important. Different choices of spread may result in non-isomorphic designs (see Section 3).

It is also interesting to observe that the symmetric design in both constructions is Hadamard, and therefore constructible from a Hadamard matrix. The constructions relate Hadamard matrices to affine 2-designs with class number 4. Their relationship with affine 2-designs with class number 2 is, of course, well-known.

We shall need the following results due to Kantor (see [3, pp.839]) and Dembowski and Wagner (see [3, pp.812]).

Result 2.3 *An affine 2-design of class number m is isomorphic to the design of points and hyperplanes of some affine geometry $AG(n,m)$ or to the design of points and lines of an affine plane of order m if, and only if, the intersection of any two non-parallel blocks is contained in $m+1$ blocks.*

Result 2.4 *A symmetric 2-(v,k,λ) design is isomorphic to the design of points and hyperplanes of some projective geometry $PG(n,m)$ or to the design of points and lines of a projective plane of order m, where $m=(v-1)/k$, if, and only if, the intersection of any two distinct blocks is contained in exactly $m+1$ distinct blocks (or, dually, every line has exactly $m+1$ points).*

Given any set $X=\{a,b,c\}$ of three distinct mutually skew lines in $PG(3,2)$, there are exactly three transversals to the three lines of X (a *transversal* of X is a line meeting each line of X in a point). The three lines of X form a *regulus* and the three transversals form the *opposite regulus*. Thus, any line spread $\mathcal{L}$ in $PG(3,2)$ is *regular* in the sense that any three lines of $\mathcal{L}$ form a regulus. Moreover, any regulus in $PG(3,2)$ is contained in a unique line spread of five lines. See Hirschfeld [8] for details of results concerning reguli.

Definition 2.5 *A spread $\mathcal{L}$ of lines in $PG(n,2)$, $n\geq 3$, is* normal *if for any two distinct lines in $\mathcal{L}$ the intersection of all hyperplanes containing both lines contains three further lines of $\mathcal{L}$.*

In Constructions 2.1 and 2.2, the affine design has $m=4$, while the symmetric design has $m=2$ and is therefore Hadamard. In what follows, we use the notation $AG_t(n,q)$ (resp. $PG_t(n,q)$) for the design having as points and blocks the points and t-dimensional subspaces of $AG(n,q)$ (resp. $PG(n,q)$). The aim of this section is to prove the following theorem.

Theorem 2.6 *With the notation of Constructions 2.1 and 2.2, the following statements are equivalent:*
(a) Γ *is isomorphic to the design of points and hyperplanes* $AG_{n-1}(n,4)$ *of the affine geometry* $AG(n,4)$*, where* $n \geq 2$*.*
(b) Π *is isomorphic to the design of points and hyperplanes* $PG_{2n-2}(2n-1,2)$ *of the projective geometry* $PG(2n-1,2)$*, where* $n \geq 2$*. The line spread* $\mathcal{L}$ *in the dual* Π^* *of* Π *is normal.*

The proof proceeds through a series of lemmas.

Lemma 2.7 *Let* Γ *be the design* $AG_{n-1}(n,4)$*, where* $n \geq 2$*. Let* $A = \{A_1, A_2, A_3, A_4\}$ *and* $B = \{B_1, B_2, B_3, B_4\}$ *be distinct parallel classes of* Γ*. Let* Δ *be the design whose points are the sixteen affine subspaces* $A_i \cap B_j$ $(i,j = 1,2,3,4)$ *and whose blocks are the hyperplanes of* Γ *parallel to these subspaces. Then* Δ *is isomorphic to the affine plane* $AG_1(2,4)$ *of order 4.*

Moreover, if $i,j \in \{2,3,4\}$*, the points* $A_1 \cap B_1$*,* $A_i \cap B_1$*,* $A_1 \cap B_j$ *and* $A_i \cap B_j$ *are the four points of a subplane* Δ_0 *of order 2 whose parallelism is induced by that of* Γ*.*

Proof. The proof is straightforward and is omitted. ■

Lemma 2.8 *Let* Γ *be the design* $AG_{n-1}(n,4)$*, where* $n \geq 2$ *and let* Π *be constructed as in Construction 2.1. Then* Π*, and hence* Π^* *also, is isomorphic to* $PG_{2n-2}(2n-1,2)$*. The spread* $\mathcal{L}$ *is a normal line spread of* Π^**. Furthermore, given any two distinct lines of* $\mathcal{L}$*, the intersection of all hyperplanes of* Π^* *containing them is a 3-dimensional subspace on which* $\mathcal{L}$ *induces a spread of five lines.*

Proof. With the notation of Lemma 2.7, given the two non-parallel blocks $A \cup A_i$ and $B \cup B_j$ of Π, where $w \in A \cap B$, let $C \cup C_k$ be the unique block of Γ containing the two points $A_1 \cap B_1$ and $A_i \cap B_j$ of Δ_0, where $w \in C$. Then C contains $A \cap B$ and C_k contains $A_i \cap B_j$.

It follows that in Π the intersection of any two distinct blocks is contained in a third block. So $\Pi \cong PG_{2n-2}(2n-1,2)$ by Result 2.4. This proves the first part of the lemma.

Let L_A and L_B be distinct lines of the line spread $\mathcal{L}$, where $L_A = \{A \cup A_1 - \{w\}, A \cup A_2 - \{w\}, A \cup A_3 - \{w\}\}$, $L_B = \{B \cup B_1 - \{w\}, B \cup B_2 - \{w\}, B \cup B_3 - \{w\}\}$, $\{A, A_1, A_2, A_3\}$ and $\{B, B_1, B_2, B_3\}$ are parallel classes of Γ and $w \in A \cap B$.

A point $P \neq w$ in Γ is on L_A and L_B, considered as blocks of Γ, if, and only if, P, considered as a hyperplane of Π^*, contains both L_A and L_B, considered as lines of the line spread $\mathcal{L}$ of Π^* or, equivalently, P, as a point of Π, is on all blocks of L_A and L_B. That is, $P \in A \cap B$ in Γ and $P \neq w$.

Now, from Lemma 2.7, since Δ is an affine plane of order 4, there are exactly five hyperplanes X of Γ containing $A \cap B$. A and B are two of the five hyperplanes. So, in Π^*, the five lines L_X are in the unique subspace S which is the intersection of all hyperplanes containing L_A and L_B. Hence, the line spread $\mathcal{L}$ of Π^* is normal. ■

Lemma 2.9 *Let Π be the design $PG_{2n-2}(2n-1,2)$ and suppose that $\mathcal{L}$ is a normal line spread of Π^*. Let A and B be points of Π^* on different lines of $\mathcal{L}$. Then the intersection of all hyperplanes of Π^* that contain A and B but neither of the lines of $\mathcal{L}$ on A and B consists of exactly five points of Π^*.*

Proof. Let A and B be points of Π on lines a and b, respectively, of $\mathcal{L}$, where $a \neq b$. Let A also represent the homogeneous coordinates of A in $\Pi^* \cong PG_{2n-2}(2n-1,2)$ and similarly for the other points of Π^*.

Then $C = A + B$ is the third point of the line AB and C is on the line c of $\mathcal{L}$, say. Let S be the 3-dimensional subspace of Π^* generated by the lines a and b.

Thus, a, b and c form a regulus in $S \cong PG_2(3,2)$ Let the opposite regulus of three lines be $\{A, B, C\}$, $\{A', B', C'\}$ and $\{A'', B'', C''\}$. Now, if d and e denote the other two lines in the line spread of S induced by $\mathcal{L}$ containing a, b and c (see Lemma 2.8), then it is not difficult to see, without loss of generality, that $d = \{A + B', B + C', C + A'\}$, $e = \{A + C', B + A', C + B'\}$, and the plane containing the lines c and AB meets d at $B + C'$ and meets e at $A + C'$.

Any hyperplane H of Π^* containing the line AB but neither a nor b must meet S in a plane containing A, B and C but neither A' nor B' . The intersection of H with d is the point $B + C'$; for suppose for instance that $A + B'$ is on H. Then H will contain $A + A + B' = B'$ as well as B. H will therefore contain b, which is a contradiction. The other cases are dealt with similarly. By a similar argument, we can show that H meets e at the point $A + C'$.

It follows that the intersection of all hyperplanes of Π^* containing A and B but neither a nor b consists of the five points A, B, C, $A + C'$ and $B + C'$. This completes the proof. ■

Remark 2.10 With the notation of the proof of Lemma 2.9, it follows that H contains the point $A + (A + C') = C'$. Since C is also on H, H must contain the line c. Therefore, $A + C'$ and $B + C'$ are the points where the plane containing the lines AB and c meets e and d, respectively.

Now we can prove Theorem 2.6.

Proof of Theorem 2.6. That (a) implies (b) follows immediately from Lemma 2.8.

We prove that (b) implies (a). Assume that Π is isomorphic to $PG_{2n-2}(2n-1,2)$ with $n \geq 2$ and that $\mathcal{L}$ is a normal line spread of Π^*.

We show that in the corresponding affine design Γ, the intersection of two non-parallel blocks is contained in exactly five blocks. Hence, by Result 2.3, it will follow that (b) implies (a).

Let A and B be distinct non-parallel blocks of Γ.

Case 1: $A, B \in \mathcal{L}$ and $A \neq B$.
From Lemma 2.8 and the fact that $\mathcal{L}$ is normal, it follows that the intersection of all hyperplanes of Π^* containing A and B contains three further lines of $\mathcal{L}$. Thus, in Γ these correspond to three further blocks (apart from A and B) containing $A \cap B$.

Case 2: $A \in \mathcal{L}$, $B \in \mathcal{B}$ and, in Π, $B \notin A$.
Then a point P in Γ is on A and B if, and only if, P as a hyperplane in Π^* contains A and meets the line L of $\mathcal{L}$ on B in B only.

In Π^*, let z be the plane containing the line A and the point B. Then, z cannot contain L, for otherwise A and L would meet. So, z meets L only at B.

There are five lines of $\mathcal{L}$ in the 3-dimensional subspace of Π^* generated by A and L; denote them by A, L, L_1, L_2 and L_3. For each i, the plane z does not contain L_i since A and L_i do not meet. Hence, z meets L_i in a single point Y_i, say, $(i = 1, 2, 3)$.

Any hyperplane H of Π^* containing the line A and the point B must contain the plane z. If H does not contain L then H cannot contain any of L_1, L_2 or L_3, by the definition of a normal line spread since H contains A.

It follows that the intersection of all hyperplanes containing A and meeting L only at B consists of the line A and the four points B, Y_1, Y_2 and Y_3. Hence, in Γ there are five blocks containing the intersection of A and B.

Case 3: $A, B \in \mathcal{B}$ and $A \neq B$.
From Lemma 2.9, it follows that there are exactly five blocks of Γ containing the intersection of A and B. This completes the proof of Theorem 2.6. ■

Next we consider the issue of isomorphism arising from the constructions. Recall that a *dilatation* α of an affine design is an automorphism fixing every parallel class (as a set). It is *central* if it fixes a point, and such a fixed point is called a *centre* of α. If α is central and is not the trivial automorphism, then

α fixes a unique point. The proofs of the following theorem and corollary are routine and are omitted.

Theorem 2.11 *With the notation of Constructions 2.1 and 2.2, extended in the obvious way, let* Γ *and* Γ' *be affine* 2-$(16\mu, 4\mu, \frac{1}{3}(4\mu-1)$ *designs and let* Π *and* Π' *be the corresponding symmetric Hadamard* 2-$(16\mu-1, 8\mu-1, 4\mu-1)$ *designs. Then there is an isomorphism* $\Gamma \to \Gamma'$ *mapping* w *to* w' *if, and only if, there is an isomorphism* $\Pi \to \Pi'$ *mapping the spread* $\mathcal{L}$ *onto the spread* $\mathcal{L}'$ *(as sets).*

Corollary 2.12 *Any automorphism of* Γ *fixing the point* w *induces a unique automorphism* α *of* Π *fixing* $\mathcal{L}$ *as a set, and conversely,* α *is a central dilatation of* Γ *with centre* w *if, and only if,* α *fixes each line in* $\mathcal{L}$.

3 Line spreads in $PG(5,2)$ and related affine designs

In this section, we give some computational results concerning line spreads in $PG(5,2)$ and the related affine 2-$(64,16,5)$ designs.

As proved in the preceding section, a normal line spread in $PG(5,2)$ yields an affine 2-$(64,16,5)$ design that is isomorphic to the classical design $AG_2(3,4)$ of points and planes in $AG(3,4)$. However, there are many projectively inequivalent line spreads that produce non-isomorphic affine designs with these parameters.

The most interesting among these designs is a non-geometric design $\mathcal{D}$ that has incidence matrix of the same 2-rank as the classical design $AG_2(3,4)$. This design provides a counter-example to the well-known Hamada conjecture [6], which states that the design of the points and subspaces of a given dimension in $AG(n,p^m)$ (p a prime) is characterized as having the minimum p-rank of its incidence matrix amongst those designs of the same parameters (see [1, p.134]). This design $\mathcal{D}$ was originally discovered recently in [7] as a design that contains a symmetric subnet invariant under an elementary Abelian group of order four. However, this new construction by means of spreads suggests that the same method may be used to find more such examples and to provide some insight into the nature of affine designs of minimum rank.

Here is a short outline of the algorithm used for the computations. The points of $PG(5,2)$ are the 6-bit nonzero $(0,1)$-vectors ordered lexicographically:

$$\mathbf{1} = 000001,\ \mathbf{2} = 000010, \ldots, \mathbf{63} = 111111.$$

A line in $PG(5,2)$ is a set of three linearly dependent points (as vectors in

$GF(2)^6$). There are 651 lines that are listed explicitly in Table 3.2. We define a graph G having the 651 lines as vertices, where two vertices are adjacent if the corresponding lines are disjoint. A line spread in $PG(5, 2)$ is just a 21-clique in G. Using a clique finding program written by the third author, over $30,000$ different line spreads were found. Automorphism group considerations suggest that this collection of line spreads is just a small portion of the total number of line spreads in $PG(5, 2)$. Because of their huge number, we did not attempt to classify these $30,000$ spreads up to a projective equivalence. Instead, we classified the resulting affine 2-$(64, 16, 5)$ designs according to the 2-rank of their incidence matrices (clearly, affine designs of different 2-rank correspond to projectively inequivalent spreads).

Affine designs were found of all 2-ranks in the range from 16 to 22. Examples of line spreads that yield designs for each rank between 16 and 22 are listed in Table 3.3. In that table, a line spread is a set of 21 labels of lines as in Table 3.2.

The line spread No. 1 is a normal spread, and the corresponding affine design is isomorphic to the classical design $AG_2(3, 4)$. The spread No. 2 yields a non-geometric (i.e. not isomorphic to $AG_2(3, 4)$) affine design of (supposedly) minimum 2-rank 16. This design can be distinguished from the design $AG_2(3, 4)$ by the order of its full automorphism group, 368640. This exceptional design is isomorphic to the design obtained from net 36 in [7], with a compete list of blocks available at www.math.mtu.edu/∼tonchev/Z2Z2nets.

Remark 3.1 In [7], two exceptional non-geometric affine 2-$(64, 16, 5)$ designs of 2-rank 16 were found. One of these two designs yields a Hadamard design which is not isomorphic to the classical design $PG_4(5, 2)$. It cannot therefore be obtained from line spreads in $PG(5, 2)$.

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The authors also wish to acknowledge an observation by W. M. Kantor that Corollary 2.12 implies an elementary embedding of $\Gamma L(n, 4)$ into $GL(2n, 2)$ and that this can be used to show that the line spread of Theorem 2.6 is unique up to projective equivalence. He also informed us of a nice construction for the normal line spread in the dual of the symmetric design Π of the points and hyperplanes of $PG(2n - 1, 2)$, which is due to R. C. Bose (see [4], for example).

Consider the n-dimensional vector space $V_n(4)$ over $GF(4)$. Then $V_n(4)$ can also be considered as a $2n$-dimensional vector space $V_{2n}(2)$ over $GF(2)$. To each hyperplane H of $V_{2n}(2)$ (i.e. a $(2n - 1)$-dimensional subspace) we can associate a triple of hyperplanes aH, $a \in GF(4)^*$, whose intersection is a hyperplane of $V_n(4)$. Conversely, any hyperplane of $V_n(4)$ arises in this way.

Considering Π as the design whose points and blocks are the 1-dimensional and $(2n-1)$-dimensional subspaces, respectively, of $V_{2n}(2)$, it is easy to see that each of the triples of hyperplanes described above is a line of size 3 in Π^* and the set of such lines is a normal line spread in Π^*.

Kantor also drew our attention to a paper by Lunardon [5] on normal spreads, which gives a very good account of their applications, and a paper by Barlotti and Cofman [2]. A proof that statement (b) implies statement (a) in Theorem 2.6 is implicit in [2] and uses the fact that $PG(2m-1,q)$ may be regarded as a hyperplane of $PG(2m,q)$.

Our approach in this paper and our proofs, which are relatively self-contained, are in contrast to those of [2] and [5] more synthetic in character. Moreover, we have shown that the Rahilly construction generalizes to a construction in design theory which, in the geometric case, is essentially the Bose construction.

Table 3.2 *Lines in* $PG(5,2)$

1	1 2 3	2	1 4 5	3	1 6 7	4	1 8 9	5	1 10 11	6	1 12 13
7	1 14 15	8	1 16 17	9	1 18 19	10	1 20 21	11	1 22 23	12	1 24 25
13	1 26 27	14	1 28 29	15	1 30 31	16	1 32 33	17	1 34 35	18	1 36 37
19	1 38 39	20	1 40 41	21	1 42 43	22	1 44 45	23	1 46 47	24	1 48 49
25	1 50 51	26	1 52 53	27	1 54 55	28	1 56 57	29	1 58 59	30	1 60 61
31	1 62 63	32	2 4 6	33	2 5 7	34	2 8 10	35	2 9 11	36	2 12 14
37	2 13 15	38	2 16 18	39	2 17 19	40	2 20 22	41	2 21 23	42	2 24 26
43	2 25 27	44	2 28 30	45	2 29 31	46	2 32 34	47	2 33 35	48	2 36 38
49	2 37 39	50	2 40 42	51	2 41 43	52	2 44 46	53	2 45 47	54	2 48 50
55	2 49 51	56	2 52 54	57	2 53 55	58	2 56 58	59	2 57 59	60	2 60 62
61	2 61 63	62	3 4 7	63	3 5 6	64	3 8 11	65	3 9 10	66	3 12 15
67	3 13 14	68	3 16 19	69	3 17 18	70	3 20 23	71	3 21 22	72	3 24 27
73	3 25 26	74	3 28 31	75	3 29 30	76	3 32 35	77	3 33 34	78	3 36 39
79	3 37 38	80	3 40 43	81	3 41 42	82	3 44 47	83	3 45 46	84	3 48 51
85	3 49 50	86	3 52 55	87	3 53 54	88	3 56 59	89	3 57 58	90	3 60 63
91	3 61 62	92	4 8 12	93	4 9 13	94	4 10 14	95	4 11 15	96	4 16 20
97	4 17 21	98	4 18 22	99	4 19 23	100	4 24 28	101	4 25 29	102	4 26 30
103	4 27 31	104	4 32 36	105	4 33 37	106	4 34 38	107	4 35 39	108	4 40 44
109	4 41 45	110	4 42 46	111	4 43 47	112	4 48 52	113	4 49 53	114	4 50 54
115	4 51 55	116	4 56 60	117	4 57 61	118	4 58 62	119	4 59 63	120	5 8 13
121	5 9 12	122	5 10 15	123	5 11 14	124	5 16 21	125	5 17 20	126	5 18 23
127	5 19 22	128	5 24 29	129	5 25 28	130	5 26 31	131	5 27 30	132	5 32 37
133	5 33 36	134	5 34 39	135	5 35 38	136	5 40 45	137	5 41 44	138	5 42 47
139	5 43 46	140	5 48 53	141	5 49 52	142	5 50 55	143	5 51 54	144	5 56 61
145	5 57 60	146	5 58 63	147	5 59 62	148	6 8 14	149	6 9 15	150	6 10 12
151	6 11 13	152	6 16 22	153	6 17 23	154	6 18 20	155	6 19 21	156	6 24 30
157	6 25 31	158	6 26 28	159	6 27 29	160	6 32 38	161	6 33 39	162	6 34 36

163	6 35 37	164	6 40 46	165	6 41 47	166	6 42 44	167	6 43 45	168	6 48 54
169	6 49 55	170	6 50 52	171	6 51 53	172	6 56 62	173	6 57 63	174	6 58 60
175	6 59 61	176	7 8 15	177	7 9 14	178	7 10 13	179	7 11 12	180	7 16 23
181	7 17 22	182	7 18 21	183	7 19 20	184	7 24 31	185	7 25 30	186	7 26 29
187	7 27 28	188	7 32 39	189	7 33 38	190	7 34 37	191	7 35 36	192	7 40 47
193	7 41 46	194	7 42 45	195	7 43 44	196	7 48 55	197	7 49 54	198	7 50 53
199	7 51 52	200	7 56 63	201	7 57 62	202	7 58 61	203	7 59 60	204	8 16 24
205	8 17 25	206	8 18 26	207	8 19 27	208	8 20 28	209	8 21 29	210	8 22 30
211	8 23 31	212	8 32 40	213	8 33 41	214	8 34 42	215	8 35 43	216	8 36 44
217	8 37 45	218	8 38 46	219	8 39 47	220	8 48 56	221	8 49 57	222	8 50 58
223	8 51 59	224	8 52 60	225	8 53 61	226	8 54 62	227	8 55 63	228	9 16 25
229	9 17 24	230	9 18 27	231	9 19 26	232	9 20 29	233	9 21 28	234	9 22 31
235	9 23 30	236	9 32 41	237	9 33 40	238	9 34 43	239	9 35 42	240	9 36 45
241	9 37 44	242	9 38 47	243	9 39 46	244	9 48 57	245	9 49 56	246	9 50 59
247	9 51 58	248	9 52 61	249	9 53 60	250	9 54 63	251	9 55 62	252	10 16 26
253	10 17 27	254	10 18 24	255	10 19 25	256	10 20 30	257	10 21 31	258	10 22 28
259	10 23 29	260	10 32 42	261	10 33 43	262	10 34 40	263	10 35 41	264	10 36 46
265	10 37 47	266	10 38 44	267	10 39 45	268	10 48 58	269	10 49 59	270	10 50 56
271	10 51 57	272	10 52 62	273	10 53 63	274	10 54 60	275	10 55 61	276	11 16 27
277	11 17 26	278	11 18 25	279	11 19 24	280	11 20 31	281	11 21 30	282	11 22 29
283	11 23 28	284	11 32 43	285	11 33 42	286	11 34 41	287	11 35 40	288	11 36 47
289	11 37 46	290	11 38 45	291	11 39 44	292	11 48 59	293	11 49 58	294	11 50 57
295	11 51 56	296	11 52 63	297	11 53 62	298	11 54 61	299	11 55 60	300	12 16 28
301	12 17 29	302	12 18 30	303	12 19 31	304	12 20 24	305	12 21 25	306	12 22 26
307	12 23 27	308	12 32 44	309	12 33 45	310	12 34 46	311	12 35 47	312	12 36 40
313	12 37 41	314	12 38 42	315	12 39 43	316	12 48 60	317	12 49 61	318	12 50 62
319	12 51 63	320	12 52 56	321	12 53 57	322	12 54 58	323	12 55 59	324	13 16 29
325	13 17 28	326	13 18 31	327	13 19 30	328	13 20 25	329	13 21 24	330	13 22 27
331	13 23 26	332	13 32 45	333	13 33 44	334	13 34 47	335	13 35 46	336	13 36 41
337	13 37 40	338	13 38 43	339	13 39 42	340	13 48 61	341	13 49 60	342	13 50 63
343	13 51 62	344	13 52 57	345	13 53 56	346	13 54 59	347	13 55 58	348	14 16 30
349	14 17 31	350	14 18 28	351	14 19 29	352	14 20 26	353	14 21 27	354	14 22 24
355	14 23 25	356	14 32 46	357	14 33 47	358	14 34 44	359	14 35 45	360	14 36 42
361	14 37 43	362	14 38 40	363	14 39 41	364	14 48 62	365	14 49 63	366	14 50 60
367	14 51 61	368	14 52 58	369	14 53 59	370	14 54 56	371	14 55 57	372	15 16 31
373	15 17 30	374	15 18 29	375	15 19 28	376	15 20 27	377	15 21 26	378	15 22 25
379	15 23 24	380	15 32 47	381	15 33 46	382	15 34 45	383	15 35 44	384	15 36 43
385	15 37 42	386	15 38 41	387	15 39 40	388	15 48 63	389	15 49 62	390	15 50 61
391	15 51 60	392	15 52 59	393	15 53 58	394	15 54 57	395	15 55 56	396	16 32 48
397	16 33 49	398	16 34 50	399	16 35 51	400	16 36 52	401	16 37 53	402	16 38 54
403	16 39 55	404	16 40 56	405	16 41 57	406	16 42 58	407	16 43 59	408	16 44 60
409	16 45 61	410	16 46 62	411	16 47 63	412	17 32 49	413	17 33 48	414	17 34 51
415	17 35 50	416	17 36 53	417	17 37 52	418	17 38 55	419	17 39 54	420	17 40 57
421	17 41 56	422	17 42 59	423	17 43 58	424	17 44 61	425	17 45 60	426	17 46 63
427	17 47 62	428	18 32 50	429	18 33 51	430	18 34 48	431	18 35 49	432	18 36 54
433	18 37 55	434	18 38 52	435	18 39 53	436	18 40 58	437	18 41 59	438	18 42 56
439	18 43 57	440	18 44 62	441	18 45 63	442	18 46 60	443	18 47 61	444	19 32 51
445	19 33 50	446	19 34 49	447	19 35 48	448	19 36 55	449	19 37 54	450	19 38 53

451	19 39 52	452	19 40 59	453	19 41 58	454	19 42 57	455	19 43 56	456	19 44 63
457	19 45 62	458	19 46 61	459	19 47 60	460	20 32 52	461	20 33 53	462	20 34 54
463	20 35 55	464	20 36 48	465	20 37 49	466	20 38 50	467	20 39 51	468	20 40 60
469	20 41 61	470	20 42 62	471	20 43 63	472	20 44 56	473	20 45 57	474	20 46 58
475	20 47 59	476	21 32 53	477	21 33 52	478	21 34 55	479	21 35 54	480	21 36 49
481	21 37 48	482	21 38 51	483	21 39 50	484	21 40 61	485	21 41 60	486	21 42 63
487	21 43 62	488	21 44 57	489	21 45 56	490	21 46 59	491	21 47 58	492	22 32 54
493	22 33 55	494	22 34 52	495	22 35 53	496	22 36 50	497	22 37 51	498	22 38 48
499	22 39 49	500	22 40 62	501	22 41 63	502	22 42 60	503	22 43 61	504	22 44 58
505	22 45 59	506	22 46 56	507	22 47 57	508	23 32 55	509	23 33 54	510	23 34 53
511	23 35 52	512	23 36 51	513	23 37 50	514	23 38 49	515	23 39 48	516	23 40 63
517	23 41 62	518	23 42 61	519	23 43 60	520	23 44 59	521	23 45 58	522	23 46 57
523	23 47 56	524	24 32 56	525	24 33 57	526	24 34 58	527	24 35 59	528	24 36 60
529	24 37 61	530	24 38 62	531	24 39 63	532	24 40 48	533	24 41 49	534	24 42 50
535	24 43 51	536	24 44 52	537	24 45 53	538	24 46 54	539	24 47 55	540	25 32 57
541	25 33 56	542	25 34 59	543	25 35 58	544	25 36 61	545	25 37 60	546	25 38 63
547	25 39 62	548	25 40 49	549	25 41 48	550	25 42 51	551	25 43 50	552	25 44 53
553	25 45 52	554	25 46 55	555	25 47 54	556	26 32 58	557	26 33 59	558	26 34 56
559	26 35 57	560	26 36 62	561	26 37 63	562	26 38 60	563	26 39 61	564	26 40 50
565	26 41 51	566	26 42 48	567	26 43 49	568	26 44 54	569	26 45 55	570	26 46 52
571	26 47 53	572	27 32 59	573	27 33 58	574	27 34 57	575	27 35 56	576	27 36 63
577	27 37 62	578	27 38 61	579	27 39 60	580	27 40 51	581	27 41 50	582	27 42 49
583	27 43 48	584	27 44 55	585	27 45 54	586	27 46 53	587	27 47 52	588	28 32 60
589	28 33 61	590	28 34 62	591	28 35 63	592	28 36 56	593	28 37 57	594	28 38 58
595	28 39 59	596	28 40 52	597	28 41 53	598	28 42 54	599	28 43 55	600	28 44 48
601	28 45 49	602	28 46 50	603	28 47 51	604	29 32 61	605	29 33 60	606	29 34 63
607	29 35 62	608	29 36 57	609	29 37 56	610	29 38 59	611	29 39 58	612	29 40 53
613	29 41 52	614	29 42 55	615	29 43 54	616	29 44 49	617	29 45 48	618	29 46 51
619	29 47 50	620	30 32 62	621	30 33 63	622	30 34 60	623	30 35 61	624	30 36 58
625	30 37 59	626	30 38 56	627	30 39 57	628	30 40 54	629	30 41 55	630	30 42 52
631	30 43 53	632	30 44 50	633	30 45 51	634	30 46 48	635	30 47 49	636	31 32 63
637	31 33 62	638	31 34 61	639	31 35 60	640	31 36 59	641	31 37 58	642	31 38 57
643	31 39 56	644	31 40 55	645	31 41 54	646	31 42 53	647	31 43 52	648	31 44 51
649	31 45 50	650	31 46 49	651	31 47 48						

Table 3.3 *Affine 2*-$(64, 16, 5)$ *designs from line spreads*

No.	Line spread	2-rank
1	1 92 122 151 177 396 414 431 445 468 486 503 517 536 554 571 585 592 610 627 641	16
2	1 92 122 151 177 396 414 431 445 469 487 502 516 538 552 569 587 595 609 624 642	16
3	1 92 122 151 177 396 414 431 445 468 486 503 517 536 554 571 585 593 611 626 640	17
4	1 92 122 151 177 396 414 431 445 468 486 503 517 536 554 571 585 592 611 625 642	18
5	1 92 122 151 177 396 414 431 445 468 486 503 517 536 555 569 586 593 611 626 640	19
6	1 92 122 151 177 396 414 431 445 468 486 503 517 536 555 569 586 592 611 625 642	20
7	1 92 122 151 177 396 414 431 445 468 486 503 520 537 554 560 587 593 611 626 645	21
8	1 92 122 151 177 396 414 431 445 468 486 503 520 537 554 560 587 594 609 627 645	22

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