On ordered k-paths and rims for certain families of Kazhdan-Lusztig cells of S_n

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31st May, 2020

Abstract

For a composition λ of n we consider the Kazhdan-Lusztig cell in the symmetric group S_n containing the longest element of the standard parabolic subgroup of S_n associated to λ. In this paper we extend some of the ideas and results in [Beiträge zur Algebra und Geometrie, **59** (2018), no. 3, 523–547]. In particular, by introducing the notion of an ordered k-path, we are able to obtain alternative explicit descriptions for some additional families of cells associated to compositions. This is achieved by first determining the rim of the cell, from which reduced forms for all the elements of the cell are easily obtained.

Key words: symmetric group; Kazhdan-Lusztig cell; reduced form
2010 MSC Classification: 05E10; 20C08; 20C30

1 Introduction

In [KL79] Kazhdan and Lusztig introduced, via certain preorders, the left cells, the right cells and the two-sided cells of a Coxeter group as a means of investigating the representation theory of the Coxeter group and its associated Hecke algebra.

In the case of the symmetric group S_n, the cell to which an element belongs can be determined by an application of the Robinson-Schensted process. Moreover, one can obtain all the elements in a given cell by applying the reverse Robinson-Schensted process but, unfortunately, this does not lead to some straightforward way of obtaining reduced forms for these elements. A useful observation is that each right (resp., left) cell of S_n contains a unique involution.

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The present paper, which is a continuation of the work in [MP08, MP15, MP17], is concerned with the problem of determining reduced expressions for all the elements in a given cell. As in [MP15, MP17], we again focus attention on (right) cells which have the property that the unique involution they contain is the longest element in some standard parabolic subgroup of S_n and which are thus associated to compositions λ of n. The motivation for some of the main ideas in [MP17], which we also use here, emanates from work in [Sch61] and [Gre74] on increasing and decreasing subsequences. By extending various ideas in [MP17] we are able to obtain alternative explicit descriptions for some additional families of Kazhdan-Lusztig cells via the determination of their rim. This directly leads to determining reduced forms for all the elements in these cells. These results have close connections with the work of Nguyen [Ngu12] and of Howlett and Nguyen [HN12, HN16] on W-graph ideals. Indeed, the elements of the rim generate a right W-graph ideal, and hence their inverses generate a W-graph ideal from which the W-graph representation of the cell can be constructed. An illustration of this connection is given following Theorem 2.11 below.

The paper is organized as follows: In Section 2 we recall some basic facts about Kazhdan-Lusztig cells in the symmetric group. We also recall some useful properties of paths and admissible diagrams described in [MP17]. Moreover, in Theorem 2.11 we characterize all compositions of n for which the rim of the associated cell consists of precisely one element.

In Section 3 we introduce the notion of an ordered k-path which plays a key role in proving some of the main results in this paper. In Theorem 3.13 we show that every k-path in a diagram D is equivalent to an ordered k-path. At the end of Section 3 we establish some results on extending certain ordered k-paths in a particular way which turns out to be useful in various arguments that follow.

Finally, in Section 4, using the ideas and techniques developed earlier on in the paper, we prove the main results of the section (Theorems 4.3, 4.6 and 4.8) on determining the rim and hence reduced forms for the elements in certain families of cells. Another result which is very useful in this direction is Proposition 4.2 which uses ideas in [MP17] related to the induction of cells (see [BV83]) and which allows us to 'transform' certain cells of S_n into cells of S_m with $m > n$.

Acknowledgement. The authors are grateful to the reviewer for drawing our attention to the work of Howlett and Nguyen on W-graph ideals, mentioned above, and to its relevance to our work.

2 Preliminaries and generalities

2.1 Kazhdan-Lusztig cells in the symmetric group

For any Coxeter system (W, S), Kazhdan and Lusztig [KL79] introduced three preorders $\leqslant_L$, $\leqslant_R$ and $\leqslant_{LR}$, with corresponding equivalence relations $\sim_L$, $\sim_R$ and $\sim_{LR}$, whose equivalence classes are called *left cells*, *right cells* and *two-sided cells*, respectively. Each cell of W provides a representation of W, with the C-basis of the Hecke algebra $\mathcal{H}$ of (W, S) playing an important role in the construction of this representation; see [KL79, § 1].

We refer to [GP00] and [Hum90] for basic concepts relating to Coxeter groups and Hecke algebras. In particular, for a Coxeter system (W, S), $W_J = \langle J \rangle$ denotes the standard parabolic subgroup determined by a subset J of S, w_J denotes the longest element of W_J and $\mathfrak{X}_J$ denotes the set of minimum length elements in the right cosets of W_J in W (the distinguished right coset representatives). Also recall the *prefix relation* on the elements of W: if $x, y \in W$ we say that x is a *prefix* of y if y has a *reduced form* beginning with a reduced form for x.

The following result collects some useful propositions concerning cells. For proofs of (i) and (ii), see [KL79, 2.3ac] and [Lus84, 5.26.1] respectively.

Result 1 ([KL79, Lus84])**.**

(i) If x, y, z are elements of W such that x is a prefix of y, y is a prefix of z and $x \sim_R z$ then $x \sim_R y$.

(ii) If $J \subseteq S$, then the right cell containing w_J is contained in $w_J\mathfrak{X}_J$.

In this paper we focus on the symmetric group. For the basic definitions and background concerning partitions, compositions, Young diagrams, Young tableaux and the Robinson-Schensted correspondence we refer to [Ful97] or [Sag00].

The symmetric group S_n (acting on the right) on $\{1, \dots, n\}$ is a Coxeter group with Coxeter system (W, S) where $W = S_n$, $S = \{s_1, \dots, s_{n-1}\}$, and s_i is the transposition $(i, i+1)$. An element w of W can be described in different forms: as a word in the generators $s_1, \dots, s_{n-1}$, as products of disjoint cycles on $1, \dots, n$, and in *row-form* $[w_1, \dots, w_n]$ where $w_i = iw$ for $i = 1, \dots, n$. The Coxeter *length* $l(w)$ of the element $w \in W$, that is the shortest length of a word in the elements of S representing w, has an easy combinatorial description; $l(w)$ is the number of pairs (w_i, w_j) with $i < j$ and $w_i > w_j$. The longest element w_0 in W is the permutation defined by $i \mapsto n + 1 - i$.

Let $\Phi = \{\epsilon_i - \epsilon_j : 1 \leqslant i, j \leqslant n,\ i \neq j\}$ and $\Phi^+ = \{\epsilon_i - \epsilon_j : 1 \leqslant i < j \leqslant n\}$ where $\{\epsilon_1, \dots, \epsilon_n\}$ is an orthonormal basis of an n-dimensional Euclidean space; see [Hum90, p. 41]. There is an action of S_n on Φ given by $(\epsilon_i - \epsilon_j)w = \epsilon_{iw} - \epsilon_{jw}$ $(w \in S_n)$. The Coxeter generator s_i corresponds to the reflection in the hyperplane orthogonal to $\epsilon_i - \epsilon_{i+1}$. For $w \in S_n$, we define $N^+(w) = \{\alpha \in \Phi^+ : \alpha w \in \Phi^+\}$ and $N^-(w) = \Phi^+ - N^+(w)$. Then $l(w) = |N^-(w)|$; see [Hum90, p. 14].

All our partitions and compositions will be assumed to be *proper* (that is, with no zero parts). We use the notation $\lambda \vDash n$ (respectively, $\lambda \vdash n$) to say that λ is a composition (respectively, partition) of n.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a *composition* of n with r *parts*. Recall that the *conjugate* composition $\lambda' = (\lambda'_1, \ldots, \lambda'_{r'})$ of λ is defined by $\lambda'_i = |\{j : 1 \leqslant j \leqslant r \text{ and } i \leqslant \lambda_j\}|$ for $1 \leqslant i \leqslant r'$, where r' is the maximum part of the composition λ. It is immediate that λ' is a partition of n with r' parts. We also define the subset $J(\lambda)$ of S to be $S \backslash \{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, \ldots, s_{\lambda_1+\ldots+\lambda_{r-1}}\}$. Thus, corresponding to the composition λ, there is a standard parabolic subgroup of W, also known as a Young subgroup, whose Coxeter generator set is $J(\lambda)$. The longest element $w_{J(\lambda)}$ of $W_{J(\lambda)}$ can be described in row-form by concatenating the sequences $(\widehat{\lambda}_{i+1}, \ldots, \widehat{\lambda}_i + 1)$ for $i = 0, \ldots, r-1$, where $\widehat{\lambda}_0 = 0$, $\widehat{\lambda}_r = n$, and $\widehat{\lambda}_{i+1} = \lambda_{i+1} + \widehat{\lambda}_i$.

If $\nu = (\nu_1, \ldots, \nu_r) \vdash n$ and $\mu = (\mu_1, \ldots, \mu_s) \vdash n$, write $\nu \trianglelefteq \mu$ if $\sum_{1 \leqslant i \leqslant k} \nu_i \leqslant \sum_{1 \leqslant i \leqslant k} \mu_i$, for all k with $1 \leqslant k \leqslant s$. This is the dominance order of partitions (see [Sag00, p. 58]). If $\nu \trianglelefteq \mu$ and $\nu \neq \mu$, we write $\nu \triangleleft \mu$.

In the case of the symmetric group S_n, the Robinson-Schensted correspondence gives a combinatorial method of identifying the Kazhdan-Lusztig cells. The Robinson-Schensted correspondence is a bijection of S_n to the set of pairs of standard Young tableaux $(\mathcal{P}, \mathcal{Q})$ of the same shape and with n entries, where the shape of a tableau is the partition counting the number of entries on each row. Denote this correspondence by $w \mapsto (\mathcal{P}(w), \mathcal{Q}(w))$. Then $\mathcal{Q}(w) = \mathcal{P}(w^{-1})$. The *shape* of w, denoted by $\operatorname{sh} w$, is defined to be the common shape of the Young tableaux $\mathcal{P}(w)$ and $\mathcal{Q}(w)$.

The following result in [KL79] (see also [Ari00, Theorem A] or [Gec06, Corollary 5.6]) characterises the cells in S_n: If $\mathcal{P}$ is a fixed standard Young tableau then the set $\{w \in W : \mathcal{P}(w) = \mathcal{P}\}$ is a left cell of W and the set $\{w \in W : \mathcal{Q}(w) = \mathcal{P}\}$ is a right cell of S_n. Conversely, every left cell and every right cell arises in this way. Moreover, the two-sided cells are the subsets of W of the form $\{w \in W : \operatorname{sh} w \text{ is a fixed partition}\}$.

Great care is needed in describing the connection between the Kazhdan-Lusztig left and right cells of S_n and the tableaux arising from the Robinson-Schensted process since it is affected by how the elements of the (abstract) Coxeter group act on the set $\{1, \ldots, n\}$, whether on the right or on the left. We give an elementary illustration.

The elements of one of the left cells in S_4 are $s_2s_3s_2s_1$, $s_1s_2s_1s_3$, and $s_1s_2s_3s_2s_1$. With permutations acting on the left, these elements correspond to the permutations $[4,1,3,2]$, $[3,2,4,1]$, and $[4,2,3,1]$, respectively, and the corresponding tableaux pairs arising from the Robinson-Schensted process with row-insertion are $\begin{bmatrix} 12 & 13 \\ 3 \;, & 2 \\ 4 & 4 \end{bmatrix}$, $\begin{bmatrix} 14 & 13 \\ 2 \;, & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 13 & 13 \\ 2 \;, & 2 \\ 4 & 4 \end{bmatrix}$ with a common second component.

With permutations acting on the right, these elements correspond to the permutations $[2,4,3,1]$, $[4,2,1,3]$, and $[4,2,3,1]$, respectively, and the corresponding tableaux pairs arising from the same Robinson-Schensted process are $\begin{bmatrix} 1\,3 & 1\,2 \\ 2 \quad , & 3 \\ 4 & 4 \end{bmatrix}, \begin{bmatrix} 1\,3 & 1\,4 \\ 2 \quad , & 2 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 1\,3 & 1\,3 \\ 2 \quad , & 2 \\ 4 & 4 \end{bmatrix}$ with a common first component.

The elements $s_1s_2s_3s_2$, $s_3s_1s_2s_1$, and $s_1s_2s_3s_2s_1$ form a right cell in S_4 which, with right action of permutations, correspond to the permutations $[4,1,3,2]$, $[3,2,4,1]$ and $[4,2,3,1]$, respectively, which form the first set of three permutations above. So, the tableaux pairs corresponding to the elements of a right cell with permutation action on the right have a common second component.

2.2 Diagrams, rims and reduced forms

We recall the generalizations of the notions of diagram and tableau, commonly used in the basic theory, which we described in [MP15]. A *diagram* D is a non-empty finite subset of $\mathbb{Z}^2$. We will assume that D has no empty rows or columns. These are the principal diagrams of [MP15]. We will also assume that both rows and columns of D are indexed consecutively from 1; a node in D will be given coordinates (a, b) where a and b are the indices respectively of the row and column which the node belongs to (rows are indexed from top to bottom and columns from left to right). The *row-composition* λ_D (respectively, *column-composition* μ_D) of D is defined by setting $\lambda_{D,k}$ (respectively, $\mu_{D,k}$) to be the number of nodes on the k-th row (respectively, column) of D. If λ and μ are compositions, we will write $\mathcal{D}^{(\lambda,\mu)}$ for the set of (principal) diagrams D with $\lambda_D = \lambda$ and $\mu_D = \mu$. We also define $\mathcal{D}^{(\lambda)} = \bigcup_{\mu \vDash n} \mathcal{D}^{(\lambda,\mu)}$. A well-known diagram associated with a partition $\nu = (\nu_1, \ldots, \nu_r)$ is a *Young diagram* $V(\nu) = \{(i,j) \colon 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant \nu_i\}$. A *special diagram* is a diagram obtained from a Young diagram by permuting the rows and columns. Special diagrams are characterised in the following proposition.

Result 2 ([MP15, Proposition 3.1]. Compare [DMP10, Lemma 5.2])**.** Let D be a diagram. The following statements are equivalent. (i) D is special; (ii) $\lambda''_D = \mu'_D$; (iii) for every pair of nodes $(i,j), (i',j')$ of D with $i \neq i'$ and $j \neq j'$, at least one of (i',j) and (i,j') is also a node of D. $\square$

Clearly if $\nu \vdash n$, then $V(\nu)$ is the unique element of $\mathcal{D}^{(\nu,\nu')}$. It follows that $\mathcal{D}^{(\lambda,\mu)}$ consists of a single diagram, which is special, if λ and μ are compositions of n with $\lambda'' = \mu'$.

If D is a diagram of size n (that is, consisting of precisely n nodes), a D*-tableau* is a bijection $t : D \to \{1, \ldots, n\}$ and we refer to $(i,j)t$, where $(i,j) \in D$, as the (i,j)*-entry* of t. The group W acts on the set of D-tableaux in the obvious way—if $w \in W$, an entry i is replaced by iw and tw denotes the tableau resulting from the action of w on the tableau t. We denote by t^D and t_D the two D-tableaux obtained by filling the nodes of D with $1, \ldots, n$ by rows and by columns, respectively, and we write w_D for the element of W defined by $t^D w_D = t_D$.

Now let D be a diagram and let t be a D*-tableau.* We say t is *row-standard* if it is increasing on rows. Similarly, we say t is *column-standard* if it is increasing on columns.

We say that t is *standard* if $(i',j')t \leqslant (i'',j'')t$ for any $(i',j'),(i'',j'') \in D$ with $i' \leqslant i''$ and $j' \leqslant j''$. Note that a standard D-tableau is row-standard and column-standard, but the converse is not true, in general.

For $1 \leqslant l,m \leqslant |D|$, we write $l <_{ne} m$ (resp., $l \leqslant_{se} m$) in t if $i_l > i_m$ and $j_l < j_m$ (resp., $i_l \leqslant i_m$ and $j_l \leqslant j_m$) where, for $1 \leqslant r \leqslant |D|$, we set $r = (i_r, j_r)t$ with $(i_r, j_r) \in D$. Informally, $l <_{ne} m$ means m is strictly north-east of l in t and $l \leqslant_{se} m$ means m is weakly south-east of l in t.

The row-form of $w \in S_n$ is obtained by writing the rows of $t^D w$ in one row so that, for each i, the $(i+1)$-th row is to the right of the i-th row. It follows easily that if $k <_{ne} k+1$ in $t^D w$, then $l(ws_k) = l(w) - 1$. Moreover, we can prove the following lemma.

Lemma 2.1 (Compare [MP15, Lemma 3.4].). *Let D be a diagram of size n, let $u \in S_n$ and let $k \in \mathbb{N}$ with $1 \leqslant k \leqslant n-1$. Suppose that $t^D u$ is a standard D-tableau. Then (i) and (ii) below hold.*

- *(i) If $l(us_k) = l(u) - 1$, then $k <_{ne} k+1$ in $t^D u$, and $t^D us_k$ is a standard D-tableau.*
- *(ii) If $k+1 <_{ne} k$ in $t^D u$, then $N^-(u) \subsetneqq N^-(us_k)$ (so $l(us_k) = l(u)+1$) and $t^D us_k$ is a standard D-tableau.*

Proof. (i) If $l(us_k) = l(u) - 1$, then $k+1$ precedes k in the row-form of u. The assumption that $t^D u$ is standard now forces $k <_{ne} k+1$. It also ensures that $t^D us_k$ is standard in view of the location of k and $k+1$ in $t^D u$.

(ii) If $k+1 <_{ne} k$ in $t^D u$, then k precedes $k+1$ in the row-form of u and so $ku^{-1} < (k+1)u^{-1}$. Hence, $N^-(us_k) = N^-(u) \cup \{\epsilon_{ku^{-1}} - \epsilon_{(k+1)u^{-1}}\}$. Again the assumption that $t^D u$ is standard, together with the location of k and $k+1$ in $t^D u$, ensure that $t^D us_k$ is standard. $\square$

For a diagram D of size n, we define the subset Ψ_D of Φ by $\Psi_D = \{\epsilon_l - \epsilon_m \in \Phi \colon l \neq m$ and $l \leqslant_{se} m$ in $t^D\}$. Clearly $\Psi_D \subseteq \Phi^+$ since t^D is standard.

Lemma 2.2. *Let D be a diagram of size n and let $u \in S_n$. Suppose that $t^D u$ is a standard D-tableau. Then,*

- *(i) $\Psi_D \subseteq N^+(u)$ and, moreover, $\Psi_D = N^+(w_D)$.*
- *(ii) If $u \neq w_D$, then $N^-(u) \subsetneqq N^-(us_k) \subseteq N^-(w_D)$ for some k with $1 \leqslant k \leqslant n-1$ and, moreover, $t^D us_k$ is a standard D-tableau.*

Proof. We assume the hypothesis.

(i) Suppose that $1 \leqslant l,m \leqslant n$ with $l \neq m$ and $l \leqslant_{se} m$ in t^D. Since t^D (resp., $t^D u$) is standard, $l < m$ (resp., $lu < mu$). Hence $\epsilon_l - \epsilon_m \in N^+(u)$ showing that $\Psi_D \subseteq N^+(u)$. In particular, $\Psi_D \subseteq N^+(w_D)$ since t_D $(= t^D w_D)$ is standard.

Now suppose $\alpha = \epsilon_p - \epsilon_q \in \Phi^+ - \Psi_D$. First, $p < q$ since $\alpha \in \Phi^+$. In view of the way t^D is constructed we have $q <_{ne} p$ in t^D. We also have $qw_D < pw_D$ from the way

$t_D\,(= t^D w_D)$ is constructed. So $\alpha w_D = \epsilon_{pw_D} - \epsilon_{qw_D} \in \Phi - \Phi^+$. Hence, $\alpha \in N^-(w_D)$. Thus, $\Phi^+ - \Psi_D \subseteq N^-(w_D) = \Phi^+ - N^+(w_D)$. Hence, $N^+(w_D) \subseteq \Psi_D$. So $N^+(w_D) = \Psi_D$.

(ii) Suppose that $u \neq w_D$. Then $t^D u \neq t_D$, so there exists k with $1 \leqslant k \leqslant n-1$ such that $k+1$ appears in $t^D u$ in a column of lower index than the column k appears in. (If there is no such k, clearly this forces $t^D u = t_D$.) Since $t^D u$ is standard, we must have $k+1 <_{ne} k$ in $t^D u$. The result now follows from Lemma 2.1(ii) and item (i) of this lemma. □

We continue with D a diagram of size n and u a prefix of w_D. Beginning with the standard tableau $t_D\,(= t^D w_D)$ and applying Lemma 2.1(i) a finite number of times, we see that $t^D u$ is a standard D-tableau. Conversely, if we suppose that $t^D v$, where $v \in S_n$, is a standard D-tableau, a finite number of applications of Lemma 2.2(ii) shows that v is a prefix of w_D. (Note that by Lemma 2.2(ii) we know that $v' = w_D$ whenever $t^D v'$ is standard and satisfies $N^-(v') = N^-(w_D)$.) Hence we have,

Result 3 ([MP15, Proposition 3.5]. Compare [DJ86, Lemma 1.5])**.** Let D be a diagram. Then the mapping $u \mapsto t^D u$ is a bijection of the set of prefixes of w_D to the set of standard D-tableaux. □

The argument presented above provides an interpretation of the proof of Result 3 given in [MP15] in terms of certain subsets of the root system Φ. Writing $\tilde{N}(u) = (N^-(w_D) - N^-(u))u$ for $u \in S_n$ with $t^D u$ standard, the set $\tilde{N}(u)$ can naturally be identified with the set N_u in the proof of [MP15, Proposition 3.5]. In particular, with D, u and k as in Lemma 2.2(ii), we have $\tilde{N}(us_k)s_k \subseteq \tilde{N}(u)$ and $\tilde{N}(us_k)s_k = \tilde{N}(u) - \{\epsilon_k - \epsilon_{k+1}\}$.

Remark 2.3. (i) Lemma 2.2(ii) provides a straightforward process for completing a reduced expression of any prefix of w_D to a reduced expression for w_D. Compare also with [MP15, Algorithm 1].

(ii) Considering the root-subsystem of Φ corresponding to the parabolic subgroup $W_{J(\lambda_D)}$, we can easily observe that $\mathfrak{X}_{J(\lambda_D)} = \{w \in S_n \colon t^D w \text{ is row-standard}\}$; see also [DJ86, Lemma 1.1]. In particular, w_D and all its prefixes belong to $\mathfrak{X}_{J(\lambda_D)}$.

In general, an element of W will have an expression of the form w_D for many different diagrams D of size n. If $\lambda \vDash n$ and $d \in \mathfrak{X}_{J(\lambda)}$, a way to locate suitable diagrams $D \in \mathcal{D}^{(\lambda)}$ with $d = w_D$ is given in [MP15, Proposition 3.7]. The proof involves the construction of a very particular diagram $D = D(d, \lambda) \in \mathcal{D}^{(\lambda)}$ with $w_D = d$. This is formed by partitioning the row-form of d in parts of sizes corresponding to λ, placing these parts on consecutive rows and moving the entries on the rows minimally to make a tableau of the form t_D.

Moreover, in [MP15, Proposition 3.8] it is shown that among all diagrams $E \in \mathcal{D}^{(\lambda)}$ with $w_E = d$, diagram $D(d, \lambda)$ is the unique one with the minimum number of columns. Also described in the same proposition is the way any such diagram E relates to $D(d, \lambda)$.

Remark 2.4. Let E be a special diagram. Combining Result 2 with [MP15, Proposition 3.8], we see that $E = D(w_E, \lambda_E)$.

As in [MP17], for a composition λ of n, we define the following subsets of $\mathfrak{X}_{J(\lambda)}$ and $\mathcal{D}^{(\lambda)}$:

$$\begin{aligned}
Z(\lambda) &= \{e \in \mathfrak{X}_{J(\lambda)} \colon w_{J(\lambda)}e \sim_R w_{J(\lambda)}\},\\
Z_s(\lambda) &= \{e \in Z(\lambda) \colon e = w_D \text{ for some special diagram } D \in \mathcal{D}^{(\lambda)}\},\\
Y(\lambda) &= \{x \in Z(\lambda) \colon x \text{ is not a prefix of any other } y \in Z(\lambda)\},\\
Y_s(\lambda) &= Y(\lambda) \cap Z_s(\lambda) = \{y \in Y(\lambda) \colon D(y, \lambda) \text{ is special}\},\\
\mathcal{E}^{(\lambda)} &= \{D(y, \lambda) \colon y \in Y(\lambda)\} \text{ and } \mathcal{E}_s^{(\lambda)} = \{D \in \mathcal{E}^{(\lambda)} \colon D \text{ is special}\}.
\end{aligned}$$

In view of Result 1, $Z(\lambda)$ is closed under the taking of prefixes and $w_{J(\lambda)}Z(\lambda)$ is the right cell of W containing $w_{J(\lambda)}$. We denote this right cell by $\mathfrak{C}(\lambda)$. A knowledge of $Y(\lambda)$ leads directly to $Z(\lambda)$ by determining all prefixes. We call $Y(\lambda)$ the *rim* of the cell $\mathfrak{C}(\lambda)$. The map $y \mapsto D(y, \lambda)$ from $Y(\lambda)$ to $\mathcal{E}^{(\lambda)}$ is a bijection, so $Y(\lambda) = \{w_D \colon D \in \mathcal{E}^{(\lambda)}\}$. Hence, in order to give an explicit description of $\mathfrak{C}(\lambda)$ it is enough to locate the diagrams in $\mathcal{E}^{(\lambda)}$.

Remark 2.5. In the case that λ is a partition of n, it follows from [MP05, Lemma 3.3] that $\mathcal{E}^{(\lambda)} = \mathcal{E}_s^{(\lambda)} = \{V(\lambda)\}$.

2.3 Paths and admissible diagrams

In [MP17] we investigated how the subsequence type of a diagram D, defined in Definition 2.6 below, relates to the shape of the Robinson-Schensted tableau of the element $w_{J(\lambda_D)}w_D$. The work in [Sch61] and [Gre74], see also [MP17, Lemma 3.2], motivates the following definition.

Definition 2.6 (Compare with the definition before Remark 3.3 in [MP17]). Let D be a diagram of size n.

(i) A *path* of *length* m in D is a non-empty sequence of nodes $((a_i, b_i))_{i=1}^{m}$ of D such that $a_i < a_{i+1}$ and $b_i \leqslant b_{i+1}$ for $i = 1, \ldots, m-1$.

(ii) For $k \in \mathbb{N}$, a *k-path* in D is a sequence of k mutually disjoint paths in D; the paths in this sequence are the *constituent paths* of the k-path. The *length* of a k-path is the sum of the lengths of its constituent paths; this is the total number of nodes in the k-path. The *type* of a k-path is the sequence of lengths of its paths in non-strictly decreasing order—in particular, the type of a k-path is a k-part partition. The *support* of a k-path Π, which we denote by $s(\Pi)$, is the set of nodes occurring in its paths.

(iii) Let Π be a k-path in D and let $k' \leqslant k$. A *k'-subpath* of Π is a k'-path in D whose constituent paths are also constituent paths of Π.

(iv) A k-path and a k'-path in D are said to be *equivalent* to one another if they have the same support.

(v) The diagram D is said to be of *subsequence type* ν, where $\nu = (\nu_1, \ldots, \nu_r) \vdash n$, if the maximum length of a k-path in D is $\nu_1 + \ldots + \nu_k$ whenever $1 \leqslant k \leqslant r$. We call D *admissible* if it is of subsequence type λ'_D.

See [MP17, Remark 3.3]. Using the notion of *k-increasing subsequence* of the row form of a permutation (see, for example, [Sag00, Definition 3.5.1]), we see from [MP17, Lemma 3.2 and Remark 3.3] that there is a bijection between the set of k-paths in a diagram D and the set of k-increasing subsequences in $w_{J(\lambda_D)}w_D$, for any positive integer k. In fact, the increasing subsequences occurring in the row-form of $w_{J(\lambda_D)}w_D$ are precisely the ones which have form $((a_i, b_i)t_D)_{i=1}^{m}$ for some path $((a_i, b_i))_{i=1}^{m}$ inside D.

As the support of a path defines a unique path, we may refer to the path by just giving the support. However, in general, a set of nodes may form the support of many different k-paths if $k \geqslant 2$.

If $D, E \in \mathcal{D}^{(\lambda)}$ for some $\lambda \vDash n$, there is a natural bijection $\theta_{E,D} \colon E \to D$ given by: $(a, b)\theta_{E,D}$ is the l-th node on the i-th row of D if (a, b) is the l-th node on the i-th row of E for all nodes (a, b) of E. We write $(a, b)\theta_{E,D} = (a, (a, b)\theta''_{E,D})$.

Proposition 2.7. *Let $\lambda \vDash n$, let $D, E \in \mathcal{D}^{(\lambda)}$ and let $\theta = \theta_{E,D}$. Suppose that $t^E w_D$ is a standard E-tableau. If $\Pi = (\pi_1, \ldots, \pi_k)$ be a k-path in E then $\Pi\theta = (\pi_1\theta, \ldots, \pi_k\theta)$ is a k-path in D.*

Proof. First, fix a particular j. Let $\pi_j = ((a_{i,j}, b_{i,j}))_{i=1}^{m_j}$, for $j = 1, \ldots, k$. Since $t^E w_D$ is a standard E-tableau, and π_j is a path in E, $((a_{i,j}, b_{i,j})t^E w_D)_{i=1}^{m_j}$ is a strictly increasing sequence of integers. Since $(a_{i,j}, b_{i,j})t^E w_D = ((a_{i,j}, b_{i,j})\theta)t^D w_D = ((a_{i,j}, b_{i,j})\theta)t_D$, and t_D is a standard D-tableau, the sequence $((a_{i,j}, b_{i,j})\theta)_{i=1}^{m_j} = \pi_j\theta$ of nodes of D, which correspond in t_D to the integers of the preceding strictly increasing sequence, form a path in D.

Since θ is a bijection, $\Pi\theta = (\pi_1\theta, \ldots, \pi_k\theta)$ is a k-path in D. $\square$

The following observations can be made about paths and admissible diagrams.

Result 4 ([MP17, Proposition 3.4])**.** Let D be a diagram and write $\lambda'_D = (\lambda'_1, \ldots, \lambda'_{r'})$. If $1 \leqslant u \leqslant r'$, then a u-path in D of length $\sum_{1\leqslant j\leqslant u} \lambda'_j$ (if it exists) contains all λ_i nodes on the i-th row of D if $\lambda_i \leqslant u$ and exactly u nodes on all remaining rows.

If for each u, $1 \leqslant u \leqslant r'$, there is a u-path Π_u such that, for all i, Π_u has exactly $\min\{u, \lambda_i\}$ nodes on the i-th row, then D is an admissible diagram. $\square$

Result 5 ([MP17, Propositions 3.5 and 3.6 and Corollary 3.7])**.** Let D be a diagram of size n and let ν be a partition of n.

(i) If D is of subsequence type ν then $\mu''_D \trianglelefteq \nu \trianglelefteq \lambda'_D$. Moreover, $\mathrm{sh}\,(w_{J(\lambda_D)}w_D) = \nu$ if, and only if, D is of subsequence type ν.

(ii) $w_{J(\lambda_D)}w_D \sim_R w_{J(\lambda_D)}$ if, and only if, D is admissible. $\square$
In particular, if D is a special diagram then D is admissible.

Lemma 2.8. *Let $\lambda \vDash n$ and let $y \in Y(\lambda)$. Also let $D = D(y, \lambda)$. Suppose that $E \in \mathcal{D}^{(\lambda)}$ is an admissible diagram with $E = D(w_E, \lambda)$ and that $t^E w_D$ is a standard E-tableau. Then $E = D$.*

Proof. First observe that $w_D = y \in Y(\lambda)$. Now $w_E \in Z(\lambda)$ since E is admissible. Since $t^E w_D$ is a standard E-tableau, w_D is a prefix of w_E by Result 3. Since $w_D \in Y(\lambda)$, $w_E = w_D$. Hence $D = D(y, \lambda) = D(w_D, \lambda) = D(w_E, \lambda) = E$. $\square$

Combining Remark 2.4, Result 5(ii) and Lemma 2.8 we get the following corollary.

Corollary 2.9. *Let $\lambda \vDash n$ and let $y \in Y(\lambda)$. Also let $D = D(y, \lambda)$. Suppose that $E \in \mathcal{D}^{(\lambda)}$ is a special diagram and that $t^E w_D$ is a standard E-tableau. Then $E = D$.*

The *reverse* composition $\dot\lambda$ of a composition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the composition $(\lambda_r, \ldots, \lambda_1)$ obtained by reversing the order of the entries. For a principal diagram $D \in \mathcal{D}^{(\lambda)}$, the diagram $\dot D \in \mathcal{D}^{(\dot\lambda)}$ is the diagram obtained by rotating D through 180°. If $D \in \mathcal{D}^{(\lambda,\mu)}$, then $\dot D \in \mathcal{D}^{(\dot\lambda,\dot\mu)}$. Since rotating D through 180° maps k-paths into k-paths, for any k, diagrams D and $\dot D$ have the same subsequence type.

Remark 2.10 (See [MP17, Proposition 3.9])**.** Let $\lambda \vDash n$ and consider the (graph) automorphism of W given by $z \mapsto w_0 z w_0$ ($z \in W$) where w_0 is the longest element of W (recall that w_0^2 is the identity). Since $w_0 s_i w_0 = s_{n-i}$ for $1 \leqslant i \leqslant n-1$, a reduced form for $w_0 z w_0$ can be obtained from a reduced form for z by replacing s_i with s_{n-i} for $1 \leqslant i \leqslant n-1$. Moreover, if $D \in \mathcal{D}^{(\lambda)}$ then $t^{\dot D}$ (resp., $t_{\dot D}$) can be obtained from $t^D w_0$ (resp., $t_D w_0$) by rotating through 180°. Since $t^D w_0(w_0 w_D w_0) = t_D w_0$, we get $w_{\dot D} = w_0 w_D w_0$. We can thus easily determine the prefixes of $w_{\dot D}$ if we know the prefixes of w_D.
A consequence of the above remarks is that the map $z \mapsto w_0 z w_0$ induces an injection $Z(\lambda) \to Z(\dot\lambda)$ which restricts to injections $Y(\lambda) \to Y(\dot\lambda)$ and $Y_s(\lambda) \to Y_s(\dot\lambda)$. Since the

reverse composition of $\dot{\lambda}$ is the composition λ itself, we see that the above injections are in fact bijections. We conclude that the map $D \mapsto \dot{D}$ from $\mathcal{D}^{(\lambda,\mu)}$ to $\mathcal{D}^{(\dot{\lambda},\dot{\mu})}$ induces a bijection between the sets $\mathcal{E}^{(\lambda)}$ and $\mathcal{E}^{(\dot{\lambda})}$.

Note that if ρ is the representation of S_n corresponding to the cell $\mathfrak{C}(\lambda)$, then the representation corresponding to $\mathfrak{C}(\dot{\lambda})$ is given by $s_i \mapsto s_{n-i}\rho$ for all i.

We now see that partitions and their reverses are the only compositions for which the corresponding cells consist of prefixes of a single element.

Theorem 2.11. *Let $\lambda \vDash n$. Then $|Y(\lambda)| = 1$ if, and only if, either λ or $\dot{\lambda}$ is a partition.*

Proof. The 'if' part follows from Remarks 2.5 and 2.10. For the 'only if' part, let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and suppose that neither λ nor $\dot{\lambda}$ is a partition. In view of Remark 2.10, we may replace λ by $\dot{\lambda}$ if necessary and assume that either (i) $\lambda_b < \lambda_a \leqslant \lambda_c$ or (ii) $\lambda_c \leqslant \lambda_a < \lambda_b$ for some a, b and c with $1 \leqslant a < b < c \leqslant r$. Let l be the number of parts of λ' and, for convenience, let $\mu = \lambda'$, $d = \lambda_a$, $e = \lambda_b$, and $f = \lambda_c$. Let F_1 be the unique diagram in $\mathcal{D}^{(\lambda,\mu)}$, and let F_2 be the unique diagram in $\mathcal{D}^{(\lambda,\dot{\mu})}$. By Result 5(ii), $w_{F_1}, w_{F_2} \in Z(\lambda)$ since F_1, F_2 (which belong to $\mathcal{D}^{(\lambda)}$) are special diagrams.

Suppose now that w_{F_1} and w_{F_2} are prefixes of a single $y \in Y(\lambda)$. Then $y = w_G$, where $G \in \mathcal{D}^{(\lambda)}$ is admissible (see Result 5(ii)). By Result 3, $t^G w_{F_1}$ and $t^G w_{F_2}$ are standard G-tableaux.

Let $\{(a,p_1), \ldots, (a,p_d)\}$, $\{(b,q_1), \ldots, (b,q_e)\}$, $\{(c,r_1), \ldots, (c,r_f)\}$ be the sets of nodes of G on its a-th row, b-th row, and c-th row, respectively, where $p_1 < \cdots < p_d$, $q_1 < \cdots < q_e$, and $r_1 < \cdots < r_f$.

Case (i): $e < d \leqslant f$.

Since $(b,e)t_{F_1} < (a,e+1)t_{F_1}$, we get $((b,e)\theta_{F_1,G})t^G w_{F_1} < ((a,e+1)\theta_{F_1,G})t^G w_{F_1}$. Hence $(b,q_e)t^G w_{F_1} < (a,p_{e+1})t^G w_{F_1}$. As $t^G w_{F_1}$ is standard, $q_e < p_{e+1}$.

Similarly, $(c,l-e)t_{F_2} < (b,l-e+1)t_{F_2}$. So, $((c,l-e)\theta_{F_2,G})t^G w_{F_2} < ((b,l-e+1)\theta_{F_2,G})t^G w_{F_2}$. Hence $(c,r_{f-e})t^G w_{F_2} < (b,q_1)t^G w_{F_2}$. As $t^G w_{F_2}$ is standard, $r_{f-e} < q_1$.

Since G has subsequence type λ' and $e+1 \leqslant l$, there exists an $(e+1)$-path Π in G of length $\lambda'_1 + \ldots + \lambda'_{e+1}$. By Result 4, Π contains exactly $e+1$ nodes on each of the a-th and c-th rows of G and contains all e nodes on the b-th row of G.

Let Π' be the e-subpath of Π containing all the nodes on the b-th row of G. The remaining path π of Π necessarily contains one node on each of the a-th and c-th rows. Since $q_e < p_{e+1}$, the nodes of Π' on the a-th row are in columns $\leqslant q_e$. Hence they are the nodes (a,p_i) with $1 \leqslant i \leqslant e$. So, the node of π on the a-th row is $(a,p_{i'})$ for some $i' \geqslant e+1$. Since each one of the constituent paths of Π' contains precisely one node on each of rows b and c of G and Π' contains all nodes on row b of G, the nodes of Π' on the c-th row of G are in columns $j \geqslant q_1$. Hence they are the nodes (c,r_j) with $f-e+1 \leqslant j \leqslant f$ in view of the fact that $r_{f-e} < q_1$. So, the node of π on the c-th row is $(c,r_{j'})$ for some $j' \leqslant f-e$. Since $r_{j'} \leqslant r_{f-e} < q_1 \leqslant q_e < p_{e+1} \leqslant p_{i'}$, this is impossible.

Case (ii): $f \leqslant d < e$.

Since $(c, f)t_{F_1} < (b, f + 1)t_{F_1}$, we get $((c, f)\theta_{F_1,G})t^G w_{F_1} < ((b, f + 1)\theta_{F_1,G})t^G w_{F_1}$. Hence $(c, r_f)t^G w_{F_1} < (b, q_{f+1})t^G w_{F_1}$. Since $t^G w_{F_1}$ is standard, $r_f < q_{f+1}$.

Similarly, $(b, l{-}d)t_{F_2} < (a, l{-}d{+}1)t_{F_2}$. So, $((b, l{-}d)\theta_{F_2,G})t^G w_{F_2} < ((a, l{-}d{+}1)\theta_{F_2,G})t^G w_{F_2}$. Hence $(b, q_{e-d})t^G w_{F_2} < (a, p_1)t^G w_{F_2}$. As $t^G w_{F_2}$ is standard, $q_{e-d} < p_1$.

Next we let $\tilde{\Pi}$ be an f-path in G of length $\lambda'_1 + \ldots + \lambda'_f$. By Result 4, $\tilde{\Pi}$ must contain precisely f nodes on each one of rows a, b, c of G; in particular it contains all nodes on row c of G. It follows that each of the constituent paths of $\tilde{\Pi}$ contains precisely one node on each one of rows a, b and c of G.

Since $r_f < q_{f+1}$, the nodes (b, q_j) for $j \geqslant f + 1$ cannot belong to $\tilde{\Pi}$. So the nodes in row b of G which belong to $\tilde{\Pi}$ are precisely the nodes (b, q_j), $1 \leqslant j \leqslant f$. Now let p_{j_0} be the minimum of the column indices of the nodes in $\tilde{\Pi}$ that belong to row a of G. We then have, $q_1 \leqslant q_{e-d} < p_1 \leqslant p_{j_0}$. It follows that node (b, q_1) does not belong to $\tilde{\Pi}$, giving the desired contradiction.

As neither of the cases is possible, the assumption that w_{F_1} and w_{F_2} are prefixes of a single element of $Y(\lambda)$ is false. So $|Y(\lambda)| \geqslant 2$. □

We are grateful to the reviewer for providing the following alternative proof of the 'only if' part of the proof of Theorem 2.11 which illustrates the use of W-graph ideals.

Proof. By [Ngu12, Lemma 4.5], $Z(\lambda)^{-1}$ is the unique maximal left cell of $\mathcal{X}^{-1}_{J(\lambda)}$. By [Ngu12, Corollary 4.9], it is an S_n-graph ideal. Suppose now that $|Y(\lambda)| = 1$ for some composition λ. Then $\mathcal{E}^{(\lambda)}$ has a unique diagram D and w_D^{-1} generates an S_n-graph ideal. It follows from [Ngu15, Theorem 4.16], that D is a skew diagram with shape μ/ν. Since cell representations of S_n are irreducible (over $\mathbb{Q}$), either $\nu = \emptyset$, or else $\nu \neq \emptyset$, and the diagram of μ has exactly one removable node. Hence, the shape λ of D must be either a partition or the reverse of a partition. □

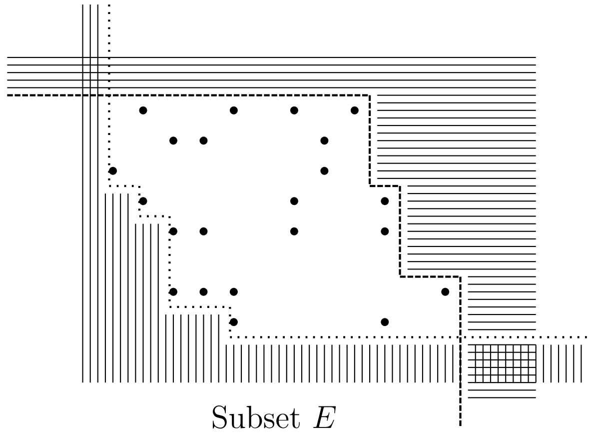
3 Ordered k-paths

In this section we introduce the notion of an ordered k-path. This will play a key role in determining the rim for certain families of Kazhdan-Lusztig cells (see Section 4).

We first consider a relation between non-empty subsets of a diagram D.

Definition 3.1. Let D be a diagram and let D_1, D_2 be non-empty subsets of D. We write "$D_1 \prec D_2$" if whenever (a_1, b_1) and (a_2, b_2) are nodes of D_1 and D_2 respectively with $a_1 \leq a_2$, then $b_1 < b_2$; otherwise we write "$D_1 \not\prec D_2$". Note that $D_1 \not\prec D_2$ if $D_1 \cap D_2 \neq \varnothing$. In particular, $E \not\prec E$ for any non-empty subset E of D.

Example 3.2. In this picture of a non-empty subset E of a diagram D, only the nodes of D which belong to E are included and they are represented by small black discs. The distance between the two nodes of lower column index in the second row of E is one unit.



Subset E

An arbitrary node (a, b) of D satisfies $\{(a, b)\} \prec E$ if, and only if, (a, b) is to the left and below the dotted line (⋯⋯), which partitions $\mathbb{Z}^2$ into two infinite regions. Similarly, an arbitrary node (a, b) of D satisfies $E \prec \{(a, b)\}$ if, and only if, (a, b) is to the right and above the dashed line (- - - - -), which partitions $\mathbb{Z}^2$ into two infinite regions.

We make this more precise in the following definitions and lemma.

Definition 3.3. Let E be a non-empty subset of a diagram D. Let $c_D = \sup\{b \in \mathbb{N} \colon (a, b) \in D\}$ and, for each $n \in \mathbb{N}$, let $E(\leqslant n) = \{b \in \mathbb{N} \colon (a, b) \in E$ for some $a \leqslant n\}$ and $E(\geqslant n) = \{b \in \mathbb{N} \colon (a, b) \in E$ for some $a \geqslant n\}$. Define $\gamma_E(n) = \sup E(\leqslant n)$ if $E(\leqslant n) \neq \varnothing$ (resp., $\gamma_E(n) = 0$ if $E(\leqslant n) = \varnothing$) and $\delta_E(n) = \inf E(\geqslant n)$ if $E(\geqslant n) \neq \varnothing$ (resp., $\delta_E(n) = 1 + c_D$ if $E(\geqslant n) = \varnothing$). We also define the *right side* $\mathcal{R}(E)$ and the *left side* $\mathcal{L}(E)$ of E to be the sets $\mathcal{R}(E) = \{(n, m) \in \mathbb{N}^2 \colon m > \gamma_E(n)\}$ and $\mathcal{L}(E) = \{(n, m) \in \mathbb{N}^2 \colon m < \delta_E(n)\}$.

Lemma 3.4. *If D_1 and D_2 are non-empty subsets of a diagram D, then the following statements are equivalent:*

(i) $D_1 \subseteq \mathcal{L}(D_2)$, *(ii)* $D_2 \subseteq \mathcal{R}(D_1)$, *(iii)* $D_1 \prec D_2$.

Proof. (i)⇒(ii): From (i), if $(a, b) \in D_1$ then $(a, b) \in \mathcal{L}(D_2)$. So $b < \delta_{D_2}(a)$. That is, $b < d$ if $(c, d) \in D_2$ and $a \leq c$.

Hence, given $(c, d) \in D_2$, $d > b$ for all nodes $(a, b) \in D_1$ with $a \leq c$. That is, $d > \gamma_{D_1}(c)$. So, $(c, d) \in \mathcal{R}(D_1)$. Thus, $D_2 \subseteq \mathcal{R}(D_1)$.

(ii)⇒(iii): From (ii), if $(c, d) \in D_2$ then $(c, d) \in \mathcal{R}(D_1)$. So $d > \gamma_{D_1}(c)$. That is, $b < d$ if $(a, b) \in D_1$ and $a \leq c$. Hence, if $(a, b) \in D_1$ and $(c, d) \in D_2$ and $a \leq c$ then $b < d$. That is, $D_1 \prec D_2$.

(iii)⇒(i): Let $(a, b) \in D_1$. If $(c, d) \in D_2$ with $a \leq c$, then by (iii), $b < d$. Hence, $b < \delta_{D_2}(a)$. So, $(a, b) \in \mathcal{L}(D_2)$. Hence, $D_1 \subseteq \mathcal{L}(D_2)$. □

Remark 3.5. We collect some immediate observations regarding the relation $\prec$. Let D be a diagram and let D_1, D_2, D_3 be non-empty subsets of D.

(i) If D_1' and D_2' are non-empty subsets of D_1 and D_2, respectively, and $D_1 \prec D_2$ then $D_1' \prec D_2'$.

(ii) If $D_1 \prec D_3$ and $D_2 \prec D_3$, then $(D_1 \cup D_2) \prec D_3$.

(iii) If $D_1 \prec D_2$ and $D_2 \prec D_3$, it is not true in general that $D_1 \prec D_3$. For example, take $D_1 = \{(2,3)\}$, $D_2 = \{(1,1)\}$ and $D_3 = \{(3,2)\}$.

(iv) It is possible for both $D_1 \prec D_2$ and $D_2 \prec D_1$ to be true. For example, take $D_1 = \{(1,1)\}$ and $D_2 = \{(2,2)\}$.

We now give the definition of an ordered k-path in a diagram D.

Definition 3.6. Let $\Pi = (\pi_1, \ldots, \pi_k)$ be a k-path in a diagram D. We say that Π is *ordered* if $s(\pi_i) \prec s(\pi_j)$ whenever $i < j$ $(1 \leqslant i, j \leqslant k)$. It is then immediate that Π is ordered if, and only if, $\bigcup_{i=1}^{j-1} s(\pi_i) \prec s(\pi_j)$ for $2 \leq j \leq k$.

Remark 3.7. Keeping the setup and notation of Proposition 2.7, it does not follow, in general, that if Π is an ordered k-path then $\Pi\theta$ is an ordered k-path. Consider for example

$$t^E w_D = \begin{array}{cccccc} 1 & 4 & & & & \\ 2 & & & 5 & 8 & \\ 3 & & 7 & & 9 & 10 \\ & 6 & & & & \end{array} \qquad t_D = \begin{array}{ccccc} 1 & 4 & & & \\ 2 & 5 & & 8 & \\ 3 & & 7 & 9 & 10 \\ & 6 & & & \end{array}$$

The partition $\{\{1,2,3,6\},\{4,7\},\{5,9\},\{8,10\}\}$ of the entries in $t^E w_D$ gives an ordered 4-path in E, while the corresponding 4-path in D is not ordered.

However, we can prove the following lemma. We will need a definition first.

Definition 3.8. Let $\lambda \vDash n$ and let $D \in \mathcal{D}^{(\lambda)}$. Suppose that $D = s(\Pi)$ for some ordered k-path Π with $\Pi = (\pi_1, \ldots, \pi_k)$. We denote by $D(\Pi)$ the diagram in $\mathcal{D}^{(\lambda)}$ constructed from D by replacing each node of π_j by a node on the same row but in the column j, for $j = 1, \ldots, k$. Observe that carrying out the same operation on the tableau t_D results in the $D(\Pi)$-tableau $t^{D(\Pi)} w_D$.

Lemma 3.9. *Let $\lambda \vDash n$ and let $D \in \mathcal{D}^{(\lambda)}$. Suppose that $D = s(\Pi)$ for some ordered k-path Π. Then $t^{D(\Pi)} w_D$ is a standard $D(\Pi)$-tableau.*

Proof. Let $\Pi = (\pi_1, \ldots, \pi_k)$ and set $E = D(\Pi)$. Let (a,j) and (g,j') be nodes of E with $a \leqslant g$ and $j \leqslant j'$, and let (a,b) and (g,h) be the corresponding nodes of D. (Note that $(a,b) = (a,j)\theta_{D,E}^{-1}$ and $(b,h) = (g,j')\theta_{D,E}^{-1}$, with the bijection $\theta_{D,E} : D \to E$ as defined in Section 2.3.) Then (a,b) is on π_j and (g,h) is on $\pi_{j'}$. We claim that $b \leqslant h$. To see this, we can use the fact that π_j is a path for the case $j = j'$, and the fact that the k-path Π is ordered, and so $s(\pi_j) \prec s(\pi_{j'})$, for the case $j < j'$. Hence, $(a,j)t^E w_D = (a,b)t_D \leqslant (g,h)t_D = (g,j')t^E w_D$. This shows that $t^E w_D$ is a standard E-tableau. $\square$

It will be convenient to prove some further elementary results concerning the relation $\prec$.

Lemma 3.10. *Let π be a path in a diagram D and let (a', b') be a node of D which is not in $s(\pi)$ and such that $s(\pi)$ does not have two nodes (a_1, b') and (a_2, b') with $a_1 < a' < a_2$. Let π' be the path of length one with $s(\pi') = \{(a', b')\}$. Then either $s(\pi') \prec s(\pi)$ or $s(\pi) \prec s(\pi')$.*

Proof. Suppose that $s(\pi') \not\prec s(\pi)$. Then there is a node $(c, d) \in s(\pi)$ such that $a' \leqslant c$ and $b' \geqslant d$. Let $(a, b) \in s(\pi)$ with $a \leqslant a'$. Since $a \leqslant c$, $b \leqslant d \leqslant b'$. If $b = b'$ then $b' = d$. So $a \neq a'$ and $a' \neq c$. Since (a', b') is between the nodes (a, b') and (c, b') of π, this contradicts the hypothesis. Hence $b < b'$ and we have shown that $s(\pi) \prec s(\pi')$. □

Lemma 3.11. *Let (π_1, π_2) be an ordered 2-path in a diagram D and let π' be a path of length one in D such that $s(\pi') \cap (s(\pi_1) \cup s(\pi_2)) = \varnothing$ and $s(\pi_1) \not\prec s(\pi')$. Then $s(\pi') \prec s(\pi_2)$.*

Proof. Suppose that $s(\pi') \not\prec s(\pi_2)$. Let $s(\pi') = \{(a', b')\}$. There is a node $(c, d) \in s(\pi_2)$ such that $a' \leqslant c$ and $b' \geqslant d$. Also, by hypothesis, there is a node $(a, b) \in s(\pi_1)$ such that $a \leqslant a'$ and $b \geqslant b'$. Hence, $a \leqslant c$ and $b \geqslant d$. However, since (π_1, π_2) is an ordered 2-path and $a \leqslant c$, we get $b < d$. This contradiction establishes the result. □

Remark 3.12. Let Π be a path with at least k nodes in a diagram D. Then Π is equivalent to an ordered k-path Π' in D. For example, if $s(\Pi) = \{(a_1, b_1), \ldots, (a_l, b_l)\}$ where $a_1 < \cdots < a_l$ and $k \leqslant l$, let $\Pi' = (\pi_1, \ldots, \pi_k)$ with $s(\pi_k) = \{(a_1, b_1), \ldots, (a_{l-k+1}, b_{l-k+1})\}$ and $s(\pi_i) = \{(a_{l-i+1}, b_{l-i+1})\}$ for $1 \leqslant i < k$.

Theorem 3.13. *Let $k \geq 1$ and suppose Π is a k-path in a diagram D. Then Π is equivalent to an ordered k-path in D.*

Proof. In view of Remark 3.12, it is enough to construct an ordered k'-path $\Pi' = (\rho_{k'}, \ldots, \rho_1)$ in D where $1 \leq k' \leq k$, with $s(\Pi) = s(\Pi')$.

Our first task is to construct a path ρ_1 and we do this in a sequence of steps.

First construction:

Set $\Pi_0 = s(\Pi)$, $\pi_0 = \varnothing$; also $\Pi_0 \neq \varnothing$ by the definition of a path. We construct two sequences $\{\Pi_r\}_{r\geq 0}$ and $\{\pi_r\}_{r\geq 0}$ of sets of nodes of D.

Let $r \geq 1$ and assume that Π_{r-1} and π_{r-1} have been constructed.

- (A0) If $\Pi_{r-1} = \varnothing$, we terminate both sequences.
- (A1) If $\Pi_{r-1} \neq \varnothing$, let i_r be the least index of a row in D with a node in Π_{r-1}.
- (A2) Let $x_r = (i_r, j_r)$ be the node in Π_{r-1} with greatest column index j_r.
- (A3) If π_{r-1} has no node with column index greater than j_r, let $\pi_r = \pi_{r-1} \cup \{x_r\}$; otherwise, let $\pi_r = \pi_{r-1}$.
- (A4) Obtain Π_r from Π_{r-1} by removing all nodes in it with row-index i_r.
- (A5) Replace r be $r + 1$ and repeat (A0)-(A5).

In this process, $|\Pi_r| < |\Pi_{r-1}|$ if $|\Pi_{r-1}| > 0$. Hence, the process must terminate. Also, since $\Pi_0 \neq \varnothing$, the node $x_1 = (i_1, j_1)$ exists and $\pi_1 = \{x_1\} \neq \varnothing$. Let $\rho_1 = \bigcup_{r\geq 1} \pi_r$.

By the first construction, ρ_1 is a path in Π_0. If $\Pi_0 = s(\rho_1)$, we can set $\Pi' = (\rho_1)$ and obtain the result. So we may assume $\Pi_0 - s(\rho_1) \neq \varnothing$. We show that $\Pi_0 - s(\rho_1) \prec s(\rho_1)$.

Let $(c_1, d_1) \in \Pi_0 - s(\rho_1)$ and $(c_2, d_2) \in s(\rho_1)$ with $c_1 \leq c_2$. If $c_1 = c_2$, then $d_1 < d_2$ from the construction. If $c_1 < c_2$, there exists a node (c_0, d_0) in ρ_1 with $c_0 \leq c_1 (< c_2)$ and $d_0 > d_1$ from the construction. Now (c_0, d_0), (c_2, d_2) are nodes of ρ_1 with $c_0 < c_2$. Since ρ_1 is a path, $d_0 \leq d_2$. But $d_1 < d_0$, so $d_1 < d_2$ in this case also. We conclude that $\Pi_0 - s(\rho_1) \prec s(\rho_1)$.

Second construction:

Now let $P_0 = P_1 = \Pi_0$ and recall that $s(\rho_1) \subseteq P_1$. We construct two sequences $\{P_q\}_{q\geq 1}$ and $\{\rho_q\}_{q\geq 1}$ where P_q is a set of nodes of D and ρ_q is a path in D with $s(\rho_q) \subseteq P_q$, for $q \geq 1$.
Let $r > 1$ and assume that P_{r-1} and ρ_{r-1} have been constructed so that $P_{r-1} \subseteq P_{r-2}$ and ρ_{r-1} is a path with $s(\rho_{r-1}) \subseteq P_{r-1}$.

- (B0) Let $P_r = P_{r-1} - s(\rho_{r-1})$.
- (B1) If $P_r = \varnothing$ then let $p = r - 1$ and let Π'' be the p-path $(\rho_p, \rho_{p-1}, \ldots, \rho_1)$.
- (B2) If $P_r \neq \varnothing$ then $P_r \prec s(\rho_{r-1})$ and we construct the path ρ_r with $s(\rho_r) \subseteq P_r$ using the first construction.
- (B3) Replace r by $r + 1$ and repeat steps (B0)-(B3).

In this process, if $|P_{r-1}| > 0$ then $|P_r| < |P_{r-1}|$ and, since $s(\rho_{q-1}) = P_{q-1} - P_q$ for $1 < q \leqslant r$, the paths $s(\rho_1), \ldots, s(\rho_{r-1})$ are mutually disjoint. Also, for $1 < q < r$, $s(\rho_{r-1}) \subseteq P_{r-1} \subseteq P_q$ and, since $P_q \prec s(\rho_{q-1})$, we get $s(\rho_{r-1}) \prec s(\rho_{q-1})$. Since the sizes of the sets in the sequence $\{P_r\}$ are strictly decreasing, the process must terminate and the p-path Π'' is an ordered p-path which is equivalent to Π.

If $p \leqslant k$, the remarks at the start of the proof complete the proof. So we can assume that $p > k$. We can easily deduce from this assumption that $P_{k+1} \neq \varnothing$ or, equivalently, that P_k is not the support of a path. It follows that there are nodes (a_k, b_k) and (a_{k+1}, b_{k+1}) in P_k such that $a_k \leqslant a_{k+1}$ and $b_k > b_{k+1}$. Now node (a_k, b_k) is not a node of ρ_{k-1}. Hence, by the first construction, there is a node (a_{k-1}, b_{k-1}) of path ρ_{k-1} such that $a_{k-1} \leqslant a_k$ and $b_{k-1} > b_k$, since (a_k, b_k) was not picked in forming ρ_{k-1}. Be repeating this argument we can find, for $k - 2 \geqslant l \geqslant 1$, a node (a_l, b_l) in path ρ_l with $a_l \leqslant a_{l+1}$ and $b_l > b_{l+1}$. Now, from the way they are located, the $k + 1$ nodes (a_{k+1}, b_{k+1}), $\ldots$, (a_1, b_1) cannot belong to k or fewer paths but clearly belong to the k-path Π. We have thus reached the desired final contradiction and completed the proof. $\square$

Example 3.14. This is an example of a 7-path which is not ordered and equivalent ordered k-paths. Let Π be the 7-path

$$
\begin{array}{llll}
(& ((1,1),(4,2),(6,4)), & ((1,2),(3,3),(4,3),(6,5)), & ((1,3),(3,4),(4,4)), \\
 & ((3,5),(4,5)), & ((1,4),(2,4),(5,4),(6,6)), & ((1,5),(3,6),(4,7)), \\
 & ((2,1),(5,3)) \quad), & &
\end{array}
$$

let Π'' be the 5-path

$$
\begin{array}{lll}
(& ((1,1),(2,1),(4,2)), & ((1,2),(3,3),(4,3),(5,3),(6,4)), \\
& ((1,3),(3,4),(4,4),(5,4),(6,5)), & ((1,4),(2,4),(3,5),(4,5),(6,6)), \\
& ((1,5),(3,6),(4,7)) \quad), &
\end{array}
$$

and let Π''' be the 7-path

$$
\begin{array}{llll}
(& ((2,1),(5,3)), & ((1,1),(4,2),(6,4)), & ((1,2),(3,3),(4,3),(6,5)), \\
& ((5,4),(6,6)), & ((1,3),(3,4),(4,4)), & ((1,4),(2,4),(3,5),(4,5)), \\
& ((1,5),(3,6),(4,7)) \quad). & &
\end{array}
$$

These three k-paths are described diagrammatically in Table 1, where the nodes on each path of a k-path are represented by the index of that path. Π'' is the equivalent ordered k'-path (here $k' = 5$) produced by the algorithm of Theorem 3.13. There are other equivalent ordered k-paths. Π''' is one such k-path with $k = 7$.

$$
\begin{array}{ccccccc}
1 & 2 & 3 & 5 & 6 & & \\
7 & & & 5 & & & \\
& & 2 & 3 & 4 & 6 & \\
& 1 & 2 & 3 & 4 & & 6 \\
& & 7 & 5 & & & \\
& & & 1 & 2 & 5 & \\
& & & \Pi & & &
\end{array}
\qquad
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & & \\
1 & & & 4 & & & \\
& & 2 & 3 & 4 & 5 & \\
& 1 & 2 & 3 & 4 & & 5 \\
& & 2 & 3 & & & \\
& & & 2 & 3 & 4 & \\
& & & \Pi'' & & &
\end{array}
\qquad
\begin{array}{ccccccc}
2 & 3 & 5 & 6 & 7 & & \\
1 & & & 6 & & & \\
& & 3 & 5 & 6 & 7 & \\
& 2 & 3 & 5 & 6 & & 7 \\
& & 1 & 4 & & & \\
& & & 2 & 3 & 4 & \\
& & & \Pi''' & & &
\end{array}
$$

Table 1: constituent paths indicated by path indices

Finally for this section we establish some results which will play some part in the arguments in Section 4.

Lemma 3.15. *Let $\Pi = (\pi_1, \ldots, \pi_k)$ be an ordered k-path in a diagram D, and let (a', b') be a node of D which is not in $s(\Pi)$.*

- *(i) If there is a path π_j, $1 \leqslant j \leqslant k$, with a pair of nodes (a_1, b') and (a_2, b') with $a_1 < a' < a_2$, then Π may be extended to an ordered k-path $\Pi' = (\pi_1, \ldots, \pi_{j-1}, \pi'_j, \pi_{j+1}, \ldots, \pi_k)$ where $s(\pi'_j) = s(\pi_j) \cup \{(a', b')\}$.*
- *(ii) If there is no path π_j, $1 \leqslant j \leqslant k$, with a pair of nodes (a_1, b') and (a_2, b') with $a_1 < a' < a_2$, then Π may be extended to an ordered $(k+1)$-path $\Pi' = (\pi_1, \ldots, \pi_{k'}, \pi', \pi_{k'+1}, \ldots, \pi_k)$ for some k', $0 \leqslant k' \leqslant k$, where $s(\pi') = \{(a', b')\}$.*

Proof. (i) In this case, it is immediate that π'_j is a path in D. If $i < j$ and $(a, b) \in s(\pi_i)$ satisfies $a \leqslant a'$, then $a < a_2$. So, $b < b'$. Hence, $s(\pi_i) \prec s(\pi'_j)$. However, if $j < i$ and $(a, b) \in s(\pi_i)$ satisfies $a' \leqslant a$, then $a_1 < a$. So, $b' < b$. Hence, $s(\pi'_j) \prec s(\pi_i)$. It follows that $\Pi' = (\pi_1, \ldots, \pi_{j-1}, \pi'_j, \pi_{j+1}, \ldots, \pi_k)$ is an ordered k-path.

(ii) Let π' be the path with $s(\pi') = \{(a', b')\}$. Let l be the maximum index, $0 \leqslant l \leqslant k$, such that $s(\pi_j) \prec s(\pi')$ for all j with $1 \leqslant j \leqslant l$; $l = 0$ indicates $s(\pi_1) \not\prec s(\pi')$. If $l = k$ then we may take $\Pi' = (\pi_1, \ldots, \pi_k, \pi')$. If $l < k$ then $s(\pi_{l+1}) \not\prec s(\pi')$. By Lemma 3.11, we get $s(\pi') \prec s(\pi_j)$ for $l + 2 \leqslant j \leqslant k$. Also, by Lemma 3.10, we get $s(\pi') \prec s(\pi_{l+1})$. So we may take $\Pi' = (\pi_1, \ldots, \pi_l, \pi', \pi_{l+1}, \ldots, \pi_k)$. $\square$

Corollary 3.16. *Let $\Pi = (\pi_1, \ldots, \pi_k)$ be an ordered k-path in a diagram D, and let (a'_i, b'_i), $1 \leqslant i \leqslant l$, be l distinct nodes of D which are not in $s(\Pi)$. If no path π_j, $1 \leqslant j \leqslant k$, contains a pair of nodes of the form $(a_{i,j,1}, b'_i)$, $(a_{i,j,2}, b'_i)$ with $a_{i,j,1} < a'_i < a_{i,j,2}$ for any i satisfying $1 \leqslant i \leqslant l$, then the paths $((a'_i, b'_i))$ may be inserted into the sequence Π to give an ordered $(k+l)$-path.*

Proof. The proof is a repeated application of Lemma 3.15(ii). □

We will refer to the process described in Corollary 3.16 as *extending an ordered k-path by paths of length one.*

Proposition 3.17. *Let $\lambda \vDash n$ and let $y \in Y(\lambda)$. Set $D = D(y, \lambda)$. Suppose that $D = s(\Pi)$ for some ordered k-path Π. Then (i) and (ii) below hold.*

(i) If $D(\Pi)$ is admissible and $D(\Pi) = D(w_{D(\Pi)}, \lambda)$, then $D = D(\Pi)$.

(ii) If Π has type λ', then $D = D(\Pi)$ and D is special.

Proof. (i) This is immediate from Lemmas 2.8 and 3.9.

(ii) If Π has type λ', then k equals the number of parts of λ' and, moreover, $D(\Pi) \in \mathcal{D}^{(\lambda)}$ is a special diagram. (In fact, $D(\Pi)$ is the unique element of $\mathcal{D}^{(\lambda,\mu)}$ where $\mu = (\mu_1, \ldots, \mu_k) \vDash n$ with $\mu_i =$ length of π_i for $1 \leqslant i \leqslant k$.) Combining item (i) of this proposition with Remark 2.4 and Result 5(ii) we get $D = D(\Pi)$ and hence the desired result. □

4 Determining the rim for certain families of cells

For an arbitrary composition $\lambda = (\lambda_1, \ldots, \lambda_r) \vDash n$, let $\lambda_* = (\lambda_1, \ldots, \lambda_r, 1) \vDash n+1$. In [MP17, Section 4], there is a well-defined mapping ψ from the set of admissible diagrams in $\mathcal{D}^{(\lambda)}$ to the set of admissible diagrams in $\mathcal{D}^{(\lambda_*)}$, which induces an injective mapping $\theta_* \colon Y(\lambda) \to Y(\lambda_*)$. For a given admissible diagram D in $\mathcal{D}^{(\lambda)}$, the diagram $D\psi$ is obtained by examining all diagrams constructed from D by appending an $(r+1)$-th row with a single node to D and selecting the diagram which is admissible and such that the column index of the new node is minimal. The mapping ψ induces an injection $\mathcal{E}^{(\lambda)} \to \mathcal{E}^{(\lambda_*)}$ and the mapping θ_* is then given by $w_D \mapsto w_{D\psi}$. In Proposition 4.2, we obtain a condition on λ which ensures that the mapping θ_* is a bijection.

Lemma 4.1. *Let $r \geqslant 2$, let $n \geqslant 2$, let $\lambda = (\lambda_1, \ldots, \lambda_r) \vDash n$ be an r-part composition with $\lambda_r = 1$. Also let $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{r-1}) \vDash n-1$ and let λ', the conjugate of λ, be given by $\lambda' = (\lambda'_1, \ldots, \lambda'_{r'})$.*

Suppose D is an admissible diagram in $\mathcal{D}^{(\lambda)}$. Then the diagram $\tilde{D}$, obtained from D by removing the r-th row, is an admissible diagram in $\mathcal{D}^{(\tilde{\lambda})}$. Moreover, for $1 \leqslant k \leqslant r'$, every k-path in D of length $\lambda'_1 + \cdots + \lambda'_k$ contains the node on the r-th row of D.

Proof. By definition, D has subsequence type λ'. Let ν be the subsequence type of $\tilde{D}$. By Result 5(i), $\nu \trianglelefteq (\tilde{\lambda})' = (\lambda'_1 - 1, \lambda'_2, \ldots, \lambda'_{r'})$. Let N denote the node on the r-th row of D, let Π be a k-path in D of length $\lambda'_1 + \cdots + \lambda'_k$, where $1 \leqslant k \leqslant r'$. If Π did not contain N, then Π would also be a k-path in $\tilde{D}$. However, k-paths in $\tilde{D}$ have length at most $\lambda'_1 + \cdots + \lambda'_k - 1$. Hence, Π must contain N. (Alternatively, this follows from Result 4.) Moreover, $s(\Pi) - \{N\}$ is the support of a k-path in $\tilde{D}$ of length $\lambda'_1 + \cdots + \lambda'_k - 1$. Thus $\nu = (\tilde{\lambda})'$ and so $\tilde{D}$ is admissible. $\square$

Proposition 4.2. *Let $r \geqslant 2$, let $n \geqslant 2$, let $\lambda = (\lambda_1, \ldots, \lambda_r) \vDash n$ be an r-part composition with $\lambda_r = 1$. Then the mapping $\theta_* \colon Y(\lambda) \to Y(\lambda_*)$, described in [MP17, Section 4], is a bijection.*

Proof. Since θ_* is injective by [MP17, Theorem 4.3], we need only prove that it is surjective. Let $y \in Y(\lambda_*)$ and let $D_* = D(y, \lambda_*)$. So $y = w_{D_*}$ and D_* is admissible. Thus, writing $\lambda' = (\lambda'_1, \ldots, \lambda'_{r'})$, we see that D_* has subsequence type $(\lambda_*)' = (\lambda'_1 + 1, \lambda'_2, \ldots, \lambda'_{r'})$. Let $N = (r, a)$ and $N_* = (r+1, b)$ be the nodes on the r-th and $(r+1)$-th rows of D_*, and let $D = D_* - \{N_*\}$. By Lemma 4.1, D is an admissible diagram in $\mathcal{D}^{(\lambda)}$ and if Π_* is a k-path in D_* of length $\lambda'_1 + \cdots + \lambda'_k + 1$, then Π_* contains N_*, $1 \leqslant k \leqslant r'$. Again by Lemma 4.1, $s(\Pi_*) - \{N_*\}$ is the support of a k-path in D which contains N. In particular, as D_* has a 1-path of length $\lambda'_1 + 1$ containing both N and N_*, $a \leqslant b$.

We now construct a diagram $\bar{D}$ from D by adding the node $\bar{N} = (r+1, a)$ as the single node on the $(r+1)$-th row. Since every k-path of length $\lambda'_1 + \cdots + \lambda'_k$ in D contains N, $1 \leqslant k \leqslant r'$, each may be extended to a k-path of length $\lambda'_1 + \cdots + \lambda'_k + 1$ in $\bar{D}$ by adding the node $\bar{N}$. Hence, the subsequence type ν of $\bar{D}$ satisfies $(\lambda_*)' \trianglelefteq \nu$. By Result 5(i), $\nu \trianglelefteq (\lambda_*)'$. Hence, $\nu = (\lambda_*)'$ and $\bar{D}$ is admissible. From [MP17, Section 4], or equivalently from the paragraph preceding Lemma 4.1, $\bar{D} = D\psi$.

If $a < b$ then $w_{\bar{D}} \neq w_{D_*}$ since no column of D is empty. Since $t^{\bar{D}} w_{D_*}$ is a standard $\bar{D}$-tableaux and $\bar{D}$ is admissible, it follows from Result 3 that $w_{D_*} \notin Y(\lambda_*)$. Since this is contrary to hypothesis, $a = b$, $D_* = \bar{D} = D\psi$. So $y = w_{D_*} = w_D \theta_*$. This concludes the proof. $\square$

Recall that the rim $Y(\lambda)$ of the right cell $\mathfrak{C}(\lambda)$ is given by $Y(\lambda) = \{w_D \colon D \in \mathcal{E}^{(\lambda)}\}$. Thus, informally, we see that the elements of $Y(\lambda_*)$, with λ as in Proposition 4.2, are obtained from the elements of $Y(\lambda)$ by constructing the diagrams in $\mathcal{E}^{(\lambda)}$, then forming the diagrams in $\mathcal{E}^{(\lambda_*)}$ by appending to each diagram in $\mathcal{E}^{(\lambda)}$ a new node in the column of the node on the last row, and taking the corresponding 'w' of the new diagrams. In [MP17, Remark 4.4] it is described how this process relates to the induction of cells (see [BV83, Proposition 3.15]).

We turn to deal with some special compositions and we begin with the case of compositions in which at most the first two parts are greater than 1.

Theorem 4.3. *Let $r \geqslant 3$ and $s \geqslant t \geqslant 1$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a composition where (λ_1, λ_2) is a permutation of (s, t) and $\lambda_i = 1$ if $i > 2$.*

(i) *If* $(\lambda_1, \lambda_2) = (s, t)$*, then* $\mathcal{E}^{(\lambda)} = \mathcal{E}_s^{(\lambda)} = \{V(\lambda)\}$.

(ii) *If* $(\lambda_1, \lambda_2) = (t, s)$*, then* $\mathcal{E}^{(\lambda)} = \{D_{t,s,u} \colon 1 \leqslant u \leqslant s-t+1\}$*, where,* $D_{t,s,u} = \{(1,u)\} \cup \{(1,i) \colon s-t+2 \leqslant i \leqslant s\} \cup \{(2,i) \colon 1 \leqslant i \leqslant s\} \cup \{(i,u) \colon 3 \leqslant i \leqslant r\}$. *Hence,* $\mathcal{E}^{(\lambda)} = \mathcal{E}_s^{(\lambda)}$*. Moreover,* $|\mathcal{E}^{(\lambda)}| = s-t+1$.

Proof. If $(\lambda_1, \lambda_2) = (s, t)$ then λ is a partition and this case is covered in Remark 2.5.

Now let $s \geqslant t \geqslant 1$ and suppose that $\lambda = (t, s, 1)$. Then λ' is the partition $3^1 2^{t-1} 1^{s-t}$. Let $d \in Y(\lambda)$ and let $D = D(d, \lambda)$ so that $d = w_D$. Since $d \in Z(\lambda)$, D is an admissible diagram. It follows that D contains a t-path Π' of length $2t+1$. We can assume that Π' is ordered in view of Theorem 3.13.

Since D has exactly 3 rows, all constituent paths of Π' have length $\leqslant 3$. Moreover, at least one of the constituent paths of Π' has length 3 (otherwise the length of Π' would be $\leqslant 2t$). Since D has exactly one node in the third row, exactly one of the constituent paths of Π' has length 3 (alternatively this can be deduced from the fact that D is of subsequence type λ'). Let π be the unique path in Π' of length 3. Then π contains one node in each one of the three rows of D. It follows that every one of the remaining $t-1$ paths of Π' has length 2 and contains one node in each of the first two rows of D.

The remaining $s-t$ nodes of D (the size of D is $s+t+1$) all belong to the second row of D. Clearly the nodes of π which are located in the first and third rows of D cannot both have column index which equals the column index of any of these $s-t$ nodes. Hence, by Corollary 3.16, we may extend Π' to an ordered s-path Π by the $s-t$ paths of length 1 whose nodes are the remaining nodes on the second row of D. Clearly Π has type λ' and the support of Π is the diagram D. By Proposition 3.17(ii), D is special. In particular, since D is a rearrangement of a Young diagram, every column of D contains a node located in the second row of D (so the nodes in the second row of D are the nodes $(2, r)$ for $1 \leqslant r \leqslant s$).

Suppose that the first node on the first row of $D\,(= D(d, \lambda))$ is $(1, u)$. Since the first row of D has t nodes, $1 \leqslant u \leqslant s-t+1$. Form a diagram F whose first row nodes are $(1, u)$, $(1, s-t+2)$, …, $(1, s)$, whose second row nodes are the same as D, and whose single third row node is $(3, u)$. Then $F \in \mathcal{D}^{(\lambda)}$, F is special, hence admissible, and $t^F w_D$ is a standard F-tableau. By Corollary 2.9, $D = F$. We may refer to the admissible diagram F just constructed as F_u. It is also clear the w_{F_u}, for $1 \leqslant u \leqslant s-t+1$ are mutually non-prefixes of one another. It follows that $\mathcal{E}^{(\lambda)} = \mathcal{E}_s^{(\lambda)} = \{F_u \colon 1 \leqslant u \leqslant s-t+1\}$. In particular $|\mathcal{E}^{(\lambda)}| = s-t+1$.

To complete the proof for $r > 3$, we have only to use Proposition 4.2 and the remarks following its proof. □

Remark 4.4. Combining Theorem 4.3 with Remark 2.10 we can determine $\mathcal{E}^{(\lambda)}$ for $\lambda = (1^r, a, b)$ where $r, a, b \geqslant 1$.

Before we consider all compositions with three parts, we describe some special (and hence admissible) diagrams which we will use in Theorem 4.6.

Example 4.5. Let $s \geqslant t \geqslant u \geqslant 1$.

(i): If $\lambda = (s, u, t)$ and $C \subseteq \{1, \ldots, t\}$ with $|C| = u$, then $F_C = \{(1, i)\colon 1 \leqslant i \leqslant s\} \cup \{(2, i)\colon i \in C\} \cup \{(3, i)\colon 1 \leqslant i \leqslant t\}$ is a special diagram.

If $\lambda = (8, 3, 5)$ and $C = \{2, 3, 4\}$, then F_C is the diagram
$$\begin{array}{cccccccc} \times & \times & \times & \times & \times & \times & \times & \times \\ & \times & \times & \times & & & & \\ \times & \times & \times & \times & \times & & & \end{array}$$

(ii): If $\lambda = (t, s, u)$ and $C \subseteq \{1, \ldots, s - t + u\}$ with $|C| = u$, then $G_C = \{(1, i)\colon i \in (C \cup \{s - t + u + 1, \ldots, s\})\} \cup \{(2, i)\colon 1 \leqslant i \leqslant s\} \cup \{(3, i)\colon i \in C\}$ is a special diagram.

If $\lambda = (5, 8, 3)$ and $C = \{2, 4, 5\}$, then G_C is the diagram
$$\begin{array}{cccccccc} & \times & & \times & \times & & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \\ & \times & & \times & \times & & & \end{array}$$

(iii): If $\lambda = (t, u, s)$ and $C \subseteq \{s-t+1, \ldots, s\}$ with $|C| = u$, then $H_C = \{(1, i)\colon s-t+1 \leqslant i \leqslant s\} \cup \{(2, i)\colon i \in C\} \cup \{(3, i)\colon 1 \leqslant i \leqslant s\}$ is a special diagram.

If $\lambda = (5, 3, 8)$ and $C = \{5, 6, 8\}$, then H_C is the diagram
$$\begin{array}{cccccccc} & & & \times & \times & \times & \times & \times \\ & & & & \times & \times & & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \end{array}$$

(iv): If $\lambda = (u, s, t)$ and $C \subseteq \{t - u + 1, \ldots, s\}$ with $|C| = u$, then $K_C = \{(1, i)\colon i \in C\} \cup \{(2, i)\colon 1 \leqslant i \leqslant s\} \cup \{(3, i)\colon i \in (\{1, \ldots, t - u\} \cup C)\}$ is a special diagram.

If $\lambda = (3, 8, 5)$ and $C = \{3, 5, 7\}$, then K_C is the diagram
$$\begin{array}{cccccccc} & & \times & & \times & & \times & \\ \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & & \times & & \times & \end{array}$$

(v): If $\lambda = (u, t, s)$ then $L = \{(1, i)\colon s - u + 1 \leqslant i \leqslant s\} \cup \{(2, i)\colon s - t + 1 \leqslant i \leqslant s\} \cup \{(3, i)\colon 1 \leqslant i \leqslant s\}$ is a special diagram.

If $\lambda = (3, 5, 8)$, then L is the diagram
$$\begin{array}{cccccccc} & & & & & \times & \times & \times \\ & & & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \end{array}$$

Theorem 4.6. *Let $s \geqslant t \geqslant u \geqslant 1$ and let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a composition which is a permutation of (s, t, u). Then, with $|C| = u$ and diagrams G_C and H_C as described in Example 4.5,*
$$\mathcal{E}^{(\lambda)} = \begin{cases} \{V(\lambda)\} & \textit{if} \quad \lambda = (s, t, u), \\ \{G_C\colon C \subseteq \{1, \ldots, s - t + u\}\} & \textit{if} \quad \lambda = (t, s, u), \\ \{H_C\colon C \subseteq \{s - t + 1, \ldots, s\}\} & \textit{if} \quad \lambda = (t, u, s), \\ \{\dot{M}\colon M \in \mathcal{E}^{(\dot{\lambda})}\} & \textit{if} \quad \lambda = (u, t, s),\ (u, s, t) \textit{ or } (s, u, t). \end{cases}$$
Moreover, in all cases, $\mathcal{E}^{(\lambda)} = \mathcal{E}_s^{(\lambda)}$ and the value of $|\mathcal{E}^{(\lambda)}|$ is given in Table 2.

Proof. Choose distinct $i_1, i_2, i_3 \in \{1, 2, 3\}$ so that $\lambda_{i_1} = s$, $\lambda_{i_2} = t$ and $\lambda_{i_3} = u$. Also let λ be as in the statement of the theorem.

Let $d \in Y(\lambda)$ and let $D = D(d, \lambda)$. Then D is an admissible diagram; so D has subsequence type λ'. Since $\lambda' = 3^u 2^{t-u} 1^{s-t}$ and D is admissible, it has a t-path $\Pi' = (\pi'_1, \ldots, \pi'_t)$ of length $2t + u$, which we may assume to be ordered by Theorem 3.13. For $1 \leqslant i \leqslant 3$, let z'_i be the number of paths of length i in Π'. Since each path of length 3 has a node on each row, $z'_3 \leqslant u$. Simple counting gives $z'_1 + z'_2 + z'_3 = t$ and $z'_1 + 2z'_2 + 3z'_3 = 2t + u$. Hence, $z'_2 + 2z'_3 = t + u$. Also, $z'_2 + z'_3 \leqslant t$. Hence, $z'_3 \geqslant u$. So, $z'_3 = u$, $z'_1 = 0$ and $z'_2 = t - u$.

The u paths of Π' of length 3 together contain all nodes on row i_3. The remaining $t-u$ paths of Π' have length 2 but have no nodes on row i_3. Hence, these paths contain all remaining $t-u$ nodes on row i_2.

The $s-t$ remaining nodes of D are on row i_1. None of these may be used to extend a path of length 2 in Π' to a path of length 3; otherwise, D would have a t-path of length $\geqslant 2t+u+1$. Moreover, if $i_1 = 2$ and π is a path of length 3 in Π', it is clear that the nodes of π in rows i_2 and i_3 cannot both have column index which equals the column index of any of these $s-t$ nodes of D on row i_1. Hence, by Corollary 3.16, we may extend the ordered t-path Π' by the $s-t$ paths of length one, whose nodes are these remaining nodes on row i_1, to an ordered s-path $\Pi = (\pi_1, \dots, \pi_s)$ of length $s+t+u$. Clearly, Π has z_i paths of length i where $z_1 = s-t$, $z_2 = t-u$, and $z_3 = u$. Since the support of Π is the whole of D and Π has type λ', D is special by Proposition 3.17(ii). So, $\mathcal{E}^{(\lambda)} = \mathcal{E}^{(\lambda)}_s$.

Let A and B be the sets of indices of the columns in D containing nodes on rows i_2 and i_3, respectively. Since D is special, the nodes on row i_1 are in columns $1, \dots, s$ and, in addition, $B \subseteq A \subseteq \{1, \dots, s\}$.

Consider first the case $\lambda = (t, s, u)$. Let C be the set of u smallest indices in A. Since $|A - C| = t - u$, $C \subseteq \{1, \dots, s-t+u\}$. Then $t^{G_C} w_D$, which is obtained from t_D by moving the entries on the third row to the left into the corresponding positions in G_C and moving the last $t-u$ entries on the first row to the right into the last $t-u$ columns, is a standard G_C-tableau. Since G_C is special, $D = G_C$ by Corollary 2.9.

Every u-subset C of $\{1, \dots, s-t+u\}$ gives rise to an admissible diagram $G_C \in \mathcal{E}^{(\lambda)}_s$. Moreover, if C_1 and C_2 are distinct u-subsets of $\{1, \dots, s-t+u\}$, then it is immediate that $t^{G_{C_1}} w_{G_{C_2}}$ is not a standard tableau. Hence, $\mathcal{E}^{(\lambda)} = \mathcal{E}^{(\lambda)}_s = \{G_C \colon C$ a u-subset of $\{1, \dots, s-t+u\}\}$ and $|\mathcal{E}^{(\lambda)}| = \begin{pmatrix} s-t+u \\ u \end{pmatrix}$.

Now consider $\lambda = (t, u, s)$. Let $\tilde{A} = \{s-t+1, \dots, s\}$ and let C be the subset of $\tilde{A}$ whose elements occupy the same positions in $\tilde{A}$ as those of B occupy in A. Then $t^{H_C} w_D$, which is obtained from t_D by moving the entries on the first row to the right into the columns given by $\tilde{A}$ and the entries in the second row to the columns given by C, is a standard H_C-tableau. By Corollary 2.9, $D = H_C$ since H_C is special.

Every u-subset C of $\{s-t+1, \dots, s\}$ gives rise to an admissible diagram $H_C \in \mathcal{E}^{(\lambda)}_s$. Moreover, if C_1 and C_2 are distinct u-subsets of $\{s-t+1, \dots, s\}$, then it is immediate that $t^{H_{C_1}} w_{H_{C_2}}$ is not a standard tableau. Hence, $\mathcal{E}^{(\lambda)} = \mathcal{E}^{(\lambda)}_s = \{H_C \colon C$ a u-subset of $\{s-t+1, \dots, s\}\}$ and $|\mathcal{E}^{(\lambda)}| = \begin{pmatrix} t \\ u \end{pmatrix}$.

Next consider $\lambda = (s, t, u)$. Then λ is a partition and this case is covered in Remark 2.5.

Finally, the diagrams in $\mathcal{E}^{(\lambda)}$ for $\lambda = (u, s, t)$, (s, u, t) and (u, t, s) are obtained by rotating through 180° those in $\mathcal{E}^{(\mu)}$ for $\mu = (t, s, u)$, (t, u, s) and (s, t, u), respectively (see Remark 2.10). Since $\mathcal{E}^{(\lambda)} = \mathcal{E}_s^{(\lambda)}$ for these cases also, $|\mathcal{E}^{(\lambda)}| = \binom{s-t+u}{u}$ or $\binom{t}{u}$ or 1 according as $\lambda = (u, s, t)$ or (s, u, t) or (u, t, s). This completes the proof. $\square$

Direct arguments can also be given for the cases of $\lambda = (s, u, t)$, (u, s, t) and (u, t, s). The diagrams in $\mathcal{E}^{(\lambda)}$ arising in these cases are described in Examples 4.5 (i), (iv) and (v).

λ	(s,t,u)	(s,u,t)	(t,s,u)	(t,u,s)	(u,s,t)	(u,t,s)
$\lvert\mathcal{E}^{(\lambda)}\rvert$	1	$\binom{t}{u}$	$\binom{s-t+u}{u}$	$\binom{t}{u}$	$\binom{s-t+u}{u}$	1

Table 2: $s \geqslant t \geqslant u \geqslant 1$. (Theorem 4.6)

We conclude this section with a theorem in which $\mathcal{E}^{(\lambda)}$ and $\mathcal{E}_s^{(\lambda)}$ are determined for the family of compositions $\lambda = (1, 2^{r-2}, 1)$ and, as it turns out, $\mathcal{E}^{(\lambda)} \neq \mathcal{E}_s^{(\lambda)}$ for $r > 3$. We begin by identifying certain admissible diagrams for such compositions.

Example 4.7. Let $r \geqslant 3$ and let $\lambda = (1, 2^{r-2}, 1)$. Define $P^{(0)} = \{(i, 1) \colon 1 \leqslant i \leqslant r\} \cup \{(i, 2) \colon 2 \leqslant i \leqslant r - 1\}$, $P^{(r-2)} = \{(i, 1) \colon 2 \leqslant i \leqslant r - 1\} \cup \{(i, 2) \colon 1 \leqslant i \leqslant r\}$ and, for $1 \leqslant v \leqslant r - 3$, define $P^{(v)} = \{(i, 1) \colon 2 \leqslant i \leqslant v + 1\} \cup \{(i, 2) \colon 1 \leqslant i \leqslant r\} \cup \{(i, 3) \colon v + 2 \leqslant i \leqslant r - 1\}$. Thus,

$$P^{(0)} = \begin{array}{cc} \times & \\ \times & \times \\ \vdots & \vdots \\ \times & \times \\ \times & \times \\ \vdots & \vdots \\ \times & \times \\ \times & \end{array}, \quad P^{(r-2)} = \begin{array}{cc} & \times \\ \times & \times \\ \vdots & \vdots \\ \times & \times \\ \times & \times \\ \vdots & \vdots \\ \times & \times \\ & \times \end{array}, \quad P^{(v)} = \begin{array}{ccc} & \times & \\ \times & \times & \\ \vdots & \vdots & \\ \times & \times & \\ & \times & \times \\ & \vdots & \vdots \\ & \times & \times \\ & \times & \end{array} \qquad \text{for } 1 \leqslant v \leqslant r-3.$$

Since $\lambda' = (r, r - 2)$ and clearly each of these diagrams $P^{(v)}$, $0 \leqslant v \leqslant r - 2$, has a path of length r and a 2-path containing all $2r - 2$ nodes, they are all admissible. Moreover, $P^{(v)} = D(w_{P^{(v)}}, \lambda)$ for $0 \leqslant v \leqslant r - 2$. The tableaux $t_v = t_{P^{(v)}}$ are given by

$$t_0 = \begin{array}{cc} 1 & \\ 2 & r{+}1 \\ \vdots & \vdots \\ i & r{+}i{-}1 \\ i{+}1 & r{+}i \\ \vdots & \vdots \\ r{-}1 & 2r{-}2 \\ r & \end{array}, \quad t_{r-2} = \begin{array}{cc} & r-1 \\ 1 & r \\ \vdots & \vdots \\ i & r+i-1 \\ i+1 & r+i \\ \vdots & \vdots \\ r-2 & 2r-3 \\ & 2r-2 \end{array}, \quad t_v = \begin{array}{ccc} & v{+}1 & \\ 1 & v{+}2 & \\ \vdots & \vdots & \\ v & 2v{+}1 & \\ & 2v{+}2 & v{+}r{+}1 \\ & \vdots & \vdots \\ & v{+}r{-}1 & 2r{-}2 \\ & v{+}r & \end{array} \qquad \text{for } 1 \leqslant v \leqslant r-3.$$

The permutations $g_v = w_{P^{(v)}}$, $0 \leqslant v \leqslant r - 2$, are given by

$g_0 = [1, 2, r + 1, 3, r + 2, \ldots, r - 1, 2r - 2, r]$,

$g_{r-2} = [r - 1, 1, r, 2, r + 1, \ldots, r - 2, 2r - 3, 2r - 2]$, and for $1 \leqslant v \leqslant r - 3$,

$g_v = [v{+}1, 1, v{+}2, 2, \ldots, v, 2v{+}1, 2v{+}2, v{+}r{+}1, \ldots, v{+}r{-}1, 2r{-}2, v{+}r]$.

Since $P^{(v)}$ is admissible, $g_v \in Z(\lambda)$, $0 \leqslant v \leqslant r-2$, and it is immediate from a consideration of the tableaux t_v, that g_v is not a prefix of $g_{v'}$ if $v \neq v'$.

Theorem 4.8. *Let $r \geqslant 3$, $n = 2r-2$ and let λ be the composition $(1, 2^{r-2}, 1)$ of n. Then $\mathcal{E}^{(\lambda)} = \{P^{(v)} \colon 0 \leqslant v \leqslant r-2\}$, where $P^{(v)}$ is as described in Example 4.7, and $\mathcal{E}_s^{(\lambda)} = \{P^{(0)}, P^{(r-2)}\}$. So, $|\mathcal{E}_s^{(\lambda)}| = 2$ and $|\mathcal{E}^{(\lambda)}| = r-1$.*

Proof. Let $y \in Y(\lambda)$ and let $D = D(y, \lambda)$ so that $y = w_D$. Then $D = \{(1, j_1)\} \cup \{(i, j) \colon 2 \leqslant i \leqslant r-1,\ j = j_i, j'_i\} \cup \{(r, j'_r)\}$, where we write $j_i < j'_i$ if $2 \leqslant i \leqslant r-1$. Since $\lambda' = (r, r-2)$ and D is admissible, it has a path π of length r and a 2-path $\Pi = (\pi_1, \pi_2)$ containing all its nodes. By Theorem 3.13, we may assume that this 2-path is ordered. Then $(i, j_i) \in \pi_1$ and $(i, j'_i) \in \pi_2$ for $2 \leqslant i \leqslant r-1$. So $j_2 \leqslant \cdots \leqslant j_{r-1}$ and $j'_2 \leqslant \cdots \leqslant j'_{r-1}$. Since π has a node on each row, $(1, j_1), (r, j'_r) \in \pi$, $j_1 \leqslant j'_r$, $j_1 \leqslant j'_2$, and $j_{r-1} \leqslant j'_r$.

If $j_1 \leqslant j_2$, we may assume that $(1, j_1), (r, j'_r)$ belong to π_1. Then $\Pi = (\pi_1, \pi_2)$ is ordered and has type $\lambda' = (r, r-2)$. Since $D = s(\Pi)$ and $D(\Pi) = P^{(0)}$, invoking Proposition 3.17(ii) we get $D = P^{(0)}$.

If $j'_{r-1} \leqslant j'_r$, we may assume that $(1, j_1), (r, j'_r)$ belong to π_2. Then $\Pi = (\pi_1, \pi_2)$ is ordered and has type λ'. Since $D = s(\Pi)$ and $D(\Pi) = P^{(r-2)}$, Proposition 3.17(ii) now ensures that $D = P^{(r-2)}$.

Now suppose that $j_2 < j_1$ and $j'_r < j'_{r-1}$. This cannot occur for $r = 3$ since in this case the node $(2, j_2)$ would be to the left of $(1, j_1)$ and the node $(2, j'_2)$ would be to the right of $(3, j'_3)$ contradicting the fact that D has a path of length 3. Hence, $r \geqslant 4$ and $(2, j'_2), (r-1, j_{r-1})$ belong to π. So, for some v, with $2 \leqslant v \leqslant r-2$, we have $j'_v \leqslant j_{v+1}$.

Let ρ_1, ρ_2, ρ_3 be the paths in D with $s(\rho_1) = \{(2, j_2), (3, j_3), \ldots, (v, j_v)\}$, $s(\rho_3) = \{(v+1, j'_{v+1}), (v+2, j'_{v+2}), \ldots, (r-1, j'_{r-1})\}$ and $s(\rho_2) = D - (s(\rho_1) \cup s(\rho_3))$. Note that the condition $j'_v \leqslant j_{v+1}$ ensures that ρ_2 is indeed a path. It follows that $D = s(\tilde{\Pi})$, where $\tilde{\Pi} = (\rho_1, \rho_2, \rho_3)$ is an ordered 3-path in D with $D(\Pi) = P^{(v-1)}$. By Proposition 3.17(i), $D = P^{(v-1)}$ in view of the fact that $P^{(v-1)}$ is admissible and $P^{(v-1)} = D(w_{P^{(v-1)}}, \lambda)$. (In particular, this shows that $j'_v \leqslant j_{v+1}$ for precisely one v with $0 \leqslant v \leqslant r-2$.)

This establishes that $\mathcal{E}^{(\lambda)} \subseteq \{P^{(v)} \colon 0 \leqslant v \leqslant r-2\}$. However, since each $g_v \in Z(\lambda)$ and no g_v is a prefix of any other one, each $g_v \in Y(\lambda)$. So, $\mathcal{E}^{(\lambda)} = \{P^{(v)} \colon 0 \leqslant v \leqslant r-2\}$. □

Corollary 4.9. *Let $r \geqslant 3$, $s \geqslant 1$, $t \geqslant 1$, $n = 2r+s+t-4$ and let λ be the composition $(1^s, 2^{r-2}, 1^t)$ of n. Then $|\mathcal{E}_s^{(\lambda)}| = 2$ and $|\mathcal{E}^{(\lambda)}| = r-1$.*

Proof. Apply Proposition 4.2 $t-1$ times to the case of the composition $(1, 2^{r-2}, 1)$, then apply Remark 2.10, then apply Proposition 4.2 $s-1$ times, and finally apply Remark 2.10. The diagrams in $\mathcal{E}^{(\lambda)}$ are obtained from the diagrams in Example 4.7 by extending the long column of nodes upward by $s-1$ nodes and downward by $t-1$ nodes. □

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