On subsequences and certain elements which determine various cells in S_n *

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Abstract

We study the relation between certain increasing and decreasing subsequences occurring in the row form of certain elements in the symmetric group, following Schensted (Canad. J. Math., 13, 1961, 179–191) and Greene (Advances in Math., 14, 1974, 254–265), and the Kazhdan-Lusztig cells (Invent. Math., 53, 1979, 165–184) of the symmetric group to which they belong. We show that, in the two-sided cell corresponding to a partition λ, there is an explicitly defined element d_λ, each of whose prefixes can be used to obtain a left cell by multiplying the cell containing the longest element of the parabolic subgroup associated with λ on the right. Furthermore, we show that the elements of these left cells are those which possess increasing and decreasing subsequences of certain types. The results in this paper lead to efficient algorithms for the explicit descriptions of many left cells inside a given two-sided cell, and the authors have implemented these algorithms in GAP.

1 Introduction

In this paper, we will consider the symmetric group S_n, where n is an arbitrary positive integer, viewing it as a permutation group on the symbols

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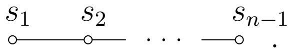
$\{1, \ldots, n\}$, acting on the right, and as a Coxeter group with Coxeter system (W, S) where $W = S_n$ and $S = \{s_1, \ldots, s_{n-1}\}$, where s_i is the transposition $(i, i+1)$. The corresponding Coxeter graph is 

. Moreover, W has the presentation $\langle s_i : s_i^2 = 1, (s_i s_{i+1})^3 = 1$ and $(s_i s_j)^2 = 1$ for all $i, j \in \{1, \ldots, n-1\}$ with $|i - j| > 1\rangle$.

For each subset $J \subseteq S$, the subgroup W_J generated by J is called a *standard parabolic subgroup* of W. It has a Coxeter system (W_J, J). Its length function l_J is that induced from l. It has a unique longest element w_J. By tradition, w_0 is written for w_S.

Throughout the paper, λ will denote an arbitrary partition $(\lambda_1, \ldots, \lambda_r)$ of n with r parts, whose parts satisfy $\lambda_1 \geq \ldots \geq \lambda_r$. Corresponding to λ, there is a standard parabolic subgroup of W whose Coxeter generator set $J(\lambda)$ is given by $J(\lambda) = S \backslash \{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, \ldots, s_{\lambda_1+\ldots+\lambda_{r-1}}\}$.

Kazhdan and Lusztig [10] introduced three equivalence relations $\sim_L$, $\sim_R$ and $\sim_{LR}$, the equivalence classes of which are called *left cells*, *right cells* and *two-sided cells*, respectively.

It is observed in [4, Lemma 1.2] and [11, Lemma 3.3] that the left and right cells containing $w_{J(\lambda)}$ has a particularly simple description. We extend this simple description to a selection of left and right cells in the same two-sided cell as $w_{J(\lambda)}$. Indeed, since for any left cell $\mathcal{C}$, the set $\{x^{-1} : x \in \mathcal{C}\}$ is a right cell, and conversely, we need only state the result for left cells. Blessenohl and Jöllenbeck [2] describe a process, which they call the crochet procedure, linking the left cells in a two-sided cell of S_n together. We show that for certain explicitly defined left cells $\mathcal{C}$ and $\mathcal{D}$ in a two-sided cell, this link takes the form of a right translation; that is, $\mathcal{D} = \mathcal{C}x$ for some explicitly defined $x \in W$.

These results lead to efficient algorithms for the explicit descriptions of many left cells inside a given two-sided cell. These algorithms have been implemented in GAP [6] by the authors, from whom they may be obtained on request.

2 Basic combinatorics of the symmetric group

In this section, we collect various basic definitions and results concerning W. We will describe an element w of W in different forms: as a word in the generators $s_1, \ldots, s_{n-1}$, as products of disjoint cycles on $1, \ldots, n$, and in *two-row form* in which an element in the lower row is the image under w of an element in the upper row. If, in the latter form, the upper row consists of the numbers $1, \ldots, n$ in their natural increasing order, we will refer to the lower row as the *row form* of the permutation and write it in the form $[w_1, \dots, w_n]$; that is, $iw = w_i$. If $U = (u_1, \dots, u_t)$ is any sequence with entries in $\{1, \dots, n\}$, let Uw be the sequence $(u_1 w, \dots, u_t w)$. We call Uw the image of U via the action by w.

The longest element w_0 of W is the permutation defined by $i \mapsto n+1-i$. More generally, the longest element $w_{J(\lambda)}$ of $W_{J(\lambda)}$ can be described in two-row form by

$$w_{J(\lambda)} = \begin{pmatrix} \dots & \widehat{\lambda}_{i-1}+1 & \dots & \widehat{\lambda}_i & \widehat{\lambda}_i+1 & \dots & \widehat{\lambda}_{i+1} & \dots \\ \dots & \widehat{\lambda}_i & \dots & \widehat{\lambda}_{i-1}+1 & \widehat{\lambda}_{i+1} & \dots & \widehat{\lambda}_i+1 & \dots \end{pmatrix}$$

where $\widehat{\lambda}_0 = 0$, $\widehat{\lambda}_r = n$, and $\widehat{\lambda}_i = \lambda_i + \widehat{\lambda}_{i-1}$ for $i = 1, \dots, r-1$. The *conjugate* partition λ' of λ is defined by $\lambda'_i = |\{j : \lambda_j \geq i\}|$ for $i \geq 1$. We will denote the number of parts of λ' by r'. Thus, $r' = \lambda_1$ and $r = \lambda'_1$. We use the notions of λ-*diagram* and λ-*tableaux* for a partition λ and associated terminology, in common use—see, for example, Fulton [5] or Sagan [12]. In particular, a λ-tableau is *row-standard* if it is increasing on rows, *column-standard* if it is increasing on columns, and *standard* if it is increasing on rows and columns. Also, if $\mathcal{T}$ is a λ-tableau, we refer to λ as the *shape* of $\mathcal{T}$ and denote it by $\operatorname{sh}\mathcal{T}$.

S_n acts on the set of λ-tableaux in the obvious way—if $w \in S_n$, an entry i is replaced by iw and tw denotes the tableau resulting from the action of w on the tableau t. This action on λ-tableaux is the action by letter permutations of Dipper and James [3, p.21]. If $x, y \in W$, we say that x is a *prefix* of y if $y = u_1 u_2 \dots u_p$ where $u_i \in S$ for $i = 1, \dots, p$, $p = l(y)$ and $x = u_1 u_2 \dots u_r$, for some $r \leq p$. The prefix relation corresponds to the *weak Bruhat order* in [3].

Proposition 2.1 ([8, Proposition 2.1.1 and Lemma 2.2.1]) *There is a special set of right coset representatives X_J associated with each parabolic subgroup W_J. An element of X_J is the unique element of minimum length in its coset. Moreover, if $w = vx$ where $v \in W_J$ and $x \in X_J$ then $l(w) = l(v) + l(x)$. Also, $X_J = \{w \in W : L(w) \subseteq S - J\}$ where $L(w) = \{s \in S : l(sw) < l(w)\}$ and, if d_J denotes the longest element in X_J, then X_J is the set of prefixes of d_J.* □

Dipper and James [3] characterise $X_{J(\lambda)}$ as follows:

Lemma 2.2 ([3, Lemma 1.1]) $X_{J(\lambda)} = \{w \in W : t^\lambda w \text{ is row-standard}\}$. □

Two special λ-tableaux t^λ and t_λ are constructed as follows. Let t^λ be obtained by filling in the λ-diagram with $1, \dots, n$ by rows and let t_λ be obtained by filling in the λ-diagram with $1, \ldots, n$ by columns. We define an element $w_\lambda \in S_n$ by $t^\lambda w_\lambda = t_\lambda$.

A further λ-tableau $\tilde{t}_\lambda$ is constructed as follows. Fill the λ-diagram with the integers n, $n-1$, ..., 2, 1, filling the columns in order beginning at the first, but filling each column from bottom to top. Then reverse the order of the entries in each row. An element d_λ of W is defined by $t^\lambda d_\lambda = \tilde{t}_\lambda$. We define the element e_λ by $w_\lambda = d_\lambda e_\lambda^{-1}$.

For example, if $\lambda = (4, 2, 2, 1)$, these tableaux are

$$t^\lambda = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & & \\ 7 & 8 & & \\ 9 & & & \end{matrix} \qquad t_\lambda = \begin{matrix} 1 & 5 & 8 & 9 \\ 2 & 6 & & \\ 3 & 7 & & \\ 4 & & & \end{matrix} \qquad \tilde{t}_\lambda = \begin{matrix} 1 & 2 & 3 & 6 \\ 4 & 7 & & \\ 5 & 8 & & \\ 9 & & & \end{matrix}$$

$w_\lambda = [1, 5, 8, 9, 2, 6, 3, 7, 4]$, $d_\lambda = [1, 2, 3, 6, 4, 7, 5, 8, 9]$ and $e_\lambda = [1, 4, 5, 9, 2, 7, 8, 3, 6]$.

Let $Y_{J(\lambda)}$ denote the set of prefixes of w_λ. Dipper and James [3] give the following characterisation of $Y_{J(\lambda)}$.

Lemma 2.3 ([3, Lemma 1.5].) *The mapping $u \mapsto t^\lambda u$ is a bijection of the set of prefixes of w_λ and the set of standard λ-tableaux.* □

Corollary 2.4 *d_λ is a prefix of w_λ.* □

We can modify the Dipper-James proof of Lemma 2.3 to obtain a similar characterisation of the set $Z_{J(\lambda)}$ of prefixes of d_λ.

Lemma 2.5 *The mapping $u \mapsto t^\lambda u$ is a bijection of the set of prefixes of d_λ and the set of those standard λ-tableaux which, when their row entries are reversed, are column-standard.*

Proof. Let $t_1 = t^\lambda w_{J(\lambda)}$. Let $w \in W$ and suppose that $t^\lambda w$ is standard and $t_1 w$ is column-standard. Let s_i be in the right descent set of w; that is, $l(ws_i) < l(w)$. Then $(i+1)w^{-1} < iw^{-1}$.

Let i and $i+1$ occur in the positions (a, b) and (a', b'), respectively, of $t^\lambda w$. That is, i is on row a and column b. Since $t^\lambda w$ is standard, $a' > a$ or $b' > b$. Since $(i+1)w^{-1}$ and iw^{-1} occur in the positions (a', b') and (a, b), respectively, of t^λ, either $a' = a$ and $b' < b$ or $a' < a$. Hence, $a' < a$ and $b' > b$. So, $t^\lambda w s_i$ is also standard.

In $t_1 w$, i and $i+1$ are in positions $(a, \lambda_a + 1 - b)$ and $(a', \lambda_{a'} + 1 - b')$, respectively. Since $t_1 w$ is column-standard and decreasing on rows, $\lambda_a + 1 - b > \lambda_{a'} + 1 - b'$. So, $t_1 w s_i$ is also column-standard.

By construction, $t^\lambda d_\lambda$ is standard and $t_1 d_\lambda$ is column-standard. If u is a proper prefix of d_λ, then us is a prefix of d_λ for some $s \in S$ with $l(us) =$

$l(u)+1$. By induction, $t^\lambda us$ is standard and t_1us is column-standard. By the preceding paragraph, $t^\lambda u$ is standard and t_1u is column-standard.

Now suppose that $u \in W$, $u \neq d_\lambda$, $t^\lambda u$ is standard and t_1u is column-standard. Then there are integers i and j with $1 \leq i, j \leq n$ and $j - i \geq 2$ such that i and j appear consecutively on a column of t_1u. Let (a_l, b_l) be the position containing l, for $l = i, \ldots, j$. Then $b_i = b_j \neq b_l$ if $i < l < j$. Hence, for some k, $i \leq k < j$, $b_k < b_{k+1}$. Since t_1u is column-standard and decreasing on rows, $a_k < a_{k+1}$. Hence, t_1us_k is column-standard.

In $t^\lambda u$, k and $k+1$ are in the positions $(a_k, \lambda_{a_k}+1-b_k)$ and $(a_{k+1}, \lambda_{a_{k+1}} + 1 - b_{k+1})$, respectively. Since $\lambda_{a_k} - b_k > \lambda_{a_{k+1}} - b_{k+1}$ and $t^\lambda u$ is standard, $t^\lambda us_k$ is also standard.

Since $a_k < a_{k+1}$, $l(us_k) = l(u)+1$. By the inductive hypothesis, us_k is a prefix of d_λ. Hence, u is a prefix of d_λ. □

It is immediate that the *transpose* t' of a λ-tableau t is a λ'-tableau. Moreover, t' is standard if, and only if, t is standard. Furthermore, we have the elementary lemma:

Lemma 2.6 *$\left(t^\lambda\right)' = t_{\lambda'}$, $(t_\lambda)' = t^{\lambda'}$ and $w_\lambda = w_{\lambda'}^{-1}$. Also, e_λ is a prefix of $w_{\lambda'}$.* □

We proceed to examine various reduced decompositions of the longest element w_0 of W. First note that the tableau $t^\lambda w_\lambda w_{J(\lambda')}$ is obtained by filling the λ-diagram with $1, \ldots, n$ by columns from left to right, filling each column from bottom to top. So, the tableau $t^\lambda w_\lambda w_{J(\lambda')} d_{\lambda'}$ is obtained by filling the λ-diagram with $1, \ldots, n$ by rows from bottom to top, filling each row from left to right. It is immediate that the tableaux $t^\lambda w_\lambda w_{J(\lambda')}$ and $t^\lambda w_\lambda w_{J(\lambda')} d_{\lambda'}$ are row-standard. We can now establish the following decomposition of w_0.

Lemma 2.7 *$w_0 = w_{J(\lambda)} w_\lambda w_{J(\lambda')} d_{\lambda'}$ and $l(w_0) = l(w_{J(\lambda)}) + l(w_\lambda) + l(w_{J(\lambda')}) + l(d_{\lambda'})$.*

Proof. Since $w_0 = (1,n)(2,n-1)\ldots$, it is easy to see that $(t^\lambda w_{J(\lambda)})w_0 = t^\lambda w_\lambda w_{J(\lambda')} d_{\lambda'}$. Hence, $w_0 = w_{J(\lambda)} w_\lambda w_{J(\lambda')} d_{\lambda'}$.

Now if $w \in W$ we can complete any reduced expression for w to a reduced expression for w_0. Hence, $l(w_0) = l(w_{J(\lambda)} w_\lambda w_{J(\lambda')}) + l(d_{\lambda'})$. Since $t^\lambda w_\lambda w_{J(\lambda')}$ is row-standard, it follows from Lemma 2.2 that $w_\lambda w_{J(\lambda')} \in X_{J(\lambda)}$. Hence, $l(w_{J(\lambda)} w_\lambda w_{J(\lambda')}) = l(w_{J(\lambda)}) + l(w_\lambda w_{J(\lambda')})$. Now t_λ is row-standard. So $w_\lambda \in X_{J(\lambda)}$ from Lemma 2.2. Moreover, using Lemma 2.2 we see that $w_\lambda^{-1} \in X_{J(\lambda')}$. Hence $l(w_\lambda w_{J(\lambda')}) = l(w_\lambda) + l(w_{J(\lambda')})$. This completes the proof. □

For example let $\lambda = (4,2,2,1)$.

$$\begin{array}{l}4\,3\,2\,1\\6\,5\\8\,7\\9\end{array} \xrightarrow{w_{J(\lambda)}} \begin{array}{l}1\,2\,3\,4\\5\,6\\7\,8\\9\end{array} \xrightarrow{w_\lambda} \begin{array}{l}1\,5\,8\,9\\2\,6\\3\,7\\4\end{array} \xrightarrow{w_{J(\lambda')}} \begin{array}{l}4\,7\,8\,9\\3\,6\\2\,5\\1\end{array} \xrightarrow{d_{\lambda'}} \begin{array}{l}6\,7\,8\,9\\4\,5\\2\,3\\1\end{array}$$

Corollary 2.8 *w_λ is a distinguished $W_{J(\lambda)} - W_{J(\lambda')}$-double coset representative.* □

Since w_0 is an involution, the following result is an easy consequence of Lemma 2.7.

Lemma 2.9 *If $w_0 = d^{-1} w_{J(\lambda)} w_\lambda w_{J(\lambda')} e$, for $e, d \in W$ with $l(w_0) = l(d) + l(w_{J(\lambda)}) + l(w_\lambda) + l(w_{J(\lambda')}) + l(e)$, then (i) $d = w_0 d_{\lambda'}^{-1} e w_0$; (ii) e is a prefix of $d_{\lambda'}$; and (iii) d is a prefix of d_λ. In particular, $d_\lambda = w_0 d_{\lambda'}^{-1} w_0$.*

*Conversely, if d is a prefix of d_λ, we can find $e \in W$ such that $w_0 = d^{-1} w_{J(\lambda)} w_\lambda$
$w_{J(\lambda')} e$, with $l(w_0) = l(d) + l(w_{J(\lambda)}) + l(w_\lambda) + l(w_{J(\lambda')}) + l(e)$.*

Proof. First note that $l(d) + l(e) = l(d_{\lambda'})$. From Lemma 2.7, $w_{J(\lambda)} w_\lambda w_{J(\lambda')} = w_0 d_{\lambda'}^{-1}$. Hence, $w_0 = d^{-1} w_0 d_{\lambda'}^{-1} e$. So $d^{-1} = w_0 e^{-1} d_{\lambda'} w_0$, from which (i) follows.

Inverting the equation in Lemma 2.7, we get $w_0 = d_{\lambda'}^{-1} w_{J(\lambda')} w_{\lambda'} w_{J(\lambda)}$. Replacing λ with λ', we get $w_0 = d_\lambda^{-1} w_{J(\lambda)} w_\lambda w_{J(\lambda')}$. Hence, from (i), $d_\lambda = w_0 d_{\lambda'}^{-1} w_0$.

From $d w_0 = w_0 d_{\lambda'}^{-1} e$, we get $l(d) = l(e^{-1} d_{\lambda'})$. Since $l(d) = l(d_{\lambda'}) - l(e)$, e is a prefix of $d_{\lambda'}$. This establishes (ii).

Inverting the given equation, we get $w_0 = e^{-1} w_{J(\lambda')} w_{\lambda'} w_{J(\lambda)} d$. By applying (ii), we see that d is a prefix of d_λ.

For the converse, recall that $w_0 = d_\lambda^{-1} w_{J(\lambda)} w_\lambda w_{J(\lambda')}$ and $l(w_0) = l(d_\lambda) + l(w_{J(\lambda)}) + l(w_\lambda) + l(w_{J(\lambda')})$. Let $e = w_{J(\lambda')} w_{\lambda'} w_{J(\lambda)} d w_0$. Since d is a prefix of d_λ, $l(d^{-1}) + l(w_{J(\lambda)}) + l(w_\lambda) + l(w_{J(\lambda')}) = l(d^{-1} w_{J(\lambda)} w_\lambda w_{J(\lambda')}) = l(w_0 e^{-1}) = l(w_0) - l(e)$, as required. □

The proofs of Lemmas 2.3 and 2.5 can be used to find reduced expressions of all prefixes of w_λ and d_λ. We use the ideas in these proofs to determine the lengths of w_λ and d_λ by establishing straightforward algorithms for producing reduced expressions for these elements. We illustrate the algorithms with examples in which $\lambda = (4, 3, 3)$. These algorithms have been implemented in GAP.

We first determine the length of d_λ.

Proposition 2.10 $l(d_\lambda) = \frac{1}{2} \sum_{j=1}^{r'} (j-1) \lambda'_j (\lambda'_j - 1)$.

Proof. Let t_1 be the tableau described in the proof of Lemma 2.5. In this proof, we will order the positions in a tableau of shape λ by defining the k-th position, for $1 \le k \le n$, to be the position of k in the tableau $t_1 d_\lambda$. For each such k, let i_k and j_k denote the numbers of the row and column, respectively, in which the k-th position occurs.

For any tableau t of shape λ and any k with $1 \leq k \leq n$, we form the following partition of the positions of t: $X(k) = \{(i_k, j_k)\}$, that is the k-th position; $R_1(k) = \{(i', j') : (i', j') \text{ is a position of } t \text{ and } j' > j_k \text{ or } j' = j_k \text{ and } i' < i_k\}$, $R_2(k) = \{(i', j') : j' < j_k \text{ and } i' < i_k\}$, and $R_3(k) = \{(i', j') : (i', j') \text{ is a position of } t \text{ and } i' > i_k \text{ and } j' \leq j_k \text{ or } i' = i_k \text{ and } j' < j_k\}$. This division into regions is illustrated in Figure 1. Let $r_1(k)$, $r_2(k)$ and $r_3(k)$ denote the number of positions in regions $R_1(k)$, $R_2(k)$ and $R_3(k)$, respectively.

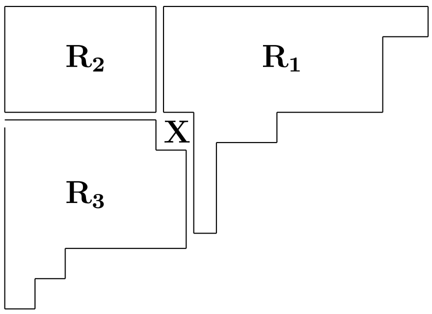


Figure 1

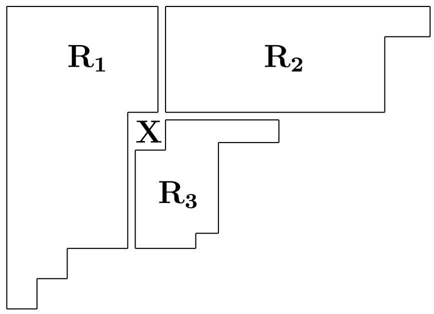


Figure 2

For each k, $1 \leq k \leq n$, we construct a sequence of tableaux $t^{k,p}$ of shape λ, where p runs from $r_2(k), \ldots, 1, 0$. The tableau $t^{k,p}$ is obtained by filling the first $k-1$ positions with $1, \ldots, k-1$ in that order, putting $k+p$ into the k-th position, and filling the positions in $R_2(k) \cup R_3(k)$ with the elements of the sequence $k, k+1, \ldots, n$, with $k+p$ removed, filling the rows from top to bottom, and filling each row from right to left. As constructed, each $t^{k,p}$ is a column-standard tableau and reversing the elements in each row clearly produces a standard tableau.

Associated with each tableau $t^{k,p}$, there is a permutation $g^{k,p}$ defined by $t_1 g^{k,p} = t^{k,p}$. By Lemma 2.5, each $g^{k,p}$ is a prefix of d_λ. Also, $g^{n,0} = d_\lambda$ since $t^{n,0} = t_1 d_\lambda$ and $g^{1,0} = 1$ since $t^{1,0} = t_1$.

The row-form of $g^{k,p}$ is obtained by writing in a single row the reversed rows of $t^{k,p}$, and placing the $(i+1)$-th reversed row immediately to the right of the i-th reversed row, for $i = 1, 2, \ldots$

Since $t^{k,p} s_{k+p-1} = t^{k,p-1}$, for $1 \leq p \leq r_2(k)$, it is immediate that $l(g^{k,p-1}) = l(g^{k,p}) + 1$, as the entries $k+p-1$ and $k+p$ are the only differences between the row-forms of $g^{k,p-1}$ and $g^{k,p}$, and $k+p-1$ occurs before $k+p$ in the latter. Hence, $l(g^{k,0}) = l(g^{k,r_2(k)}) + r_2(k)$.

In tableau $t^{k,0}$, the first k positions are filled with $1, \ldots, k$ in that order. If the $(k+1)$-th position is in the same column as the k-th position, then $j_{k+1} = j_k$ and $i_{k+1} = i_k + 1$. Since the positions of $R_2(k) \cup R_3(k)$ are filled by the sequence $k+1, \ldots, n$, the entry in the $(k+1)$-th position is $k+r_2(k)+j_k = k+1+i_k(j_k-1) = k+1+(i_{k+1}-1)(j_{k+1}-1) = k+1+r_2(k+1)$.

Hence, $t^{k,0} = t^{k+1,r_2(k+1)}$ in this case. However, if the $(k+1)$-th position is in a different column from the k-th position, then $j_{k+1} = j_k - 1$ and $i_{k+1} = 1$. Also, the entry in the $(k+1)$-th position is $k+1 = k+1+r_2(k+1)$, since $r_2(k+1) = 0$ in this case. So, we have $t^{k,0} = t^{k+1,r_2(k+1)}$ in this case also.

Hence, $l(d_\lambda) = l(g^{n,0}) = l(g^{n,r_2(n)}) + r_2(n) = l(g^{n-1,0}) + r_2(n) = \ldots = l(g^{1,0}) + \sum_{k=2}^{n} r_2(k) = \sum_{k=1}^{n} r_2(k)$, since $l(g^{1,0}) = 0$ and $r_2(1) = 0$. In general, $r_2(k) = (i-1)(j-1)$ if $i = i_k$ and $j = j_k$. Recalling that $r' = \lambda_1$ is the total number of columns, we see that $l(d_\lambda) = \sum_{j=1}^{r'} \sum_{i=1}^{\lambda'_j} (i-1)(j-1) = \sum_{j=1}^{r'} (j-1)\binom{\lambda'_j}{2}$. This completes the proof. □

Note that we have obtained implicitly a reduced expression for d_λ in the course of the proof of Proposition 2.10. We make this explicit in the following corollary.

Corollary 2.11 *For each k such that $1 \le k \le n$, let $p_k = s_{k+r_2(k)-1} \ldots s_k$, where $r_2(k)$ is as defined in the proof of Proposition 2.10 and $p_k = 1$ if $r_2(k) = 0$. Then $d_\lambda = p_1 p_2 \ldots p_n$ and, if those p_k which are equal to 1 are ignored, this is a reduced expression for d_λ.*

We illustrate this algorithm for $\lambda = (4,3,3)$. We display only the tableaux of the form $t^{k,0}$, where the k-th position is not on the first row or column, since for the excluded positions, with $k < n$, $t^{k,0} = t^{k+1,0}$. If the k-th position is not in the first row or column, we record the reduced word mapping $t^{k,0}$ to $t^{k+1,0}$, that is, $s_{k+r_2(k)-1} \ldots s_k$. The entry in the k-th position is given in bold-face font.

$$\begin{array}{cccc} 4 & 3 & 2 & 1 \\ 7 & 6 & \mathbf{5} & \\ 10 & 9 & 8 & \end{array} \xrightarrow{s_4 s_3} \begin{array}{cccc} 5 & 4 & 2 & 1 \\ 7 & 6 & 3 & \\ 10 & 9 & \mathbf{8} & \end{array} \xrightarrow{s_7 s_6 s_5 s_4} \begin{array}{cccc} 6 & 5 & 2 & 1 \\ 8 & \mathbf{7} & 3 & \\ 10 & 9 & 4 & \end{array} \xrightarrow{s_6} \begin{array}{cccc} 7 & 5 & 2 & 1 \\ 8 & 6 & 3 & \\ 10 & \mathbf{9} & 4 & \end{array} \xrightarrow{s_8 s_7} \begin{array}{cccc} 8 & 5 & 2 & 1 \\ 9 & 6 & 3 & \\ 10 & 7 & 4 & \end{array}$$

Hence, $d_\lambda = s_4 s_3 s_7 s_6 s_5 s_4 s_6 s_8 s_7$ and $l(d_\lambda) = 9$.

Now, we determine the length of w_λ.

Proposition 2.12 $l(w_\lambda) = \frac{1}{2}n(n+1) - \frac{1}{2}\sum_{j=1}^{r'} j\lambda'_j(\lambda'_j + 1)$.

Proof. In this proof, we will order the positions in a tableau of shape λ by defining the k-th position, for $1 \le k \le n$, to be the position of k in the tableau t_λ. For each such k, let i_k and j_k denote the numbers of the row and column, respectively, in which the k-th position occurs.

For any tableau t of shape λ and any k with $1 \le k \le n$, we form the following partition of the positions of t: $X(k) = \{(i_k, j_k)\}$, that is the k-th position; $R_1(k) = \{(i',j') : (i',j')$ is a position of t and $j' < j_k$ or $j' = j_k$ and $i' < i_k\}$, $R_2(k) = \{(i',j') : (i',j')$ is a position of t and $i' < i_k$ and $j' > j_k\}$, and $R_3(k) = \{(i',j') : (i',j')$ is a position of t and $i' > i_k$ and $j' \ge$

j_k or $i' = i_k$ and $j' > j_k\}$. This division into regions is illustrated in Figure 2. Let $r_1(k)$, $r_2(k)$ and $r_3(k)$ denote the number of positions in regions $R_1(k)$, $R_2(k)$ and $R_3(k)$, respectively.

For each k, $1 \leq k \leq n$, we construct a sequence of tableaux $t^{k,p}$ of shape λ, where p runs from $r_2(k), \ldots, 1, 0$. The tableau $t^{k,p}$ is obtained by filling the first $k-1$ positions with $1, \ldots, k-1$ in that order, putting $k+p$ into the k-th position, and filling the positions in $R_2(k) \cup R_3(k)$ with the elements of the sequence $k, k+1, \ldots, n$, with $k+p$ removed, filling the rows from top to bottom, and filling each row from left to right. As constructed, each $t^{k,p}$ is a standard tableau.

Associated with each tableau $t^{k,p}$, there is a permutation $g^{k,p}$ defined by $t^\lambda g^{k,p} = t^{k,p}$. By Lemma 2.3, each $g^{k,p}$ is a prefix of w_λ. Also, $g^{n,0} = w_\lambda$ since $t^{n,0} = t_\lambda$ and $g^{1,0} = 1$ since $t^{1,0} = t^\lambda$.

The row-form of $g^{k,p}$ is obtained by writing in a single row the rows of $t^{k,p}$, and placing the $(i+1)$-th row immediately to the right of the i-th row, for $i = 1, 2, \ldots$

Since $t^{k,p} s_{k+p-1} = t^{k,p-1}$, for $1 \leq p \leq r_2(k)$, it is immediate that $l(g^{k,p-1}) = l(g^{k,p}) + 1$, as the entries $k+p-1$ and $k+p$ are the only differences between the row-forms of $g^{k,p-1}$ and $g^{k,p}$, and $k+p-1$ occurs before $k+p$ in the latter. Hence, $l(g^{k,0}) = l(g^{k,r_2(k)}) + r_2(k)$.

In tableau $t^{k,0}$, the first k positions are filled with $1, \ldots, k$ in that order. If the $(k+1)$-th position is in the same column as the k-th position, then $j_{k+1} = j_k$ and $i_{k+1} = i_k + 1$. Since the positions of $R_2(k) \cup R_3(k)$ are filled by the sequence $k+1, \ldots, n$, the entry in the $(k+1)$-th position is $k + r_2(k+1) + 1$. Hence, $t^{k,0} = t^{k+1, r_2(k+1)}$ in this case. However, if the $(k+1)$-th position is in a different column from the k-th position, then $j_{k+1} = j_k + 1$ and $i_{k+1} = 1$. Also, the entry in the $(k+1)$-th position is $k+1 = k+1+r_2(k+1)$, since $r_2(k+1) = 0$ in this case. So, we have $t^{k,0} = t^{k+1,r_2(k+1)}$ in this case also.

Hence, $l(w_\lambda) = l(g^{n,0}) = l(g^{n,r_2(n)}) + r_2(n) = l(g^{n-1,0}) + r_2(n) = \ldots = l(g^{1,0}) + \sum_{k=2}^{n} r_2(k) = \sum_{k=1}^{n} r_2(k)$, since $l(g^{1,0}) = 0$ and $r_2(1) = 0$. In general, $r_2(k) = \sum_{\ell=1}^{i-1} (\lambda_\ell - j)$ if $i = i_k$ and $j = j_k$. Recalling that $r' = \lambda_1$ is the total number of columns, we see that $l(w_\lambda) = \displaystyle\sum_{j=1}^{r'} \sum_{i=1}^{\lambda'_j} \sum_{\ell=1}^{i-1} (\lambda_\ell - j) =$
$\displaystyle\sum_{j=1}^{r'} \sum_{i=1}^{\lambda'_j} (\lambda'_j - i)(\lambda_i - j) = \sum_{j=1}^{r'} \lambda'_j \sum_{i=1}^{\lambda'_j} (\lambda_i - j) - \sum_{j=1}^{r'} \sum_{i=1}^{\lambda'_j} i(\lambda_i - j) = \sum_{j=1}^{r'} \lambda'_j \sum_{\ell=j+1}^{r'} \lambda'_\ell -$
$\displaystyle\frac{1}{2} \sum_{j=1}^{r'} \sum_{\ell=j+1}^{r'} \lambda'_\ell(\lambda'_\ell + 1) = \frac{1}{2} n^2 - \frac{1}{2} \sum_{j=1}^{r'} \lambda'^2_j - \frac{1}{2} \sum_{j=1}^{r'} (j-1) \lambda'_j (\lambda'_j + 1) = \frac{1}{2} n(n+1) -$
$\frac{1}{2} \sum_{j=1}^{r'} j \lambda'_j (\lambda'_j + 1)$. This completes the proof. □

As in Proposition 2.10, we note that we have obtained implicitly a reduced expression for w_λ in the course of the proof of Proposition 2.12. We make this explicit in the following corollary.

Corollary 2.13 *For each k such that $1 \leq k \leq n$, let $p_k = s_{k+r_2(k)-1} \ldots s_k$, where $r_2(k)$ is as defined in the proof of Proposition 2.12 and $p_k = 1$ if $r_2(k) = 0$. Then $w_\lambda = p_1 p_2 \ldots p_n$ and, if those p_k which are equal to 1 are ignored, this is a reduced expression for w_λ.*

We illustrate this algorithm for $\lambda = (4, 3, 3)$. As with the illustration of the d_λ-algorithm, we display only the tableaux of the form $t^{k,0}$, noting that if the k-th position is on the first row or last column then $t^{k,0} = t^{k+1,0}$. If the k-th position is not in the first row or last column, we record the reduced word mapping $t^{k,0}$ to $t^{k+1,0}$, that is, $s_{k+r_2(k)-1} \ldots s_k$. The entry in the k-th position is given in bold-face font.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \mathbf{5} & 6 & 7 & \\ 8 & 9 & 10 & \end{array} \xrightarrow{s_4 s_3 s_2} \begin{array}{cccc} 1 & 3 & 4 & 5 \\ 2 & 6 & 7 & \\ \mathbf{8} & 9 & 10 & \end{array} \xrightarrow{s_7 s_6 s_5 s_4 s_3} \begin{array}{cccc} 1 & 4 & 5 & 6 \\ 2 & \mathbf{7} & 8 & \\ 3 & 9 & 10 & \end{array} \xrightarrow{s_6 s_5} \begin{array}{cccc} 1 & 4 & 6 & 7 \\ 2 & 5 & 8 & \\ 3 & \mathbf{9} & 10 & \end{array}$$

$$\xrightarrow{s_8 s_7 s_6} \begin{array}{cccc} 1 & 4 & 7 & 8 \\ 2 & 5 & \mathbf{9} & \\ 3 & 6 & 10 & \end{array} \xrightarrow{s_8} \begin{array}{cccc} 1 & 4 & 7 & 9 \\ 2 & 5 & 8 & \\ 3 & 6 & \mathbf{10} & \end{array} \xrightarrow{s_9} \begin{array}{cccc} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & \\ 3 & 6 & 9 & \end{array}$$

Hence, $w_\lambda = s_4 s_3 s_2 s_7 s_6 s_5 s_4 s_3 s_6 s_5 s_8 s_7 s_6 s_8 s_9$ and $l(w_\lambda) = 15$.

We end this section with some immediate corollaries of the two preceding propositions.

Corollary 2.14 *(i) $l(w_\lambda) = \frac{1}{2}n(n+1) - \frac{1}{2}\sum_{i=1}^{r} i\lambda_i(\lambda_i + 1)$; (ii) $l(d_\lambda) = \frac{1}{2}\sum_{i=1}^{r}(i-1)\lambda_i(\lambda_i - 1)$; and (iii) $l(e_\lambda) = \frac{1}{2}n(n+1) - \frac{1}{2}\sum_{i=1}^{r}((2i-1)\lambda_i^2 + \lambda_i)$.*

Proof. To prove (i), it is sufficient to note that $l(w_\lambda) = l(w_{\lambda'})$. Hence, $\sum_{i=1}^{r} i\lambda_i(\lambda_i + 1) = \sum_{j=1}^{r'} j\lambda'_j(\lambda'_j + 1)$. From this identity, together with $2\sum_{i=1}^{r} i\lambda_i = \sum_{j=1}^{r'} \lambda'_j(\lambda'_j + 1)$ and $\sum_{i=1}^{r} \lambda_i = n$ and the corresponding identities with λ and λ' interchanged, (ii) follows. Finally, $l(e_\lambda) = l(w_\lambda) - l(d_\lambda)$; and (iii) is immediate. □

An immediate consequence of this corollary is that $d_\lambda = 1$ if λ is a hook. It is also easily shown that $l(d_\lambda) = l(w_\lambda)$ if λ is a rectangle; and since d_λ is a prefix of w_λ, it follows that $d_\lambda = w_\lambda$ in this case.

3 Subsequences in the row form of a permutation

Let $w \in W$. A *decreasing cover of type* λ for w is a list C of r disjoint decreasing subsequences $C_1, \ldots, C_r$ appearing in the row form of w so that the union of the elements in these subsequences is $\{1, \ldots, n\}$ and the subsequences have lengths $\lambda_1, \ldots, \lambda_r$ in some order. Similarly, we define an *increasing cover of type* λ for w. For example, let $w = [6, 5, 2, 1, 4, 3] \in S_6$. Then, $C = [(6, 5, 4, 3), (2, 1)]$ and $C' = [(6, 5, 2, 1), (4, 3)]$ are decreasing covers of type $(4, 2)$ for w and $C'' = [(6, 4, 3), (5, 1), (2)]$ is a decreasing cover of type $(3, 2, 1)$ for w. Also, $[(1, 4), (2, 3), (5), (6)]$ is an increasing cover of type $(2, 2, 1, 1)$ for w.

It is clear that there is a *unique* decreasing cover of type λ for $w_{J(\lambda)}$. The subsequences occurring in this cover are called the *special decreasing subsequences* for $w_{J(\lambda)}$. The i-th such subsequence is obtained by reversing the sequence of consecutive integers from $1+\sum_{j=1}^{i-1} \lambda_j$ to $\sum_{j=1}^{i} \lambda_j$. We denote this subsequence by $P^\lambda(i)$ and its j-th member by $P^\lambda(i, j)$. The row form of $w_{J(\lambda)}$ is obtained by placing the sequences $P^\lambda(1), \ldots, P^\lambda(r)$ in a row going from left to right.

Lemma 3.1 If $d \in X_{J(\lambda)}$, $[P^\lambda(i)d : i = 1, \ldots, r]$ is a decreasing cover of type λ for $w_{J(\lambda)}d$. Consequently, $w_{J(\lambda)}d$ has row form

$$[\ \lambda_1 d\ \ldots\ 1d \,|\, (\lambda_1+\lambda_2)d\ \ldots\ (\lambda_1+1)d \,|\ \ldots\ |\, nd\ \ldots\ (n-\lambda_r+1)d\ ].$$

Proof. From Lemma 2.2, $P^\lambda(i)d$ is a decreasing sequence for $i = 1, \ldots, r$. □

For example, if $\lambda = (4, 2, 2, 1)$ then $d = [1, 2, 6, 8, 3, 5, 4, 9, 7] \in X_{J(\lambda)}$. The row form of $w_{J(\lambda)}d$ is $[\ 8\ \ 6\ \ 2\ \ 1 \,|\, 5\ \ 3 \,|\, 9\ \ 4 \,|\, 7\ ]$. The images of the special decreasing subsequences for $w_{J(\lambda)}$, via the action by d, are $(8, 6, 2, 1)$, $(5, 3)$, $(9, 4)$ and (7), and these form a decreasing cover of type $(4, 2, 2, 1)$ for $w_{J(\lambda)}d$. Note that the images of the special decreasing subsequences for $w_{J(\lambda)}$, via the action by d, are given by the rows of $t^\lambda d$, read in reverse order. Also, since $d \in Y_{J(\lambda)}$ in this case, $t^\lambda d$ is column-standard and its columns give an increasing cover of type $(4, 3, 1, 1)$ for $w_{J(\lambda)}d$.

In general, if $y, w \in W$, the row form of $y^{-1}w$ is obtained from the row form of w by letting y act on the *positions* of that row.

With $\lambda = (4, 2, 2, 1)$ and $e = [1, 4, 5, 9, 2, 6, 3, 7, 8] \in X_{J(\lambda)}$,

$$e^{-1}w_{J(\lambda)} = [\ 4\ \ 6\ \ 8\ \ 3\ \ 2\ \ 5\ \ 7\ \ 9\ \ 1\ ]$$

and the special decreasing subsequences for $w_{J(\lambda)}$ are

$$\begin{array}{|ccccccccc|}\hline 4&&&3&2&&&&1\\ &6&&&&5&&&\\ &&8&&&&7&&\\ &&&&&&&9&\\ \hline\end{array}\ .$$

Note that the position of $P^\lambda(i, j)$ in the row form of $e^{-1}w_{J(\lambda)}$ is given by the j-th entry of the i-th row of $t^\lambda e$. Also, since $e \in Y_{J(\lambda)}$ in this case, $t^\lambda e$ is column-standard. Moreover, the entries of $e^{-1}w_{J(\lambda)}$, whose positions form a column of $t^\lambda e$, is an increasing subsequence. So $e^{-1}w_{J(\lambda)}$ has an increasing cover of type $(4, 3, 1, 1)$.

Proposition 3.2 *Let $d, e \in X_{J(\lambda)}$. (i) The position of $P^\lambda(i,j)d$ in the row form of $e^{-1}w_{J(\lambda)}d$ is given by the j-th entry of the i-th row of $t^\lambda e$. (ii) $[P^\lambda(i)d : 1 \leq i \leq r]$ is a decreasing cover of type λ for $e^{-1}w_{J(\lambda)}d$. (iii) If $e \in Y_{J(\lambda)}$ and $d \in Z_{J(\lambda)}$ then the columns of $t^\lambda e$ give the positions in the row form of $e^{-1}w_{J(\lambda)}d$ corresponding to increasing subsequences and the resulting increasing cover has type λ'.*

Proof. (i) Let k be the position of $P^\lambda(i,j)d$ in the row form of $e^{-1}w_{J(\lambda)}d$; that is, $ke^{-1}w_{J(\lambda)}d = P^\lambda(i,j)d$. Then $k = P^\lambda(i,j)w_{J(\lambda)}e$ as required, since $P^\lambda(i,j)w_{J(\lambda)}$ is the j-th entry of the i-th row of t^λ.

(ii) From Lemma 3.1, $P^\lambda(i)d$ is a decreasing sequence. Since $e \in X_{J(\lambda)}$, the i-th row of $t^\lambda e$ is an increasing sequence. The image of the latter sequence under $e^{-1}w_{J(\lambda)}d$ is the former sequence. Hence, $P^\lambda(i)d$ is a decreasing sequence for $e^{-1}w_{J(\lambda)}d$. The result follows.

(iii) Since $t^\lambda e$ and $t^\lambda w_{J(\lambda)}d$ are both column-standard and $(t^\lambda e)(e^{-1}w_{J(\lambda)}d) = t^\lambda w_{J(\lambda)}d$, the result follows. □

With $\lambda = (4,2,2,1)$, $d = [1,2,6,8,3,5,4,9,7]$ and $e = [1,4,5,9,2,6,3,7,8]$ we see that $[(8,6,2,1),(5,3),(9,4),(7)]$ is a decreasing cover of type λ. Compare this with the rows of $t^\lambda d$. Compare also the positions of the members of $P^\lambda(i)$ with the rows of $t^\lambda e$.

$$e^{-1}w_{J(\lambda)}d = [\,8\ 5\ 9\ 6\ 2\ 3\ 4\ 7\ 1\,]$$

and the images of the special decreasing subsequences for $w_{J(\lambda)}$ via the action by d are

$$\boxed{\begin{array}{ccccccccc} 8 & & & 6 & 2 & & & & 1 \\ & 5 & & & & 3 & & & \\ & & 9 & & & & 4 & & \\ & & & & & & & 7 & \end{array}}\ .$$

Note that even though $d, e \in Y_{J(\lambda)}$ in this case, $e^{-1}w_{J(\lambda)}d$ does not have an increasing cover of type $(4,3,1,1)$.

If $w = [w_1, \ldots, w_n] \in W$ and $C = [C_1, \ldots, C_r]$ is an increasing or decreasing cover for w, we define $a_{w,C}(i,k)$ to be the number of elements of C_i occurring in the first k positions of w. For example, with $w = [6,5,2,1,4,3]$ and $C = [(6,5,4,3),(2,1)]$, we get $a_{w,C}(1,3) = 2$ and $a_{w,C}(2,3) = 1$.

Proposition 3.3 *Let $d \in X_{J(\lambda)}$, $e \in Y_{J(\lambda)}$ and $w = e^{-1}w_{J(\lambda)}d$. If C denotes the decreasing cover $[P^\lambda(i)d{:}i = 1, \ldots, r]$ for w, then $a_{w,C}(i,k) \geq a_{w,C}(i{+}1,k)$ for all $i = 1, \ldots, r-1$ and $k = 1, \ldots, n$.*

Proof. From Lemma 2.3, $t^\lambda e$ is a standard tableau. Now let $1 \leq k \leq n$. Since $t^\lambda e$ is standard, the positions corresponding to the entries $1, \ldots, k$ form a μ-diagram for some partition $\mu = (\mu_1, \ldots, \mu_s)$ of k and the corresponding μ-tableau is standard. Since $a_{w,C}(i,k) = \mu_i$ by Proposition 3.2(i), the result now follows. □

Lemma 3.4 *Let $d \in Z_{J(\lambda)}$. Partition the row form of $w = w_\lambda^{-1} w_{J(\lambda)} d$ into $r'(= \lambda_1)$ blocks so that, for $j = 1, \ldots, r'$, the j-th block contains the λ'_j elements from the $(1 + \sum_{k=1}^{j-1} \lambda'_k)$-th position to the $(\sum_{k=1}^{j} \lambda'_k)$-th position. Then, each of these blocks is an increasing subequence of w.*

Proof. Let $e = w_{J(\lambda')} w w_0$. From Lemma 2.9, $l(w_0) = l(d) + l(w_{J(\lambda)}) + l(w_\lambda) + l(w_{J(\lambda')}) + l(e)$ and e is a prefix from $d_{\lambda'}$. Since $d_{\lambda'} \in X_{J(\lambda')}$, we get $e \in X_{J(\lambda')}$.

From Lemma 3.1, we see that in the row form of $w_{J(\lambda')} e = w w_0$, for $i = 1, \ldots, r'$, the i-th block, containing the λ'_i elements from the $(1 + \sum_{j=1}^{i-1} \lambda'_j)$-th position to the $(\sum_{j=1}^{i} \lambda'_j)$-th position, forms a decreasing subsequence. Since postmultiplication by w_0 converts a decreasing subsequence to an increasing subsequence, the result follows. □

For example, let $\lambda = (4, 3, 3)$. Then $\lambda' = (3, 3, 3, 1)$. We may take $d = s_4 s_3 s_5 s_7$ and $e = s_3 s_2 s_6 s_5 s_4$. Then $w_{J(\lambda')} e = [\ 5\ \ 3\ \ 1 \,|\, 7\ \ 6\ \ 2 \,|\, 9\ \ 8\ \ 4 \,|\, 10\ ]$, $w_{J(\lambda')} e w_0 = [\ 6\ \ 8\ \ 10 \,|\, 4\ \ 5\ \ 9 \,|\, 2\ \ 3\ \ 7 \,|\, 1\ ]$ and $w_0 = d^{-1} w_{J(\lambda)} w_\lambda w_{J(\lambda')} e$.

Corollary 3.5 *Under the hypothesis of Lemma 3.4, $P^\lambda(i, j)d < P^\lambda(i+1, j)d$ if $1 \le i \le r - 1$ and $j \le \lambda_{i+1}$.*

Proof. Since $t_\lambda(w_\lambda^{-1} w_{J(\lambda)} d) = t^\lambda(w_{J(\lambda)} d)$, the j-th increasing subsequence of $w_\lambda^{-1} w_{J(\lambda)} d$, referred to in Lemma 3.4, is the j-th column of the tableau $t^\lambda(w_{J(\lambda)} d)$ and hence consists of $P^\lambda(1, j)d, \ldots, P^\lambda(\lambda'_j, j)d$. □

4 Translating cells

The Kazhdan-Lusztig cells of W may be characterised using the Robinson-Schensted correspondence, which is a bijection of S_n to the set of pairs of standard tableaux $(\mathcal{P}, \mathcal{Q})$ of the same shape corresponding to partitions of n. See Fulton [5] or Sagan [12] for a good description of this correspondence. Denote this correspondence by $w \mapsto (\mathcal{P}(w), \mathcal{Q}(w))$. Then $\mathcal{Q}(w) = \mathcal{P}(w^{-1})$. The following proposition characterises the cells in S_n; a proof may be found in [1] or [7].

Proposition 4.1 ([1, Theorem A] or [7, Corollary 5.6]) *If $\mathcal{P}$ is a fixed standard tableau then the set $\{w \in W : \mathcal{P}(w) = \mathcal{P}\}$ is a left cell of W and the set $\{w \in W : \mathcal{Q}(w) = \mathcal{P}\}$ is a right cell of W. Conversely, every left cell and every right cell arises in this way. Moreover, the two-sided cells are the subsets of W of the form $\{w \in W : \operatorname{sh} \mathcal{P}(w)$ is a fixed partition.$\}$* □

In [1], the reader should note that Ariki considers permutations in S_n to act on the left while, in this paper, they act on the right. This causes the left and right cells to be interchanged with a consequent interchanging of the rôles of the tableaux functions $\mathcal{P}$ and $\mathcal{Q}$ in some results.

In this section, we give an alternative characterisation of some of these cells, describing their elements explicitly as reduced words formed using the elements w_λ and d_λ defined above in Section 2. We also characterise these cells in terms of certain increasing and decreasing covers which their elements possess.

For the remainder of this section we assume that the partition λ is fixed and we write $P(i,j)$ and $P(i)$ instead of $P^\lambda(i,j)$ and $P^\lambda(i)$, respectively.

Proposition 4.2 *Let $w = e^{-1}w_{J(\lambda)}d$, where $e \in Y_{J(\lambda)}$ and $d \in Z_{J(\lambda)}$. In applying the Robinson-Schensted insertion process to the row form of w, the entry $P(i,j)d$, $1 \leq i \leq r$ and $j \leq \lambda_i$, is first inserted at the top of the i-th column. Subsequently, its position is unaffected by the insertion of $P(i',j')d$ with $i' \neq i$ and it is moved down the i-th column one place by the insertion of $P(i,j'')d$ with $j'' > j$.*

Proof. Let $t = t^\lambda w_{J(\lambda)}d$. Since $t^\lambda e$ is standard and $(t^\lambda e)w = t$, $1w = P(1,1)d$. Hence, the first step in the Robinson-Schensted insertion process is to insert $P(1,1)d$ at the top of the first column.

We proceed by induction on the number of elements inserted. Suppose that $P(i,j)d$ is the next element to be inserted, with $(i,j) \neq (1,1)$. If $i = 1$ then $j > 1$ and, since $P(1)d$ is a decreasing subsequence of w, $P(1,j-1)d$ is at the top of the first column and $P(1,j)d < P(1,j-1)d$. Hence, $P(1,j)d$ is inserted at the top of the first column, pushing each other entry of that column down one place.

If $i > 1$ then, since $P(i-1,j)d$ occurs to the left of $P(i,j)d$ in w by Proposition 3.3, $P(i-1,j)d$ has already been inserted. By induction, the entry at the top of the $(i-1)$-th column is $P(i-1,k)d$ for some $k \geq j$, and i-th column is empty or has top entry $P(i,j-1)d$ according as $j = 1$ or $j > 1$. By Proposition 3.2(ii) and Corollary 3.5, $P(i-1,k)d \leq P(i-1,j)d < P(i,j)d$ and, if $j > 1$, $P(i,j)d < P(i,j-1)d$. Hence, $P(i,j)d$ is inserted at the top of the i-th column. Also using Proposition 3.2(ii) and Corollary 3.5, for $m = 1,\ldots,j-1$, $P(i-1,k-j+m)d \leq P(i-1,m)d < P(i,m)d$. So, the insertion of $P(i,j)d$ pushes each other entry of the i-th column down one place and affects no other entries. □

Corollary 4.3 *If $e \in Y_{J(\lambda)}$, $d \in Z_{J(\lambda)}$ and $w = e^{-1}w_{J(\lambda)}d$, then (i) $\mathcal{P}(w) = t_{\lambda'}d$, (ii) $\mathcal{Q}(w) = t_{\lambda'}e$, and (iii) $\operatorname{sh}\mathcal{P}(w) = \operatorname{sh}\mathcal{Q}(w) = \lambda'$.*

Proof. For (ii), note that in the recording tableau $\mathcal{Q}(w)$, the j-th entry in the i-th column is the position of $P(i,j)d$ in the row form of w. By Proposition

3.2, this position is the j-th entry of the i-th row of $t^\lambda e$, which is the j-th entry of the i-th column of $t_{\lambda'} e$. Hence, $\mathcal{Q}(w) = t_{\lambda'} e$.

Part (i) is immediate from Proposition 4.2 and part (iii) follows from the fact that both $\mathcal{P}(w)$ and $\mathcal{Q}(w)$ have the same shape as $t_{\lambda'}$. □

We may now use Proposition 4.1 and Corollary 4.3 to extend [11, Lemma 3.3] to the left cells containing elements of the form $w_{J(\lambda)} d$ where $d \in Z_{J(\lambda)}$. We observe that all these cells are obtained from the cell containing $w_{J(\lambda)}$ by 'translating' with the element d. Using the results of Lemmas 2.3 and 2.5 and the ideas contained in the proofs of Propositions 2.10 and 2.12, it is possible to construct efficient algorithms for computing the prefixes of w_λ and d_λ, which in turn can be used to compute the elements of the cells described in the following theorem. An implementation of this computation in GAP can be obtained from the authors.

Theorem 4.4 *Let $C_{J(\lambda)} = \{e^{-1} w_{J(\lambda)} : e \text{ is a prefix of } w_\lambda\}$, and let d be a prefix of d_λ. Then the left cell containing $w_{J(\lambda)} d$ is the set $C_{J(\lambda)} d = \{cd : c \in C_{J(\lambda)}\}$.*

Proof. From Proposition 4.1 and Corollary 4.3 the left cell containing $w_{J(\lambda)} d$ has $|Y_{J(\lambda)}|$ elements and contains the set $\{e^{-1} w_{J(\lambda)} d : e \in Y_{J(\lambda)}\}$, which clearly has $|Y_{J(\lambda)}|$ elements. □

For a given value of n, the number of cells covered by [11, Lemma 3.3] is the number $\mathcal{A}_n$ of partitions of n, the total number of cells is the number $\mathcal{C}_n$ of standard tableaux whose shapes are partitions of n, that is the number of prefixes of w_λ as λ runs over the partitions of n, and the number of cells covered by Theorem 4.4 is the number $\mathcal{B}_n$ of prefixes of d_λ as λ runs over the partitions of n. We do not yet have a simple description of the number $\mathcal{B}_n$, but we have calculated it for small values of n by a simple GAP program. As an indication of the improvement provided by Theorem 4.4, we give these numbers in Table 1, together with the corresponding values of $\mathcal{A}_n$ and $\mathcal{C}_n$.

n	3	4	5	6	7	8	9	10
$\mathcal{A}_n$	3	5	7	11	15	22	30	42
$\mathcal{B}_n$	3	6	9	22	35	88	183	428
$\mathcal{C}_n$	4	10	26	76	232	764	2620	9496

Table 1

Since $l(e^{-1} w_{J(\lambda)} d) = l(e) + l(w_{J(\lambda)}) + l(d)$ for all prefixes e of w_λ, where d is a prefix of d_λ, this theorem gives reduced expressions for all the elements of certain cells. Because of Proposition 4.2, the Robinson-Schensted process takes a simple form, since these elements have a very convenient decreasing cover of type λ, namely $[P^\lambda(i) d : i = 1, \ldots, r]$, in which each subsequence produces one column of the resulting $\mathcal{P}$-tableau without affecting the other columns. Moreover, by Proposition 3.2(iii), each of these elements has an increasing cover of type λ'. In general, forming the Robinson-Schensted tableaux from decreasing and increasing sequences is a complicated process—see for example [12]. Trivially, the existence of a decreasing cover of type λ does not guarantee a $\mathcal{P}$-tableau of shape λ'. However, the existence of a decreasing cover of type λ and an increasing cover of type λ' guarantees a $\mathcal{P}$-tableau of shape λ'—see Schensted [13] and Greene [9]. Conversely, the existence of a $\mathcal{P}$-tableau of shape λ' does not guarantee an increasing cover of type λ'. For example, $w = [2, 3, 6, 1, 4, 5]$ has $\mathcal{P}$-tableau of shape $(4, 2)$ but no increasing cover of this type. Also, $w_0 w$ has $\mathcal{P}$-tableau of shape $(2, 2, 1, 1)$ but no decreasing cover of type $(4, 2)$. Hence, the cells described in Theorem 4.4 are very special indeed.

Corollary 4.5 *Let $d \in Z_{J(\lambda)}$ and let $w \in W$. Then $w \sim_L w_{J(\lambda)} d$ if, and only if, $[P^\lambda(i)d : i = 1, \ldots, r]$ is a decreasing cover of type λ for w and there is an increasing cover of type λ' for w.*

Proof. For the 'only if' part, we note that $w = e^{-1} w_{J(\lambda)} d$ for some $e \in Y_{J(\lambda)}$, by Theorem 4.4. The statements regarding covers follows from Proposition 3.2.

For the 'if' part, note that if μ is the shape of the $\mathcal{P}$-tableau for w, then μ is dominated by λ' and μ' is dominated by λ, by [12, Theorem 3.5.3]. Hence, $\mu = \lambda'$. Recall that the i-th row of $t^\lambda w_{J(\lambda)} d$ is $P(i)d$. Hence, $t^\lambda w_{J(\lambda)} d w^{-1}$ is a row-standard tableau and, hence, is $t^\lambda e$ for some $e \in X_{J(\lambda)}$. So, $w = e^{-1} w_{J(\lambda)} d$.

Suppose that $t^\lambda e$ is not standard. Then there is some column, say the j-th, on which two consecutive elements x and y satisfy $x > y$. Let x be on the i-th row and y on the $(i+1)$-th row. Then $xw = P(i, j)d$ and $yw = P(i+1, j)d$. Since $y < x$, the sequence $(P(i{+}1, 1)d, \ldots, P(i{+}1, j)d, P(i, j)d, \ldots, P(i, \lambda_i)d)$ is a decreasing sequence in w of length $\lambda_i + 1$. Hence, w has a decreasing cover $[Q(i)]$ of type ν, where $\nu_k = \lambda_k$, for $k = 1, \ldots, i-1$, and $\nu_i = \lambda_i + 1$.

Let $[R(j)]$ be an increasing cover of w of type λ'. Where convenient, we will consider the sequences $Q(i)$ and $R(j)$ as sets. It is immediate that $|Q(i) \cap R(j)| \leq 1$. Let $t = \lambda_i$. Then $|R(1) \cup \ldots \cup R(t)| = \sum_{j=1}^{t} \lambda'_j = it + \sum_{k=i+1}^{r} \lambda_k$. As $R(1) \cup \ldots \cup R(t)$ contains at most it elements in $Q(1) \cup \ldots \cup Q(i)$, $|R(1) \cup \ldots \cup R(t)| \leq it + \sum_{k=i+1}^{r} \nu_k < it + \sum_{k=i+1}^{r} \lambda_k$. With this contradiction, we see that $t^\lambda e$ is standard. Hence, $e \in Y_{J(\lambda)}$. So, $w \sim_L w_{J(\lambda)} d$ by Theorem 4.4. $\square$

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