On root subsystems and involutions in S_n

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Abstract

Given an involution z in W, where W is the symmetric group of degree n, we study the relation between the subsystems of a root system for W corresponding to certain decreasing subsequences of z and the two-sided Kazhdan-Lusztig cell of W containing z.

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1 Introduction

Kazhdan and Lusztig introduced three equivalence relations $\sim_L$, $\sim_R$ and $\sim_{LR}$ on the elements of a Coxeter group in [11] and called the equivalence classes *left cells*, *right cells* and *two-sided cells*, respectively. Each left cell and each right cell contains at least one involution. Every two-sided cell is a union of left cells and a union of right cells.

We study the finite symmetric group W which is a finite Coxeter group of type A. In this case, each left cell and each right cell contains exactly one involution, and each two-sided cell contains exactly one involution of a special form—a standard parabolic involution—which we describe below. The parabolic involutions form a larger collection of involutions, each of which may be associated with the standard parabolic involution in the two-sided cell containing it in a simple combinatorial way. The Robinson-Schensted-Knuth process provides a combinatorial technique for identifying the standard parabolic involution in the same two-sided cell as a given involution. Our aim is to provide a simpler combinatorial technique for carrying out this identification for a large proportion of the involutions. Not all involutions will be covered by the technique, since they must satisfy a certain length restriction which is not satisfied by all involutions. We have computed the proportion of involutions failing this restriction for symmetric groups of degree ≤ 12

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and found it to be < 0.007. Similarly, the proportion of involutions not covered by our combinatorial technique is < 0.016.

Moreover, if Φ is a system of roots for W, we show that in order to identify the two-sided cell of W containing an involution $z \in W$ it suffices to consider the realizations of z as the longest element of a Young subgroup $W(\Psi)$ with respect to a simple generating system of the subsystem Ψ of Φ which is contained in Φ^+. In particular, for involutions covered by our combinatorial technique, among the Ψ described above there will exist a dominant one with respect to a preorder we define below.

2 Background notation, terminology and results

Let W denote the symmetric group S_n, where n is an arbitrary positive integer, acting on $\{1,\ldots,n\}$. Then W is a Coxeter group with Coxeter system (W,S) where $S = \{s_1,\ldots,s_{n-1}\}$ and s_i is the transposition $(i,i+1)$ for $1 \le i \le n-1$. The corresponding Coxeter graph is $\overset{s_1}{\circ}\!\!-\!\!\!-\!\!\!-\overset{s_2}{\circ}\!\!-\!\!\!- \cdots -\!\!\!-\overset{s_{n-1}}{\circ}$. Moreover, W has the presentation $\langle s_i : s_i^2 = 1, (s_is_{i+1})^3 = 1 \text{ and } (s_is_j)^2 = 1 \text{ for all } i,j \in \{1,\ldots,n-1\} \text{ with } |i-j| > 1\rangle$.

For each subset $J \subseteq S$, the subgroup W_J generated by J is called a *parabolic subgroup* of W. It has a Coxeter system (W_J, J). Its length function l_J is that induced from l. It has a unique longest element w_J. By tradition, w_0 is written for w_S.

For each composition $\lambda = (\lambda_1,\ldots,\lambda_r)$ of n with r parts, with $\lambda_i > 0$, for $1 \le i \le r$, there is a standard parabolic subgroup of W whose Coxeter generator set $J(\lambda)$ is given by $J(\lambda) = S\backslash\{s_{\lambda_1}, s_{\lambda_1+\lambda_2},\ldots,s_{\lambda_1+\ldots+\lambda_{r-1}}\}$. The longest element $w_{J(\lambda)}$ of $W_{J(\lambda)}$ can be described in two-row form by

$$w_{J(\lambda)} = \begin{pmatrix} \ldots & \widehat{\lambda}_{i-1}+1 & \ldots & \widehat{\lambda}_i & \widehat{\lambda}_i+1 & \ldots & \widehat{\lambda}_{i+1} & \ldots \\ \ldots & \widehat{\lambda}_i & \ldots & \widehat{\lambda}_{i-1}+1 & \widehat{\lambda}_{i+1} & \ldots & \widehat{\lambda}_i+1 & \ldots \end{pmatrix}$$

where $\widehat{\lambda}_0 = 0$, $\widehat{\lambda}_r = n$, and $\widehat{\lambda}_i = \lambda_i + \widehat{\lambda}_{i-1}$ for $i = 1,\ldots,r-1$. The *conjugate* partition λ' of λ is defined by $\lambda'_i = |\{j : \lambda_j \ge i\}|$ for $i \ge 1$. We will denote the number of parts of λ' by r'. Thus, $r = \lambda'_1$ and $r' = \max\{\lambda_i : 1 \le i \le r\}$.

We use the notions of λ-*diagram*, λ-*tableaux* and associated terminology as in Dipper and James [2]; see also Fulton [4] and Sagan [15] for the corresponding terminology when λ is a partition. In particular, a λ-tableau is *row-standard* if it is increasing on rows, *column-standard* if it is increasing on columns, and *standard* if it is increasing on rows and columns. Also, if $\mathcal{T}$ is a λ-tableau, we refer to λ as the *shape* of $\mathcal{T}$ and denote it by $\operatorname{sh}\mathcal{T}$.

W acts on the set of λ-tableaux in the obvious way—if $w \in W$, an entry i is replaced by iw and tw denotes the tableau resulting from the action of w on the tableau t. This action on λ-tableaux is the action by letter permutations of Dipper and James [2, p.21]. If $x, y \in W$, we say that x is a *prefix* of y if $y = u_1u_2\ldots u_p$ where $u_i \in S$ for $i = 1,\ldots,p$, $p = l(y)$ and $x = u_1u_2\ldots u_r$, for some $r \le p$. The prefix relation corresponds to the *weak Bruhat order* in [2].

From the general theory of Coxeter groups, every parabolic subgroup W_J of W has a *distinguished* set of right coset representatives X_J whose properties are listed in the next result.

Result 2.1. ([7, Proposition 2.1.1 and Lemma 2.2.1]) There is a special set of right coset representatives X_J associated with each parabolic subgroup W_J. An element of X_J is the unique element of minimum length in its coset. Moreover, if $w = vx$ where $v \in W_J$ and $x \in X_J$ then $l(w) = l(v) + l(x)$. Also, $X_J = \{w \in W : L(w) \subseteq S - J\}$ where $L(w) = \{s \in S : l(sw) < l(w)\}$ and, if d_J denotes the longest element in X_J, then X_J is the set of prefixes of d_J.

We construct a special λ-tableau t^λ, where t^λ is obtained by filling in the λ-diagram with $1, \ldots, n$ by rows from top to bottom, filling each row from left to right.

In the case of the symmetric group, Dipper and James [2] characterise $X_{J(\lambda)}$ as follows:

Result 2.2. ([2, Lemma 1.1]) $X_{J(\lambda)} = \{w \in W : t^\lambda w \text{ is row-standard}\}$.

3 Covers

Let $w \in W$ and let $\nu = (\nu_1, \ldots, \nu_k)$ be a partition of n into k parts. By a *decreasing cover of type* ν for w, we mean a set of k disjoint decreasing subsequences appearing in the row-form of w so that the union of the elements in these subsequences is $\{1, \ldots, n\}$ and the lengths of the subsequences (from longest to shortest) are $\nu_1, \ldots, \nu_k$. Similarly we can define an *increasing cover of type* ν for w.

A decreasing cover of w is said to be *symmetric* if, for any $i \in \{1, \ldots, n\}$, i and iw are in the same subsequence of the cover.

In the terminology of Sagan [15], a decreasing cover of w of type ν is a k-decreasing subsequence of w involving all elements in $\{1, \ldots, n\}$. More generally, a *k-decreasing subsequence* w^I of w is the restriction of w to $I \subseteq \{1, \ldots, n\}$, where I has a partition into k disjoint subsets $I_1, \ldots, I_k$, and w^{I_j} is a decreasing subsequence of w for each j.

Example 3.1. Let $w = [\,7\,,\,8\,,\,5\,,\,9\,,\,3\,,\,6\,,\,1\,,\,2\,,\,4\,] \in S_9$. Then, $\{(7,5,3,1),\ (8,6,2),\ (9,4)\}$ and $\{(7,5,3,2),\ (9,6,1),\ (8,4)\}$ are decreasing covers of type $(4,3,2)$ for w, the first being a symmetric decreasing cover. Also, $\{(7,8,9),\ (1,2,4),\ (3,6),\ (5)\}$ is an increasing cover of type $(3,3,2,1)$ for w.

We note that only involutions, or the identity, can have symmetric decreasing covers.

Lemma 3.2. Let $w \in W$ and suppose that w has a symmetric decreasing cover. Then $w^2 = 1$.

Proof. Let $i_1, \ldots, i_r$ be a subsequence of the symmetric decreasing cover, and let $j_k = i_k w^{-1}$ for $k = 1, \ldots, r$. Then the sequence $j_1, \ldots, j_r$ is increasing and $\{i_1, \ldots, i_r\} = \{j_1, \ldots, j_r\}$. Hence, $i_k = j_{r-k+1}$ for $k = 1, \ldots, r$. So, $i_k w^2 = i_{r-k+1} w = i_k$ for $k = 1, \ldots, r$. Since this is true for all subsequences of the cover, $iw^2 = i$ for $i = 1, \ldots, n$. □

We see that the symmetric decreasing covers of an involution are optimal among all decreasing covers in the sense described in the following theorem.

Theorem 3.3. *Let $w \in W$ be an involution which has a k-decreasing subsequence w^I of length l. Then w has a symmetric k'-decreasing subsequence of length at least l for some $k' \leq k$.*

Proof. Let w^I be a k-decreasing subsequence of w of length l. Thus, $I \subseteq \{1, \ldots, n\}$, I has a partition into k disjoint non-empty subsets $I_1, \ldots, I_k$, and w^{I_j} is a decreasing subsequence of w for each j.

If $1 \leq j \leq k$, $a, b \in I_j w$ and $a < b$, then $bw < aw$. So, $w^{I_j w}$ is a decreasing subsequence of w. Hence, w^{Iw} is a k-decreasing subsequence of w.

Let $I_= = \{i \in I : i = iw\}$, $I_< = \{i \in I : i < iw\}$, and $I_> = \{i \in I : i > iw\}$. Observe that $|Iw_<| > |Iw_>|$ if and only if $|I_<| < |I_>|$, so we may assume without loss of generality that $|I_<| \geq |I_>|$

Let $J = \{j : 1 \leq j \leq k \text{ and } \{i \in I_j : i \leq iw\} \neq \emptyset\}$ and let $k' = |J|$.

For each $j \in J$, let a_j be the maximum element of $\{i \in I_j : i \leq iw\}$ and let $K_j = I_j \cap (I_< \cup I_=)$. Then w^{K_j} and $w^{K_j w}$ are decreasing subsequences of w. Also, if $i \in K_j w$ then $iw \in K_j$; so, $iw \leq a_j$ and $i \geq a_j w \geq a_j$. Hence, $K_j \cap K_j w = \{a_j\} \cap I_=$. Moreover, if we let $L_j = K_j \cup K_j w$, then w^{L_j} is a symmetric decreasing subsequence of w.

Since the k' sets K_j, $j \in J$, are pairwise disjoint, so are the k' sets $K_j w$, $j \in J$.

Suppose $r, s \in J$, $r \neq s$ and let $x \in K_r$. If $xw = x$, then $x \in K_r w$ so $x \notin K_s w$. If $x < xw$, then again $x \notin K_s w$ because $yw \leq y$ for all $y \in K_s w$. Hence $\bigcup_{j \in J} K_j$ is disjoint from $\bigcup_{j \in J} (K_j w \backslash \{a_j\})$. Letting $L = \bigcup_{j \in J} L_j$, w^L is a symmetric k'-decreasing subsequence of w.

Finally, since $L = I_< \cup I_= \cup (I_<)w$, it follows that $|L| = 2|I_<| + |I_=| \geq |I_<| + |I_=| + |I_>| = |I|$ as required. □

Recall from Section 3 of [13] that, for a partition ν of n, $w_{J(\nu)}$ has a unique decreasing cover P^ν of type ν, and this cover is also symmetric. This statement generalizes easily to compositions. Thus, for a composition λ of n, $w_{J(\lambda)}$ has a unique decreasing cover, denoted by P^λ, of type λ'', and this cover is also symmetric. Moreover, we have the following lemma.

Lemma 3.4. Let λ be a composition of n and let $\nu = \lambda''$. If $w \in W$ has a decreasing cover P of type ν, then $P = P^\nu e$ for some $e \in X_{J(\nu)}$ and there is an element $d \in X_{J(\lambda)}$ such that $P^\lambda d = P^\nu e$. Moreover, there is an element $f \in X_{J(\nu)}$ such that $w = f^{-1} w_{J(\nu)} e$.

Proof. Let $(p_{i,1}, \ldots, p_{i,\nu_i})$ and $(q_{i,1}, \ldots, q_{i,\nu_i})$ be the i-th subsequences in P^ν and P, respectively, and let $r_{i,j} = q_{i,j} w^{-1}$ for all i and j. Define e by $p_{i,j} \mapsto q_{i,j}$ for all i and j, and f by $p_{i,j} \mapsto r_{i,\nu_i+1-j}$ for all i and j. Then $w = f^{-1} w_{J(\nu)} e$. Since e is increasing on each set $\{p_{i,\nu_i}, \ldots, p_{i,1}\}$, $e \in X_{J(\nu)}$. Since f is increasing on each set $\{p_{i,\nu_i}, \ldots, p_{i,1}\}$, $f \in X_{J(\nu)}$.

Since λ is a rearrangement of ν, the preceding argument gives $P = P^\lambda d$ for some $d \in X_{J(\lambda)}$, and $w = g^{-1} w_{J(\lambda)} d$ for some $g \in X_{J(\lambda)}$. □

So, a decreasing cover for an element of W does not determine the pair (λ, d) uniquely. Note that if (λ, d) and (λ, d') correspond to a cover P of type λ'', then $d'd^{-1}$ centralizes $w_{J(\lambda)}$. In particular, if ν has parts of the same size, it is possible to find distinct elements e and e' with (ν, e) and (ν, e') corresponding to P.

Example 3.5. Let $w = [2, 1, 5, 7, 3, 6, 4]$ and $P = \{(7, 6, 4), (2, 1), (5, 3)\}$. We may take $e = [4, 6, 7, 1, 2, 3, 5]$ and $e' = [4, 6, 7, 3, 5, 1, 2]$. Then $e, e' \in X_{J(\nu)}$ and $P = P^\nu e = P^\nu e'$.

Following [2] we introduce a preorder $\preceq$ on compositions of n. We write $\lambda \preceq \mu$, for any two compositions λ and μ of n, if $\lambda'' \trianglelefteq \mu''$ where $\trianglelefteq$ denotes the usual partial order of dominance on partitions. We have a corresponding preorder $\preceq$ of decreasing covers of an element $w \in W$. Now suppose that λ and μ are partitions and P and Q are decreasing covers of w of types λ and μ, respectively. We write $P \preceq Q$ if $\lambda \trianglelefteq \mu$.

We recall some details of the Robinson-Schensted correspondence, a bijection of the set of elements of the symmetric group W to pairs $(\mathcal{P}, \mathcal{Q})$, where $\mathcal{P}$ and $\mathcal{Q}$ are ν-tableaux for some partition ν of n. See Fulton [4] and Sagan [15] for a good description of this correspondence. We write the pair of tableaux corresponding to an element $w \in W$ as $(\mathcal{P}(w), \mathcal{Q}(w))$ and recall that $\mathcal{Q}(w) = \mathcal{P}(w^{-1})$. We define the *shape* of w as the shape of $\mathcal{P}(w)$, and denote it by $\operatorname{sh} w$. For example, for a composition λ, $\operatorname{sh} w_{J(\lambda)} = \lambda'$. We will say that two elements $w_1, w_2 \in W$ are *shape-equivalent* if $\operatorname{sh} w_1 = \operatorname{sh} w_2$ and, in this case, we will write $w_1 \sim_{\text{sh}} w_2$. The $\sim_{\text{sh}}$-equivalence classes are the two-sided Kazhdan-Lusztig cells of W, as described in [11]. (See also [6, Corollary 5.6].)

For our next result, we need the following theorem of Greene [8]. We refer the reader to Sagan [15] for a proof.

Theorem 3.6. (see [15, Theorem 3.5.3]) *Let $w = [w_1, w_2, \dots, w_n] \in W$ and let $\operatorname{sh} w = \nu$. Let $\xi_0 = 0$, and for each $k \geq 1$, let ξ_k be the maximum length of a k-decreasing subsequence of w. Similarly, let $\eta_0 = 0$, and for each $k \geq 1$, let η_k be the maximum length of a k-increasing subsequence of w. Let α and β be the compositions defined by $\alpha_k = \xi_k - \xi_{k-1}$ and $\beta_k = \eta_k - \eta_{k-1}$ for $k \geq 1$, where trailing zeros are ignored. Then $\alpha = \nu'$ and $\beta = \nu$.*

We can now relate the type of any increasing or decreasing cover of an element of W to its shape.

Corollary 3.7. Let $w \in W$ and $\operatorname{sh} w = \nu$. Suppose that w has a decreasing cover of type λ and an increasing cover of type μ. Then, $\lambda \trianglelefteq \nu'$ and $\mu \trianglelefteq \nu$.

Proof. In the notation of Theorem 3.6, for each $k \geq 1$, $\sum_{i=1}^{k} \lambda_i \leq \xi_k = \sum_{i=1}^{k} \alpha_i$. Since $\alpha = \nu'$, we have $\lambda \trianglelefteq \nu'$. In a similar way we get $\mu \trianglelefteq \nu$. □

In [9, 1.4.16], it is shown that the partitions of n form a lattice with respect to the dominance order. Hence, any set of partitions of n has a supremum and an infimum. It is clear from Theorem 3.6 that, for an element $w \in W$, the conjugate of $\operatorname{sh} w$ dominates the type of every decreasing cover of w and, hence, it dominates the supremum of these types. Making use of [9, Theorem 1.4.10], we see that each neighbour of $(\operatorname{sh} w)'$, which is dominated by $(\operatorname{sh} w)'$, fails to dominate the type of at least one decreasing cover of w. Hence, $(\operatorname{sh} w)'$ is the supremum of the types of the decreasing covers of w.

Combining this result with Theorem 3.3, we obtain the following theorem.

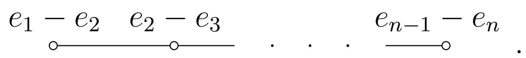
Theorem 3.8. *If $w \in W$ is an involution, then $\operatorname{sh} w$ is the conjugate of the supremum (with respect to the usual partial order of dominance on partitions) of the types of the symmetric decreasing covers of w.*

Proof. Let w be an involution and suppose that $(\operatorname{sh} w)' = (\lambda_1, \dots, \lambda_r)$. Choose k such that $1 \leq k \leq r$. We know from Theorem 3.6 that w has a k-decreasing subsequence of length $\lambda_1 + \ldots + \lambda_k$. Since $\lambda_1 + \ldots + \lambda_k$ is the length of w's longest k-decreasing subsequence, it follows from Proposition 3.3 that w has a symmetric k-decreasing subsequence of length $\lambda_1 + \ldots + \lambda_k$. We can easily 'complete' this latter symmetric k-decreasing subsequence of w into a symmetric cover of type $(\mu_1, \ldots, \mu_s)$ for w where $\mu_1 + \ldots + \mu_k = \lambda_1 + \ldots + \lambda_k$. It follows that each neighbour of $(\operatorname{sh} w)'$, which is dominated by $(\operatorname{sh} w)'$, fails to dominate the type of at least one symmetric decreasing cover of w. This is enough to complete the proof. □

4 Root subsystems

In this section we study the relation between increasing and decreasing subsequences of an element of W and certain root subsystems of a root system Φ of W.

We may take $\Phi = \{e_i - e_j : 1 \leq i, j \leq n, i \neq j\}$ where $\{e_1 \ldots, e_n\}$ is an orthogonal basis of an n-dimensional Euclidean space; see Chapter 3 of Carter [1]. The Coxeter generator s_i corresponds to the reflection in the hyperplane orthogonal to $e_i - e_{i+1}$ through the origin. The Dynkin diagram corresponding to this root system is



The subset $\Pi = \{e_i - e_{i+1} : 1 \leq i \leq n-1\}$ of Φ is a *simple* (or *fundamental*) root system for Φ; that is, it is a basis for the space generated by Φ and every element of Φ or its negative is a linear combination of Π with non-negative integers. Thus, $\Phi = \Phi^+ \cup \Phi^-$, where the set of *positive* roots Φ^+ is the set of non-negative linear combinations of Π in Φ and the set of *negative* roots Φ^- is $-\Phi^+$. In our case, $\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq n\}$.

If Σ is any simple system of roots in Φ, we write Φ_Σ for the subsystem of Φ generated by Σ. In particular, $\Phi = \Phi_\Pi$. We also write Φ_Σ^+ and Φ_Σ^- for the positive and negative roots respectively in Φ_Σ with respect to the simple system Σ. In particular, $\Phi^+ = \Phi_\Pi^+$ and $\Phi^- = \Phi_\Pi^-$. We denote by $W(\Phi_\Sigma)$ the Weyl group generated by the reflections associated with the root system Φ_Σ.

For any element $w \in W(\Phi_\Sigma)$, we define $N_\Sigma^+(w) = \{\alpha \in \Phi_\Sigma^+ : \alpha w \in \Phi_\Sigma^-\}$ and $N_\Sigma(w) = N_\Sigma^+(w) \cup -N_\Sigma^+(w)$. We write $N^+(w)$ and $N(w)$ for $N_\Pi^+(w)$ and $N_\Pi(w)$ respectively. The length of w is related to the size of $N(w)$ in the following result; see Carter [1, Theorem 2.2.2]

Result 4.1. If $w \in W$, then $2l(w) = |N(w)|$.

In the case of involutions, we have the following elementary lemma.

Lemma 4.2. If $w \in W$ is an involution, then $N(w)w = N(w)$.

Proof. Let $e_i - e_j \in N(w)$ with $i < j$. Then $iw > jw$ and $(iw)w < (jw)w$. So, $(e_i - e_j)w = e_{iw} - e_{jw} \in N(w)$. □

Let λ be a composition of n and let $\Sigma(\lambda)$ denote the simple system of roots in Π corresponding to the subset $J(\lambda)$ of S, the Coxeter generators of W; that is, $\Sigma(\lambda) =$

$\{e_i - e_{i+1} : s_i \in J(\lambda)\}$. Then $N(w_{J(\lambda)})$ is just the root subsystem $\Phi_{\Sigma(\lambda)}$ of Φ. Moreover, $|\Phi_{\Sigma(\lambda)}| = \sum_{i=1}^{r} \binom{\lambda_i}{2}$.

Let $d \in X_{J(\lambda)}$. The set $d^{-1}J(\lambda)d$ is a system of Coxeter generators for the group $d^{-1}W_{J(\lambda)}d$ corresponding to the simple system of roots $\Sigma(\lambda)d$ which generates the root subsystem $\Phi_{\Sigma(\lambda)d} = \Phi_{\Sigma(\lambda)}d$.

The following result arises from the classification of subsystems of root systems given by Dynkin [3, Theorems 5.2 and 5.3]

Result 4.3. Any subsystem Ψ of Φ is of the form $\Phi_{\Sigma(\lambda)}d$ for some composition λ of n and $d \in X_{J(\lambda)}$. Moreover, Ψ has the simple subsystem $\Sigma(\lambda)d$ which is a subset of Φ^+. Also, if $\Phi_{\Sigma(\lambda)}d = \Phi_{\Sigma(\mu)}e$ for compositions λ and μ of n with $d \in X_{J(\lambda)}$ and $e \in X_{J(\mu)}$, then λ and μ are rearrangements of one another.

Consequently, $d^{-1}J(\lambda)d$ is a system of Coxeter generators for the subgroup $W(\Phi_{\Sigma(\lambda)d})$ and d maps every root in $\Phi_{\Sigma(\lambda)} \cap \Phi^+$ to a root in Φ^+.

Example 4.4. Let $n = 8$ and $\lambda = (3, 1, 4)$. Then $J(\lambda) = \{s_1, s_2, s_5, s_6, s_7\}$. Let $d = [1, 2, 4, 5, 3, 6, 7, 8] \in X_{J(\lambda)}$. Then $\Sigma(\lambda) = \{e_1 - e_2, e_2 - e_3, e_5 - e_6, e_6 - e_7, e_7 - e_8\}$ and $\Sigma(\lambda)d = \{e_1 - e_2, e_2 - e_4, e_3 - e_6, e_6 - e_7, e_7 - e_8\}$.

Consider the involution $z = (1, 4)(3, 8)(6, 7) = d^{-1}w_{J(\lambda)}d \in W(\Phi_{\Sigma(\lambda)d})$. Observe that z maps every root in $\Sigma(\lambda)d$ to a root in $-\Sigma(\lambda)d$ and every root in $(\Phi_{\Sigma(\lambda)}d)^+ = \Phi_{\Sigma(\lambda)d} \cap \Phi^+$ to a root in $(\Phi_{\Sigma(\lambda)}d)^-$. Consequently, z is the longest element of $W(\Phi_{\Sigma(\lambda)d})$ with respect to the length function determined by its simple generating system $\Sigma(\lambda)d$.

Since the decreasing cover $P^\lambda = \{(8, 7, 6, 5), (3, 2, 1), (4)\}$ of $w_{J(\lambda)}$ is symmetric and d is a distinguished right coset representative of $W_{J(\lambda))}$, $P^\lambda d = \{(8, 7, 6, 3), (4, 2, 1), (5)\}$ is a symmetric decreasing cover of $d^{-1}w_{J(\lambda)}d$.

The following result, which shows that symmetric decreasing covers of an involution z are in bijective correspondence with root subsystems Ψ of Φ such that z is the longest element of $W(\Psi)$ with respect to the length function determined by a simple generating system of roots in Ψ contained in Φ^+, may be easily established.

Lemma 4.5. Let $z \in W$ be an involution and let $d \in X_{J(\lambda)}$ where λ denotes a composition of n. Then statements (i), (ii) and (iii) are equivalent.

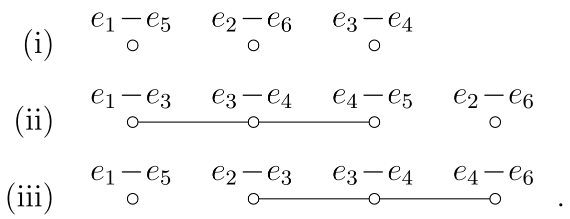
(i) $P^\lambda d$ is a symmetric decreasing cover for z.
(ii) $z \in W(\Phi_{\Sigma(\lambda)}d)$, $\Phi_{\Sigma(\lambda)}d \subseteq N(z)$ and z stabilizes $\Phi_{\Sigma(\lambda)}d$.
(iii) $z = d^{-1}w_{J(\lambda)}d$.

Let $w \in W$. It is easy to see that $P^\lambda d$ is a decreasing cover for w if, and only if, $\Phi_{\Sigma(\lambda)d} \subseteq N(w^{-1})$. In particular, if w is an involution we see that there is a bijective correspondence between decreasing covers of w and root subsystems contained in $N(w)$ since any root subsystem contains a simple system lying entirely in Φ^+.

As in the case of covers, the dominance preorder $\preceq$ of compositions induces a preorder on subsystems Ψ of Φ since any such subsystem is of the form $\Phi_{\Sigma(\lambda)d}$. Note that if λ_1 and λ_2 are compositions of n and λ_2 is not a rearrangement of λ_1, then $\Phi_{\Sigma(\lambda_2)d_2} \neq \Phi_{\Sigma(\lambda_1)d_1}$.

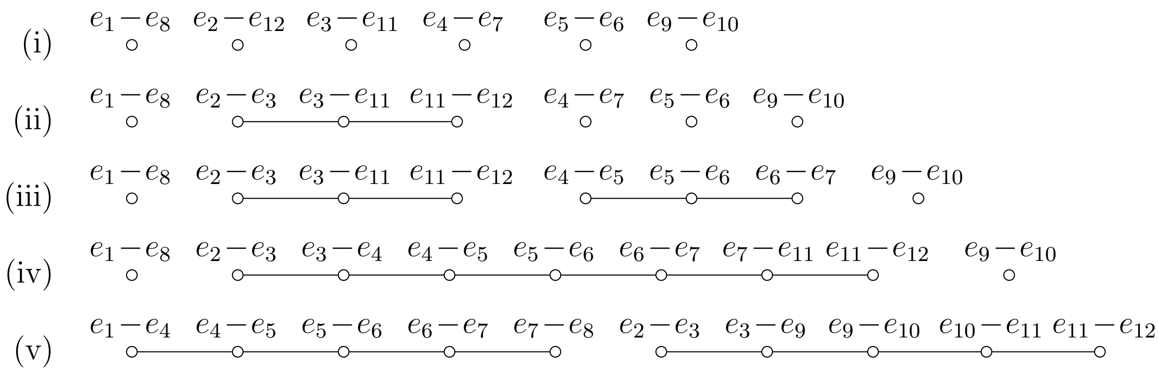
We illustrate these comments with some examples of realizations of an involution z as the longest element of a Young subgroup $W(\Psi)$, with respect to a simple generating system for Ψ which is contained in Φ^+, for various root subsystems $\Psi \subseteq N(z)$.

Example 4.6. Let $z_1 = (1,5)(2,6)(3,4)$. Among the realizations of z_1 as the longest element of a subsystem are those given by the subsystems with the following simple systems contained in Φ^+:



Note that $z_1 = [5,6,4,3,1,2]$ and that these simple systems correspond to the symmetric decreasing covers $\{(5,1),\ (6,2),\ (4,3)\}$, $\{(5,4,3,1),\ (6,2)\}$ and $\{(6,4,3,2),\ (5,1)\}$ of z_1 of shapes $(2,2,2)$, $(4,2)$ and $(4,2)$, respectively. For this example, the shape $(4,2)$ dominates the shapes of all the decreasing covers of z_1. Hence, $(2,2,1,1)$, the conjugate of $(4,2)$, is the shape of the Robinson-Schensted tableau $\mathcal{P}(z_1)$ of z_1.

Example 4.7. Let $z_2 = (1,8)(2,12)(3,11)(4,7)(5,6)(9,10)$. Among the realizations of z_2 as the longest element of a subsystem are those given by the subsystems with the following simple systems contained in Φ^+:



In this example, $z_2 = [8,12,11,7,6,5,4,1,10,9,3,2]$ and the simple systems (i)-(v) correspond to the symmetric decreasing covers $\{(8,1),\ (12,2),\ (11,3),\ (7,4),\ (6,5),\ (10,9)\}$, $\{(12,11,3,2),\ (8,1),\ (7,4),\ (6,5),\ (10,9)\}$ and $\{(12,11,3,2),\ (7,6,5,4),\ (8,1),\ (10,9)\}$ and $\{(12,11,7,6,5,4,3,2),\ (8,1),\ (10,9)\}$ and $\{(8,7,6,5,4,1),\ (12,11,10,9,3,2)\}$ of z_2 of shapes $(2,2,2,2,2,2)$, $(4,2,2,2,2)$, $(4,4,2,2)$, $(8,2,2)$ and $(6,6)$, respectively. The element z_2 has no descreasing cover with a shape which dominates the shapes of all its the decreasing covers. Simple systems (iv), of shape (8,2,2), and (v), of shape (6,6), are maximal among all simple systems contained in Φ^+ corresponding to realizations of z_2 as their longest element with respect to the preorder $\preceq$ of subsystems. Observe that $\operatorname{sh} z_2 = (2^2, 1^2)$, the conjugate of (8,4) which is the supremum of the shapes (8,2,2) and (6,6). (Compare with Theorem 3.8 and Lemma 4.5.)

Remark. In the example above, observe that z_2 has a decreasing cover of type $(7,4,1)$, namely $\{(8,7,6,5,4,3,2), (12,11,10,9), (1)\}$. However, none of the symmetric decreasing covers of z_2 has shape λ with $\lambda \trianglerighteq (7,4,1)$. It follows that our hoped-for extension of Proposition 3.3 to "whenever an involution w has a decreasing cover of type ν, then w has a symmetric decreasing cover of type λ with $\lambda \trianglerighteq \nu$" is not true.

For the remainder of this section we will look at some consequences in the special case where an involution has a dominant symmetric decreasing cover. It is easy to see that any involution in W is conjugate to a parabolic involution, see [14] for a generalization of this result to arbitrary Coxeter groups. In Corollary 4.10 we compare various realizations (that satisfy an additional length restriction) of an involution as a conjugate of a parabolic involution. First we show the following preliminary result.

Lemma 4.8. Let λ and μ be compositions of n with $\lambda \preceq \mu$. Then (i) $\left|\Phi_{\Sigma(\lambda)}\right| \leq \left|\Phi_{\Sigma(\mu)}\right|$, and (ii) $\left|W_{J(\lambda)}\right| \leq \left|W_{J(\mu)}\right|$, with equality if, and only if, λ is a rearrangement of μ.

Proof. If λ is a rearrangement of μ, it is immediate that we have equality in (i) and (ii). Now suppose that λ is not a rearrangement of μ. Then $\lambda'' \trianglelefteq \mu''$ and $\lambda'' \neq \mu''$. Using [9, Theorem 1.4.10], we see that for some $k \geq 1$ there is a sequence of partitions $\lambda'' = \alpha^{(0)} \trianglelefteq \alpha^{(1)} \trianglelefteq \ldots \trianglelefteq \alpha^{(k)} = \mu''$ where, for each $i = 1, \ldots, k$, there is a pair of integers j_1 and $j_2 > j_1$ such that $\alpha_{j_1}^{(i)} = \alpha_{j_1}^{(i-1)} + 1$, $\alpha_{j_2}^{(i)} = \alpha_{j_2}^{(i-1)} - 1$, $\alpha_j^{(i)} = \alpha_j^{(i-1)}$ for $j \neq j_1, j_2$, and either $j_1 = j_2 - 1$ or $\alpha_{j_1}^{(i-1)} = \alpha_{j_2}^{(i-1)}$. Hence, $l(w_{J(\alpha^{(i)})}) - l(w_{J(\alpha^{(i-1)})}) = \binom{\alpha_{j_1}^{(i-1)}+1}{2} + \binom{\alpha_{j_2}^{(i-1)}-1}{2} - \binom{\alpha_{j_1}^{(i-1)}}{2} - \binom{\alpha_{j_2}^{(i-1)}}{2} = \alpha_{j_1}^{(i-1)} - \alpha_{j_2}^{(i-1)} + 1 \geq 1$. Hence, $l(w_{J(\lambda'')}) < l(w_{J(\mu'')})$. Also, $|W_{J(\alpha^{(i)})})|/|W_{J(\alpha^{(i-1)})})| = (\alpha_{j_1}^{(i-1)} + 1)/\alpha_{j_2}^{(i-1)} > 1$. Hence, $|W_{J(\lambda'')}| < |W_{J(\mu'')}|$. □

Theorem 4.9. *Let $z \in W$ be an involution. Suppose that for some composition λ of n and some element $d \in X_{J(\lambda)}$, $P^\lambda d$ is a symmetric decreasing cover for z which dominates all symmetric decreasing covers for z. If Ψ is a root subsystem contained in $N(z)$ then (i) $\left|\Phi_{\Sigma(\lambda)d}\right| \geq |\Psi|$, (ii) $\left|W(\Phi_{\Sigma(\lambda)d})\right| = \left|W_{J(\lambda)}\right| \geq |W(\Psi)|$. In (i) and (ii), we get equality if, and only if, Ψ is of type λ''.*

Proof. From the hypothesis, $\Phi_{\Sigma(\lambda)d} \subseteq N(z)$. Moreover, $\operatorname{sh} z = \lambda'$ in view of Theorem 3.8. Any subsystem $\Psi \subseteq N(z)$ will be generated by a simple system Σ in $\Psi \cap \Phi^+$, Also, Σ gives a decreasing cover for z of type ν and $\nu \preceq \lambda$. The result now follows from Lemma 4.8. □

Corollary 4.10. Let z, λ and d satisfy the hypothesis of Theorem 4.9. Suppose further that (i) $z = e^{-1} w_{J(\nu)} e$ for some composition ν of n and $e \in W$ for which $l(z) = 2l(e) + l(w_{J(\nu)})$, and (ii) $l(z) = 2l(d) + l(w_{J(\lambda)})$. Then $\nu'' \trianglelefteq \lambda''$ and, if $\nu'' \neq \lambda''$, then $l(e) > l(d)$.

Proof. $P^\nu e$ is a decreasing cover for z of type ν''. From the hypothesis, λ'' dominates the shape of any decreasing cover of z. Hence, $\nu'' \trianglelefteq \lambda''$. Since $\Phi_{\Sigma(\lambda)d}$ and $\Phi_{\Sigma(\nu)e}$ are contained in $N(z)$, and since $l(d) = \frac{1}{4}\left(|N(z)| - \left|\Phi_{\Sigma(\lambda)d}\right|\right)$ and $l(e) = \frac{1}{4}\left(|N(z)| - \left|\Phi_{\Sigma(\nu)e}\right|\right)$, the last part of the corollary follows from Theorem 4.9. □

Observe that the involution z in Corollary 4.10 is the longest element of the Young subgroup $d^{-1} W_{J(\lambda)} d$ also with respect to the generating system Π.

5 (λ, μ)-diagrams

We define the notion of a *diagram* in two stages. First, let $\nu = (\nu_1, \ldots, \nu_r)$ be a partition. Then, the diagram D_ν associated with ν is the array of points $\{(i, j) : 1 \leq i \leq r, 1 \leq j \leq$

$\nu_i\}$. We refer to the elements of this array as the *nodes* of the diagram. In pictorial form, the points are listed with first index indicating the row, and second index indicating the column, on which the point occurs. The row index increases from top to bottom and the column index from left to right. A general diagram D is obtained by permuting the rows and columns of D_ν for some ν (that is, permuting the row indices and permuting the column indices). The i-th row of D is the set of nodes of D with index i. The j-th column is defined similarly. If λ_i is the number of nodes on the i-th row of D and μ_j is the number of nodes on the j-th column of D, then the compositions $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ are such that $\lambda'' = \nu = \mu'$. We refer to λ and μ as the *row-composition* and the *column-composition* of D, respectively, and we refer to D as a (λ, μ)-*diagram*. We say that D has *shape* (λ, μ) and write $\operatorname{sh} D = (\lambda, \mu)$.

Example 5.1. The diagram $D = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4), \}$ is obtained by permuting the rows and columns of $D_{(4,3,1)}$. We picture these diagrams below.

	×	×	×			×	×	×	×
		×				×	×	×	
×	×	×	×			×			
		D					$D_{(4,3,1)}$		

The row-composition and column-composition of D are $(3, 1, 4)$ and $(1, 2, 3, 2)$, respectively.

Generalizations of the notions of diagram and tableau associated with partitions similar to our generalizations have already appeared in [10, §2]. Some immediate consequences of the definition of a diagram are contained in the following lemma.

Lemma 5.2. (i) If, in a diagram D, (i_1, j_1) and (i_2, j_2) are nodes with $i_1 \neq i_2$ and $j_1 \neq j_2$, then either (i_1, j_2) or (i_2, j_1) is also a node.

(ii) If λ and μ are compositions such that $\lambda'' = \mu'$, there is a unique (λ, μ)-diagram.

Proof. For part (i), let (i_1, j_1) and (i_2, j_2) be nodes in $D_{\lambda''}$ with $i_1 < i_2$. If $j_1 > j_2$ then (i_2, j_1) is also a node of $D_{\lambda''}$. If $j_1 < j_2$ then both (i_1, j_2) and (i_2, j_1) are also nodes of $D_{\lambda''}$. For part (ii), it is enough to observe that any permutation of rows and columns of a (λ, μ)-diagram resulting in $D_{\lambda''}$ has the same effect on all (λ, μ)-diagrams. □

A (λ, μ)-*tableau* is a bijection from the (λ, μ)-diagram to $\{1, \ldots, n\}$. We say that a (λ, μ)-tableau t has *shape* (λ, μ) and write $\operatorname{sh} t = (\lambda, \mu)$. A (λ, μ)-tableau is *row-standard* if it is increasing on rows, *column-standard* if it is increasing on columns, and *standard* if it is increasing on rows and columns.

Lemma 5.3. A (λ, μ)-tableau t is standard if, and only if, for any two nodes (i_1, j_1) and (i_2, j_2) of the corresponding diagram, $i_1 \leq i_2$ and $j_1 \leq j_2$ implies that $t_{i_1,j_1} \leq t_{i_2,j_2}$.

Proof. Suppose that t is a standard tableau. Let (i_1, j_1) and (i_2, j_2) be nodes of the corresponding diagram for which $i_1 \leq i_2$ and $j_1 \leq j_2$. By Lemma 5.2 (i), either (i_1, j_2) or (i_2, j_1) is also a node. In the first case, we get $t_{i_1,j_1} \leq t_{i_1,j_2} \leq t_{i_2,j_2}$. In the second case, we get $t_{i_1,j_1} \leq t_{i_2,j_1} \leq t_{i_2,j_2}$. Hence, the condition on the tableau entries holds.

Conversely, a tableau for which the condition holds is clearly both row-standard and column-standard. Hence, it is standard. □

W acts on the set of (λ,μ)-tableaux in the obvious way; if $w \in W$, an entry i is replaced by iw and tw denotes the tableau resulting from the action of w on the tableau t. This action on (λ,μ)-tableaux is a natural extension of the action by letter permutations of Dipper and James in [2, p.21].

We construct a special standard (λ,μ)-tableau $t^{\lambda,\mu}$, where $t^{\lambda,\mu}$ is obtained by filling in the (λ,μ)-diagram with $1,\ldots,n$ by rows from top to bottom, filling each row from left to right.

For any (λ,μ)-tableau t, we define the row reversed (λ,μ)-tableau $\mathrm{rev}(t)$ to be that obtained from t by reversing the entries in its rows, and the row reflected reversal $(\lambda,\dot{\mu})$-tableau $\mathrm{refrev}(t)$ to be that obtained from $\mathrm{rev}(t)$ by reflecting the entire tableau $\mathrm{rev}(t)$ in a vertical axis. The composition $\dot{\mu}$ is the composition obtained from μ by reversing its entries.

Example 5.4. With λ and μ as in Example 5.1, let t be the (λ,μ)-tableau $\begin{array}{cccc} & 1 & 2 & 4 \\ & & 5 & \\ 3 & 6 & 7 & 8 \end{array}$.

Then $\mathrm{rev}(t) = \begin{array}{cccc} & 4 & 2 & 1 \\ & & 5 & \\ 8 & 7 & 6 & 3 \end{array}$, $\mathrm{refrev}(t) = \begin{array}{cccc} 1 & 2 & 4 & \\ & 5 & & \\ 3 & 6 & 7 & 8 \end{array}$

The following result gives a simple combinatorial technique for identifying involutions in the same two-sided cell as the standard parabolic involution $w_{J(\lambda'')}$ where λ is a composition of n. We already know from a theorem of Schűtzenberger [16] that involutions in the same two-sided cell of W are conjugate. The involutions covered by Theorem 5.5 satisfy an additional length restriction which is not in general satisfied by all involutions within a two-sided cell of W. Also note that if an involution satisfies the hypothesis of Theorem 5.5 then it also satisfies the hypothesis of Theorem 4.9 and Corollary 4.10.

Theorem 5.5. *Let t be a standard (λ,μ)-tableau for which* $\mathrm{rev}(t)$ *is column-standard. Let $d \in W$ be the element defined by $t^{\lambda,\mu}d = t$ and let $z = d^{-1}w_{J(\lambda)}d$. Then* $\mathrm{sh}\, z = \lambda'$, *$z \sim_{LR} w_{J(\lambda)}$, and $l(z) = 2l(d) + l(w_{J(\lambda)})$.*

Proof. Note that $tz = \mathrm{rev}(t)$, the rows of $\mathrm{rev}(t)$ give a decreasing cover of type λ'' for z and the columns of $\mathrm{rev}(t)$ give an increasing cover of type λ' for z. By Corollary 3.7, $\lambda'' \trianglelefteq (\mathrm{sh}\, z)'$ and $\lambda' \trianglelefteq \mathrm{sh}\, z$. It follows that $\mathrm{sh}\, z = \lambda'$.

We now show that $N(z)$ is the disjoint union of $N(d^{-1})$, $\Phi_{\Sigma(\lambda)}d$ and $N(d^{-1})z$. For $i = 1,\ldots,n$, let $(a(i), b(i))$ denote the position of i in the tableau t, where $a(i)$ denotes the row number of the position and $b(i)$ denotes the column number, and let $(a'(i), b'(i))$ denote the position of i in the tableau $\mathrm{refrev}(t)$. By hypothesis, $\mathrm{refrev}(t)$ is a standard tableau whose row-reversal is column-standard and $t^{\lambda,\dot{\mu}}d = \mathrm{refrev}(t)$.

Let $1 \le i < j \le n$. If $a(i) = a(j)$ then $id^{-1} < jd^{-1}$, since $td^{-1} = t^{\lambda,\mu}$, and id^{-1} and jd^{-1} are on the same row of $t^{\lambda,\mu}$. Hence, $e_{id^{-1}} - e_{jd^{-1}} \in \Phi_{\Sigma(\lambda)}$. So, $e_i - e_j \in \Phi_{\Sigma(\lambda)}d$. Since $jz < iz$, $e_i - e_j \in N(z)$ also. If $a(i) > a(j)$ and $b(i) < b(j)$ then $jd^{-1} < id^{-1}$. So $e_i - e_j \in N(d^{-1})$. We will see later that $e_i - e_j \in N(z)$ also in this case. If $a(i) < a(j)$ and $b(i) < b(j)$ and $e_i - e_j \in N(z)$, then $a'(iz) < a'(jz)$ and $b'(iz) > b'(jz)$ and $jz < iz$. Hence, $e_{jz} - e_{iz} \in N(d^{-1})$. So, $e_i - e_j \in N(d^{-1})z$. If $a(i) < a(j)$ and $b(i) < b(j)$ and $e_i - e_j \notin N(z)$, then $a'(iz) < a'(jz)$ and $b'(iz) > b'(jz)$ and $iz < jz$. Hence, $e_{jz} - e_{iz} \notin N(d^{-1})$. So, $e_i - e_j \notin N(d^{-1})z$. Finally, if $a(i) < a(j)$ and $b(i) \ge b(j)$ then

$a'(iz) < a'(jz)$ and $b'(iz) \leq b'(jz)$. So, $iz < jz$ and $e_i - e_j \notin N(z)$. At this point, we have established that $N(z)$ is contained in $\Phi_{\Sigma(\lambda)}d \cup N(d^{-1}) \cup N(d^{-1})z$

Suppose now that $e_i - e_j \in N(d^{-1})$. Since $i < j$, $jd^{-1} < id^{-1}$. Hence, jd^{-1} occurs on the same row of $t^{\lambda,\mu}$ as id^{-1} but before it, or on an earlier row. Consequently, j occurs on the same row of t as i but before it, or on an earlier row. Since t is standard, this is possible only if $a(i) > a(j)$ and $b(i) < b(j)$. By Lemma 5.2, t has an entry at one of the positions $(a(i), b(j))$ or $(a(j), b(i))$. Suppose that it has an entry k at position $(a(i), b(j))$. Then $i < k$ and $j < k$. So $kz < iz$ and $\mathrm{rev}(t)$ has entries jz and kz at positions $(a(j), b(j))$ or $(a(i), b(j))$, respectively. Since $\mathrm{rev}(t)$ is column-standard, $jz < kz$. Hence, $jz < iz$. We get the same result if t has an entry at position $(a(j), b(i))$. So, $e_i - e_j \in N(z)$. Hence, $N(d^{-1}) \subseteq N(z)$.

Since $N(z)z = N(z)$, $N(d^{-1})z \subseteq N(z)$. If $e_i - e_j \in N(d^{-1})z$, then $e_{iz} - e_{jz} \in N(d^{-1})$ and $jz < iz$. By an earlier argument, now applied to $\mathrm{refrev}(t)$ instead of t, $a'(jz) > a'(iz)$ and $b'(jz) < b'(iz)$. Hence, $a(j) > a(i)$ and $b(j) > b(i)$.

We have now shown that $N(z)$ contains $\Phi_{\Sigma(\lambda)}d \cup N(d^{-1}) \cup N(d^{-1})z$ and by considering the relative positions of i and j in t for the elements $e_i - e_j \in N(z)$, we see that the sets $\Phi_{\Sigma(\lambda)}d$, $N(d^{-1})$, and $N(d^{-1})z$ are mutually disjoint.

From Result 4.1, $2l(z) = |N(z)|$, $2l(d) = 2l(d^{-1}) = |N(d^{-1})|$ and $2l(w_{J(\lambda)}) = |N(w_{J(\lambda)})| = |\Phi_{\Sigma(\lambda)}|$. We conclude that $l(z) = 2l(d) + l(w_{J(\lambda)})$. □

Note that in Example 5.4, $N(d^{-1}) = \{e_3 - e_4, e_3 - e_5, e_4 - e_3, e_5 - e_3\}$ and $N(d^{-1})z = \{e_1 - e_8, e_5 - e_8, e_8 - e_1, e_8 - e_5\}$.

6 Computational Results

We determined, using programs in GAP [5] and C, the involutions which are not accounted for by Theorem 5.5. The partitions corresponding to cells which contain such involutions for the cases $n \leq 12$ are

$n = 9$: 6.3, 3^2.2.1.

$n = 10$: 7.3, 6.3.1, 4.3.2.1, $3^2.2.1^2$.

$n = 11$: 8.3, 7.4, 7.3.1, 6.3.2, $6.3.1^2$, 5.3.2.1, 4^2.2.1, $4.3.2^2$, $4.3.2.1^2$, $3^2.2^2.1$, $3^2.2.1^3$.

$n = 12$: 9.3, 8.4, 8.3.1, 7.4.1, 7.3.2, $7.3.1^2$, 6.3^2, 6.3.2.1, $6.3.1^3$, 5.4.2.1, $5.3.2.1^2$, $4^2.2^2$, $4^2.2.1^2$, $4.3.2^2.1$, $4.3.2.1^3$, 3^3.2.1, $3^2.2^2.1^2$, $3^2.2.1^4$.

We also determined the involutions z with $\mathrm{sh}\, z = \lambda'$, which cannot be written in the form $z = d^{-1}w_{J(\lambda)}d$ for some composition λ with $l(z) = 2l(d) + l(w_{J(\lambda)})$. For each n, the total number of such involutions is denoted by $N_{v,n}$. An investigation carried out in [12] had already shown that $N_{v,n} = 0$ whenever $n \leq 7$.

In the following table, we list the total number $N_{t,n}$ of involutions for $9 \leq n \leq 12$, the number of involutions $N_{b,n}$ not accounted for by Theorem 5.5 and the fraction $N_{b,n}/N_{t,n}$, together with $N_{v,n}$ and $N_{v,n}/N_{t,n}$.

n	$N_{t,n}$	$N_{b,n}$	$N_{b,n}/N_{t,n}$	$N_{v,n}$	$N_{v,n}/N_{t,n}$
9	2620	12	0.00458	4	0.00153
10	9496	58	0.00611	22	0.00232
11	35696	418	0.01171	142	0.00398
12	140152	2234	0.01594	870	0.00621

More detailed information in the form of tables can be obtained from any of the authors on request. In these tables, for each n, $9 \leq n \leq 12$ and each partition λ of n the column entries give (i) the partition λ; (ii) the number $N_{t,\lambda}$ of standard tableaux whose shape is the partition λ, and this is also the number of involutions in the corresponding two-sided cell; (iii) the number $N_{b,\lambda}$ of involutions in the two-sided cell corresponding to λ which are not accounted for by Theorem 5.5.

References

[1] R. W. Carter, "Simple Groups of Lie Type", Wiley, New York, 1989.

[2] R. Dipper and G. D. James, Representations of Hecke algebras of general linear groups, *Proc. London Math. Soc.*, (3), **52**, (1986), 20–52.

[3] E. B. Dynkin, Semisimple subalgebras of semisimple Lie algebras, *Amer. Math. Soc. Transl.*, (2), **6**, (1957), 111–244.

[4] W. Fulton, "Young Tableaux", Cambridge University Press, Cambridge, 1997.

[5] The GAP Group, *GAP–Groups, Algorithms, and Programming, Version 4.4.9*; 2006, (`http://www.gap-system.org`).

[6] M. Geck, Kazhdan-Lusztig cells and the Murphy basis, *Proc. London Math. Soc.*, (3), **93**, (2006), 635–665.

[7] M. Geck and G. Pfeiffer, "Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras" Clarendon Press, Oxford, 2000.

[8] C. Greene, An extension of Schensted's theorem, *Advances in Math.*, **14** (1974), 254–265.

[9] G. James and A. Kerber, "The Representation Theory of the Symmetric Group", Encyclopedia of Mathematics and its Applications, vol. 16. Section: Algebra, Editors: P.M. Cohn and Roger Lyndon. Addison-Wesley Publishing Co, Reading, MA, 1981.

[10] G. D. James and M. H. Peel, Specht Series for Skew Representations of Symmetric Groups *J. Algebra*, **56**, (1979) 343–364.

[11] D. A. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.*, **53** (1979) 165–184.

[12] T. P. McDonough and C. A. Pallikaros, Some investigations concerning Kazhdan-Lusztig cells in finite Coxeter groups of type A_n, *Bull. Greek Math. Soc.*, **48**, (2003), 51–59.

[13] T. P. McDonough and C. A. Pallikaros, On subsequences and certain elements which determine various cells in S_n, *Journal of Algebra*, **319**, (2008), 1249–1263.

[14] R. W. Richardson, Conjugacy classes of involutions in Coxeter groups, *Bull, Australian Math. Soc*, **26**, (1982), 1–15.

[15] B. Sagan, "The Symmetric group, Representations, Combinatorial Algorithms and Symmetric Functions", Graduate Texts in Mathematics, vol. 203. Springer-Verlag, New York, 2000.

[16] M. P. Schűtzenberger, La correspondance de Robinson, "Combinatoire et representation du groupe symmetrique", Lecture Notes in Math., 579, Editor D. Foata. Springer-Verlag, Berlin, 1977, 59–113.