On relations between the classical and the Kazhdan-Lusztig representations of symmetric groups and associated Hecke algebras*

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Abstract

Let H be the Hecke algebra of a Coxeter system (W, S), where W is a Weyl group of type A_n, over the ring of scalars $A = \mathbf{Z}[q^{1/2}, q^{-1/2}]$, where q is an indeterminate. We show that the Specht module S^λ, as defined by Dipper and James [6], is naturally isomorphic over A to the cell module of Kazhdan and Lusztig [14] associated with the cell containing the longest element of a parabolic subgroup W_J for appropriate $J \subseteq S$. We give the association between J and λ explicitly. We introduce notions of the T-basis and C-basis of the Specht module and show that these bases are related by an invertible triangular matrix over A. We point out the connection with the work of Garsia and McLarnan [9] concerning the corresponding representations of the symmetric group.

1 Introduction

In this paper we investigate the relations between the classical and the Kazhdan-Lusztig representations of the symmetric groups and associated Hecke algebras. Using elementary methods we give a self-contained proof that the Specht module S^λ, as defined by Dipper and James in [6], is H-isomorphic to a module denoted by S_{w_J} which is explicitly defined

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and is seen to afford the cell representation of [14] associated with the right cell containing w_J when the scalars consist of the ring $A = \mathbf{Z}[q^{1/2}, q^{-1/2}]$, where q is an indeterminate. We give the association between J and λ explicitly. We note that in [6], for each partition λ of $n+1$, the 'permutation' and 'monomial' modules are defined as the modules generated by the elements x_λ and y_λ of H,respectively. We construct a natural isomorphism between the cell module and the corresponding Specht module using the modules $x_\lambda H$ and $y_\mu H$ for appropriate partitions λ and μ of $n+1$.

The standard basis of S^λ is given in [6]. We show how this basis relates to Young's natural representation. We also note that there is a natural basis of S^λ related to the elements of the right cell containing w_J. The representing matrices obtained using this basis are exactly the matrices of the corresponding cell representation. In this way we are able to compare Young's natural and the cell representation and show that these representations are related by an invertible triangular matrix over A. We can give an explicit description of the coefficients of this matrix in terms of Kazhdan-Lusztig polynomials. We note that Garsia and McLarnan [9] have addressed this problem in the case of the corresponding representations of the symmetric groups.

The GAP computational system (see [18] and [8]) and, in particular, the associated CHEVIE package (see [11]) have been invaluable tools for this work.

2 Preliminary results

Let (W, S) be a Coxeter system corresponding to a Weyl group W and let l be the associated length function. We recall some basic notions concerning Weyl groups and the associated Hecke algebras. Where appropriate, we will give references to these notions in [12] or [14]. Every result involving a 'left-oriented' object connected with a Weyl group or Hecke algebra, e.g. a left transversal, a relation defined in terms of multiplication on the left or a left module, has an analogous result involving the corresponding 'right-oriented' object. We shall freely translate results from the literature involving one orientation to results involving the other.

2.1 For each element $w \in W$, the *left descent set*, $L(w)$, and the *right descent set*, $R(w)$, are defined by $L(w) := \{s \in S : l(sw) < l(w)\}$ and $R(w) := \{s \in S : l(ws) < l(w)\}$.

2.2 For each subset $J \subseteq S$, the subgroup W_J generated by J is called a *standard parabolic subgroup* of W. It has a Coxeter system (W_J, J). Its length function l_J is that induced from l. It has a unique longest element w_J. By tradition, w_0 is written for w_S.

2.3 Let $x, y \in W$. We say that x is a *prefix* of y if $y = s_1 s_2 \ldots s_p$ where $s_i \in S$ for $i = 1, \ldots, p$, $p = l(y)$ and $x = s_1 s_2 \ldots s_r$, for some $r \leq p$. The prefix relation corresponds to the *weak Bruhat order* in [6].

2.4 There is a special set of right coset representatives X_J associated with each parabolic subgroup W_J. An element of X_J is the unique element of minimum length in its coset. Moreover, if $w = vx$ where $v \in W_J$ and $x \in X_J$ then $l(w) = l(v) + l(x)$. Also, $X_J = \{w \in W : L(w) \subseteq S - J\}$ and, if d_J denotes the longest element in X_J, then X_J is the set of prefixes of d_J. (See [12, Proposition 2.1.1 and Lemma 2.2.1]).

2.5 Let $w \in W$. If $J \subseteq L(w)$ then $w = w_J x$ where $x \in X_J$. (See [12, Lemma 1.5.2]). Conversely, if $w = w_J x$ where $x \in W$ and $l(w) = l(w_J) + l(x)$, then $x \in X_J$ and $J \subseteq L(w)$. (See [4, Lemma 3.2]).

2.6 The Hecke algebra H corresponding to (W, S) and defined over the ring $A = \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, where q is an indeterminate, has a free A-basis $\{T_w : w \in W\}$ and multiplication defined by the rules (i) $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$ and (ii) $(T_s + 1)(T_s - q) = 0$ if $s \in S$. The basis $\{T_w : w \in W\}$ is called the T-*basis* of H. (See [14]).

2.7 H has a basis $\{C_w : w \in W\}$, the C-*basis*, whose terms have the form $C_y = \sum\limits_{x \le y} (-1)^{l(y)-l(x)} q^{\frac{1}{2}l(y)-l(x)} P_{x,y}(q^{-1}) T_x$, where $P_{x,y}(q)$ is a polynomial in q with integer coefficients of degree $\le \frac{1}{2}(l(y) - l(x) - 1)$ if $x < y$ and $P_{y,y} = 1$. In the preceding sentence, we use $\le$ to denote the (strong) Bruhat partial order on W and we write $x < y$ if $x \le y$ and $x \ne y$. If the degree of $P_{x,y}(q)$ is exactly $\frac{1}{2}(l(y) - l(x) - 1)$, we write $\mu(x, y)$, and $\mu(y, x)$, for its leading coefficient, which is a nonzero integer. For all other pairs $x, y \in W$, we set $\mu(x, y) = 0$.

In [14, 2.3ac], the multiplication of C-basis elements by T_s, $s \in S$ is described and is as follows,

2.8 $sx < x \Rightarrow T_s C_x = -C_x$ and $x < sx \Rightarrow T_s C_x = qC_x + q^{\frac{1}{2}} C_{sx} + \sum\limits_{z<x,\ sz<z} \mu(z, x) C_z.$

$xs < x \Rightarrow C_x T_s = -C_x$ and $x < xs \Rightarrow C_x T_s = qC_x + q^{\frac{1}{2}} C_{xs} + \sum\limits_{z<x,\ zs<z} \mu(x, z) C_z.$

There are two reflexive transitive relations (preorders), $\le_L$ and $\le_R$, defined on W using the C-basis. The preorder $\le_L$ is generated by all statements of the form: $x \le_L y$ if C_x occurs with nonzero coefficient in the expression of $T_s C_y$ in the C-basis, for some $s \in S$. The preorder $\le_R$ is defined similarly, taking $C_y T_s$ instead of $T_s C_y$ in the preceding sentence.

A third preorder $\le_{LR}$ is defined using the previous two preorders: $x \le_{LR} y$ if there is a sequence of elements $x_0 = x$, x_1, $\ldots x_r = y$ of W such that for each integer i, $0 \le i \le r - 1$, either $x_i \le_L x_{i+1}$ or $x_i \le_R x_{i+1}$.

$\sim_L$, $\sim_R$ and $\sim_{LR}$ are the equivalence relations generated by $\le_L$, $\le_R$ and $\le_{LR}$, respectively. Their equivalence classes are called *left cells*, *right cells* and *two-sided cells*, respectively. It is immediate that two-sided cells are unions of left-cells which are also unions of right cells.

We write $x <_L y$ if $x \le_L y$ and $x \not\sim_L y$. The relations $<_R$ and $<_{LR}$ are defined similarly. (See [14]).

2.9 Let $Y_J = w_J X_J$. If $x \in W$ and $x \le_R y$ for some $y \in Y_J$ then $x \in Y_J$. Moreover, $Y_J = \{w \in W : w \le_R w_J\}$ is a union of right cells. (see for example [15, 5.26.1])

For any subset $J \subseteq S$, let H_J denote the Hecke algebra corresponding to (W_J, J). From [14, Theorem 1.1 and Lemma 2.6(vi)], we see that $C_{w_J} = \left(-q^{\frac{1}{2}}\right)^{l(w_J)} \sum_{y \le w_J} (-q)^{-l(y)} T_y$.

The right H_J-module $C_{w_J}H_J$ has rank 1, since $C_{w_J}T_s = -C_{w_J}$ for all $s \in J$. The corresponding representation is described as the *alternating representation* in [6, §3] and as the *sign representation* in [3, §67]. The right H-module $C_{w_J}H$ is isomorphic to the module induced from $C_{w_J}H_J$. It has an A-basis $\{C_{w_J}T_d : d \in X_J\}$ — it is clearly spanned by these elements and they are independent over A since their 'leading terms' in the T-basis of H have the form $a_d T_{w_J d}$ where a_d is invertible in A. We will refer to this basis as the *T-basis* of $C_{w_J}H$ and to any H-module of the form $C_{w_J}H$, and any module arising from it by extending the scalars, as a *monomial* module. Note that in [2, page 314] an induced monomial representation for a group is defined as any induced representation from a one-dimensional representation of a subgroup.

Remark 2.10 There is an automorphism j of H defined by $\left(\sum_{y\in W} a_y T_y\right)j = \sum_{y\in W} \overline{a}_y \left(-q^{-1}\right)^{l(y)} T_y$, where $a \mapsto \overline{a}$ is the automorphism of A defined by $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$ (see [14, p.166]). This automorphism is used to relate the C-basis of H to another basis $\{C'_w : w \in W\}$ known as the C'-basis, which may be defined by $C'_w = (-1)^{l(w)}C_w j$.

As in the case of modules of the form $C_{w_J}H$, we see that $\{C'_{w_J}T_d : d \in X_J\}$ is an A-basis of $C'_{w_J}H$. We will refer to this basis as the *T-basis* of $C'_{w_J}H$.

In [19, Corollary 1.19], Xi obtains an A-basis for a module similar to the monomial module $C_{w_J}H$. His result is equivalent to the following lemma—though the reader should note that Xi uses the term C-basis for a basis which is different from the Kazhdan-Lusztig C-basis in [14]. Since the proof is short, we include it for completeness.

Lemma 2.11 *The module $C_{w_J}H$ has an A-basis $\{C_y : y \leq_R w_J\}$.* We will refer to this basis as the *C-basis* of $C_{w_J}H$.

Proof. Let $Y = \{C_y : y \in Y_J\}$. If $y \in Y_J$ and $s \in S$ then $C_y T_s$ is an A-linear combination of C_z, $z \leq_R y$, from the definition of $\leq_R$ and 2.8. By 2.9, any such z is in Y_J. Hence, $C_{w_J}H$ is in the A-linear span of Y.

We now show, using induction on the length of $y \in Y$, that C_y is in the A-linear span of $\{C_{w_J}T_d : d \in X_J\}$. Let $y \in Y$. Write $y = w_J e$, with $e \in X_J$. If $e = 1$, the result is trivial. Let $e \neq 1$ and write $e = fs$ with $f \in X_J$, $s \in S$ and $l(e) = l(f) + 1$. From 2.8, we get $C_{w_J f}T_s = qC_{w_J f} + q^{\frac{1}{2}}C_{w_J e} + \sum_{z<w_J e,\ zs<z} a_2(w_J e, z)C_z$, for suitable $a_2(x, z) \in A$. Each z in the right-hand sum with nonzero coefficient necessarily satisfies $z \leq_R w_J e$ and hence has the form $w_J g$, for some $g \in X_J$, by 2.9. Moreover, $l(g) < l(e)$. Hence, we can apply the inductive hypothesis to all C-basis elements in this equation, other than $C_{w_J e}$ and this has a coefficient in A which is invertible. This completes the proof. ■

Remark 2.12 In a similar way, we find that the module $C'_{w_J}H$ has an A-basis $\{C'_y : y \leq_R w_J\}$. We will refer to this basis as the *C'-basis* of $C'_{w_J}H$. Morever, if $x \leq_L w_J$, then $C'_x \in HC'_{w_J}$ so $C'_x H$ is a homomorphic image of $C'_{w_J}H$, a fact that we will need later on.

We now show that the change-of-basis matrix associated with the transition from the C-basis of $C_{w_J}H$ to the T-basis is, for a suitable ordering of the elements of the basis, a triangular matrix over A which is invertible over A. We thank the referees for pointing out to us that the polynomials $g^J_{e,d}$, which appear in the proof below, first appeared in work of Deodhar [5]. See in particular [5, Proposition 3.4].

Proposition 2.13 *For each $e \in X_J$, $C_{w_J}T_e = \sum_{d\in X_J,\ d\leq e} g^J_{e,d}C_{w_Jd}$, where $g^J_{e,d}$ denotes an element of A, $g^J_{e,e}$ is invertible in A and $g^J_{e,d} = 0$ if $d, e \in X_J$ and $d \not\leq e$.*

Proof. If $e \in X_J$, then $w_Je \leq_R w_J$ so we can express C_{w_Je} in terms of the $C_{w_J}T_d$, $d \in X_J$, in the following way: $C_{w_Je} = \sum_{d\in X_J} \bar{g}^J_{e,d}C_{w_J}T_d$. We can be more precise about the coefficients $\bar{g}^J_{e,d}$. Using 2.7, and setting $w_Je = y$, $C_y = \sum_{z\leq y}(-1)^{l(y)-l(z)}q^{\frac{1}{2}l(y)-l(z)}P_{z,y}(q^{-1})T_z$. Also, $z \leq y$ if, and only if, $z = y'd$ where $y' \in W_J$ and $d \leq e$. From [14, (2.3g)], $P_{y'd,w_Je}(q) = P_{w_Jd,w_Je}(q)$ for any $d \leq e$. Recall that $C_{w_J} = \left(-q^{\frac{1}{2}}\right)^{l(w_J)} \sum_{w\in W_J}(-q)^{-l(w)}T_w$. Hence, $C_y = \sum_{d\leq e}(-1)^{l(e)-l(d)}q^{\frac{1}{2}l(e)-l(d)}P_{w_Jd,w_Je}(q^{-1})C_{w_J}T_d$. So, we find that $\bar{g}^J_{e,d} = (-1)^{l(e)-l(d)}q^{\frac{1}{2}l(e)-l(d)}P_{w_Jd,w_Je}(q^{-1})$ if $d \leq e$ and $\bar{g}^J_{e,d} = 0$ otherwise. Note, in particular, that $\bar{g}^J_{e,d}$ is in $Z[q^{1/2}, q^{-1/2}]$ and $\bar{g}^J_{e,e}$ is invertible. By ordering the equations and the entries with respect to the Bruhat order, the equations above may be rewritten in triangular form with invertible diagonal entries, $C_{w_Je} = \sum_{d\in X_J,\ d\leq e} \bar{g}^J_{e,d}C_{w_J}T_d$, for all $e \in X_J$. Thus, we may invert this set of equations and get the equations $C_{w_J}T_e = \sum_{d\in X_J,\ d\leq e} g^J_{e,d}C_{w_Jd}$, for $e \in X_J$, where $g^J_{e,d}$ denotes an element of $Z[q^{1/2}, q^{-1/2}]$, which can be described explicitly in terms of the elements of the form $\bar{g}^J_{e,d}$. We can extend the definition of $g^J_{e,d}$ by setting $g^J_{e,d} = 0$ if $d, e \in X_J$ and $d \not\leq e$. ■

For $w \in W$, let M_w and $\hat{M}_w$ denote the H-modules with A-bases $\{C_y : y \leq_R w\}$ and $\{C_y : y <_R w\}$, respectively, and let $S_w = M_w/\hat{M}_w$. Then S_w is a Kazhdan-Lusztig cell module and affords the cell representation corresponding to the right cell containing w. Note that C_wH is a submodule of M_w. We see from Lemma 2.11 that, if $w = w_J$ for some $J \subseteq S$, then $C_wH = M_w$.

Now clearly $\hat{M}_wj$ is an H-submodule of M_wj. Define $S^\bullet_w{=}M_wj/\hat{M}_wj$. Then $S^\bullet_w$ has A-basis $\{C'_z + \hat{M}_wj : z \sim_R w\}$.

It will be convenient on occasion to extend the scalars of the algebras under consideration. Let R be any commutative ring with 1 and let $A \to R$ be a ring homomorphism. With each A-module M, we have an associated R-module $R \otimes_A M$, which we will denote briefly as M_R. In particular, we obtain an R-algebra H_R and Kazhdan-Lusztig cell modules $S_{R,w} = R \otimes S_w$. Since j can be extended easily and uniquely to an automorphism of H_R, we see that the H_R-module $S^\bullet_{R,w}$ is isomorphic to $M_{R,w}j/\hat{M}_{R,w}j$. We will use F to denote any field containing the field of fractions $\mathbf{Q}\left(q^{\frac{1}{2}}\right)$ of A, and assume that the homomorphism $A \to F$ is inclusion.

Remark 2.14 Suppose that $x \in W$ and that x is a prefix of every y such that $y \sim_R x$. Then (i) C_xH has a factor isomorphic to S_x and (ii) C'_xH has a factor isomorphic to $S^\bullet_x$. To see (i), let $y \sim_R x$. Since x is a prefix of y, it is easy to prove, by induction on the length of y, that $C_y \in C_xH + \hat{M}_x$ Hence, $(C_xH + \hat{M}_x)/\hat{M}_x$ is the A-span of $\{C_y + \hat{M}_x : y \sim_R x\} = S_x$. It follows that C_xH has a factor isomorphic to S_x. (ii) may be established in a similar way.

Proposition 2.15 *For each $w \in W$, $S^\bullet_{F,w}$ and S_{F,w_0w} are isomorphic H_F-modules.*

Proof. Let $\mathcal{C}'$ be the cell of W containing w. From [14, Corollary 3.2], we see that the set $\mathcal{C} = \{w_0 x : x \in \mathcal{C}'\}$ is also a cell. We find a matrix representation β' for the representation of H_F on $S^{\bullet}_{F,w}$ from the action of T_s on $\{C'_x : x \in \mathcal{C}'\}$. In a similar fashion we find a matrix representation β for the representation of H_F on $S_{F,w_0 w}$ from the action of T_s on $\{C_x : x \in \mathcal{C}\}$. The matrices $T_s\beta$ and $T_s\beta'$ are easily calculated and are seen to be transposes of one another for each $s \in S$—note the similarity with the situation in the proof of [15, Theorem 12.15] and [10, Corollary 2.8]. Since elements of W are conjugate to their inverses, it follows from [12, 8.2.6] that the representations β and β' have the same trace function. As H_F is split semisimple, the irreducible characters of H_F form a basis for the space of trace functions on H_F (see [12, Exercise 7.4]). It follows that the representations β and β' are equivalent. ■

3 Cell modules and Specht modules

In this section, we will suppose that W is a Weyl group of type A_n. Let $S = \{s_1, \ldots, s_n\}$ and let the corresponding Coxeter graph be $\overset{s_1}{\circ}\!-\!\!-\!\!-\!\overset{s_2}{\circ}\!-\!\!-\ \cdots\ -\!\!-\!\overset{s_n}{\circ}$. We can identify W with the symmetric group S_{n+1} by taking s_i to be the transposition $(i, i+1)$, for $i = 1, \ldots, n$. In this case, the longest element of W is the permutation in which $i \mapsto n+2-i$.

There is a natural bijection between the subsets J of S and the set of compositions of $n+1$. Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{r+1})$ is a composition of $n+1$. It will be convenient to define an auxiliary sequence $\hat\lambda_0, \hat\lambda_1, \ldots, \hat\lambda_{r+1}$. Set $\hat\lambda_0 = 0$, $\hat\lambda_{r+1} = n+1$, $\hat\lambda_i = \lambda_i + \hat\lambda_{i-1}$ for $i = 1, \ldots, r$ and $\lambda_{r+1} = n+1-\hat\lambda_r$. Then, the subset $J(\lambda)$ of S which corresponds to the composition λ of $n+1$ under this bijection is $J(\lambda) = S - \{s_{\hat\lambda_1}, s_{\hat\lambda_2}, \ldots, s_{\hat\lambda_r}\}$.

Now suppose that λ is a partition of $n+1$ with $r+1$ parts. We construct two tableaux t_λ and t^λ of shape λ. If the cells are indexed by pairs (i, j) as usual, i being the row number from the top and j the column number from the left, then $t_\lambda(i,j) := \lambda'_1 + \ldots + \lambda'_{j-1} + i = \hat\lambda'_{j-1} + i$ and $t^\lambda(i,j) := \lambda_1 + \ldots + \lambda_{i-1} + j = \hat\lambda_{i-1} + j$ for appropriate values of i and j. Let t_λ be the tableau of shape λ in which $1, 2, \ldots, n+1$ appear in order along successive columns and let t^λ be the tableau of shape λ in which $1, 2, \ldots, n+1$ appear in order along successive rows.

An element w_λ of S_{n+1} is defined by $t^\lambda(i,j)w_\lambda = t_\lambda(i,j)$ for appropriate values of i and j.

The cells of W may be described in terms of the Robinson-Schensted correspondence. See [17] for a good description of this correspondence. The correspondence is a bijection of S_{n+1} to pairs of standard tableaux (P, Q) of the same shape corresponding to partitions of $n+1$, so that if $w \mapsto (P(w), Q(w))$, then $Q(w) = P(w^{-1})$. Note, in particular, that the involutions are the elements $w \in W$ for which $Q(w) = P(w)$. If λ is a partition of $n+1$, the pair of tableaux corresponding to $w_{J(\lambda)}$ has the form $(t_{\lambda'}, t_{\lambda'})$. Hence, the tableaux corresponding to $w_{J(\lambda)}$ have shape λ', where λ' denotes the partition conjugate to λ.

3.1 If P is a fixed standard tableau then the set $\{w \in W : P(w) = P\}$ is a left cell of W and the set $\{w \in W : Q(w) = P\}$ is a right cell of W. See [14] and also [1] for an alternative proof of this result. In [1], the reader should note that Ariki considers permutations in S_n to act on the left while, in this paper, they act on the right. This causes the left and right cells to be interchanged with a consequent interchanging of the rôles of the tableaux functions P and Q in some results.

Lemma 3.2 *Let λ be a partition of $n+1$. The element of S_{n+1}, which corresponds to the pair of tableaux (t^λ, t_λ) under the Robinson-Schensted correspondence, is $w_{J(\lambda')}w_{\lambda'}$.*

Proof. Let $w^{[\lambda]}$ be the element of S_{n+1}, which corresponds to the pair of tableaux (t^λ, t_λ). Reversing the Robinson-Schensted process on the given pair of tableaux, we see that all elements of the last column are removed, then all elements from the second last, and so on.

$$w^{[\lambda]} = \begin{pmatrix} 1 & \dots & \hat{\lambda}'_1 & \hat{\lambda}'_1+1 & \dots & \hat{\lambda}'_2 & \dots\dots & \hat{\lambda}'_{\lambda_1-1}+1 & \dots & \hat{\lambda}'_{\lambda_1} \\ t^\lambda(\lambda'_1,1) & \dots & t^\lambda(1,1) & t^\lambda(\lambda'_2,2) & \dots & t^\lambda(1,2) & \dots\dots & t^\lambda(\lambda'_{\lambda_1},\lambda_1) & \dots & t^\lambda(1,\lambda_1) \end{pmatrix}.$$

So $w^{[\lambda]}w_\lambda =$

$$\begin{pmatrix} 1 & \dots & \hat{\lambda}'_1 & \hat{\lambda}'_1+1 & \dots & \hat{\lambda}'_2 & \dots\dots & \hat{\lambda}'_{\lambda_1-1}+1 & \dots & \hat{\lambda}'_{\lambda_1} \\ t_\lambda(\lambda'_1,1) & \dots & t_\lambda(1,1) & t_\lambda(\lambda'_2,2) & \dots & t_\lambda(1,2) & \dots\dots & t_\lambda(\lambda'_{\lambda_1},\lambda_1) & \dots & t_\lambda(1,\lambda_1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \dots & \hat{\lambda}'_1 & \hat{\lambda}'_1+1 & \dots & \hat{\lambda}'_2 & \dots & \dots & \hat{\lambda}'_{\lambda_1-1}+1 & \dots & \hat{\lambda}'_{\lambda_1} \\ \hat{\lambda}'_1 & \dots & 1 & \hat{\lambda}'_2 & \dots & \hat{\lambda}'_1+1 & \dots & \dots & \hat{\lambda}'_{\lambda_1} & \dots & \hat{\lambda}'_{\lambda_1-1}+1 \end{pmatrix} = w_{J(\lambda')}.$$

Since $w_{\lambda'} = w_\lambda^{-1}$, the result follows. ■

Lemma 3.3 (Compare [7, Lemma 1.2].) *Let λ be a partition of $n+1$ and let $w = w_0 w_{J(\lambda)} w_\lambda \in W$. Then (i) $w_{J(\lambda)}w_\lambda \in Y_{J(\lambda)}$, (ii) $w_{J(\lambda)}d \in Y_{J(\lambda)}$ for each prefix d of w_λ, (iii) $w_{J(\lambda)}d$ is in the same right cell as $w_{J(\lambda)}$ for each prefix d of w_λ, (iv) the right cell containing $w_{J(\lambda)}$ is $\{w_{J(\lambda)}d : d \text{ is a prefix of } w_\lambda\}$, (v) $w \sim_L w_{J(\lambda')}$, (vi) w is a prefix of every element in the right cell containing it.*

Proof. (i) From [6, Lemma 1.5], $w_\lambda \in X_{J(\lambda)}$.

(ii) If d is a prefix of w_λ, then $d \in X_{J(\lambda)}$ by [12, Lemma 2.2.1].

(iii) Take any prefix d of w_λ. From [14, 2.3ef], $w_{J(\lambda)}w_\lambda \leq_R w_{J(\lambda)}d \leq_R w_{J(\lambda)}$.

We have seen above that $w_{J(\lambda)}$ corresponds to the tableaux pair $(t_{\lambda'}, t_{\lambda'})$. Since $w_{J(\lambda)}w_\lambda$ corresponds to the tableaux pair $(t^{\lambda'}, t_{\lambda'})$, it follows from 3.1 that $w_{J(\lambda)} \leq_R w_{J(\lambda)}w_\lambda$. Hence, $w_{J(\lambda)}d \sim_R w_{J(\lambda)}$.

(iv) Note that [6, Lemma 1.5] asserts that there are as many standard tableaux of shape λ as there are prefixes of w_λ. Since this is also the number of elements in the right cell containing $w_{J(\lambda)}$, by 3.1, we obtain the desired result using (iii).

(v) Let P and Q be tableaux as in the RS correspondence. From [17, Theorem 3.2.3] and Lemma 3.2 we get $P(w) = P(w_{J(\lambda)}w_\lambda)^{tr} = P(w_{J(\lambda')})$. The result follows from 3.1.

(vi) It is immediate from (iv) and [14, Corollary 3.2] that the right cell containing w is $\{wb : b \text{ a prefix of } w_{\lambda'}\}$. From [6, Lemma 4.1], in the case that λ is a partition, we can deduce that $w_0 = w w_{\lambda'} w_{J(\lambda)}$ with $l(w_0) = l(w) + l(w_{\lambda'}) + l(w_{J(\lambda)})$. The required result now follows easily. ■

As in [6, §3], define $x_\lambda = \sum_{y \in W_{J(\lambda)}} T_y$, $y_\lambda = \sum_{w \in W_{J(\lambda)}} (-q)^{-l(w)} T_w$, the 'permutation' module $M^\lambda = x_\lambda H$ and the Specht module $S^\lambda = x_\lambda T_{w_\lambda} y_{\lambda'} H$. Then $x_\lambda = q^{\frac{1}{2} l(w_{J(\lambda)})} C'_{w_{J(\lambda)}}$ and $y_\lambda = \left(-q^{-\frac{1}{2}}\right)^{l(w_{J(\lambda)})} C_{w_{J(\lambda)}}$. Also, $C_{w_{J(\lambda)}} j = \left(-q^{-\frac{1}{2}}\right)^{l(w_{J(\lambda)})} x_\lambda = (-1)^{l(w_{J(\lambda)})} C'_{w_{J(\lambda)}}$. So $M_{w_{J(\lambda)}} j = M^\lambda$.

Lemma 3.4 *Let λ be a partition of $n+1$. Then the cell module $S_{F,w_{J(\lambda)}}$ is the unique common composition factor of the 'permutation' module $x_{\lambda'} H_F$ and the 'monomial' module $y_\lambda H_F$. In particular, $S_{F,w_{J(\lambda)}}$ is H_F-isomorphic to the Specht module $S_F^{\lambda'}$.*

Proof. First recall that $y_\lambda H_F = C_{w_{J(\lambda)}} H_F$ and $x_{\lambda'} H_F = C'_{w_{J(\lambda')}} H_F$. With w defined as in Lemma 3.3, $w_0 w = w_{J(\lambda)} w_\lambda \sim_R w_{J(\lambda)}$. So, $S_{F,w_{J(\lambda)}} = S_{F,w_{J(\lambda)} w_\lambda}$. It now follows from Proposition 2.15 that $S_{F,w_{J(\lambda)}} \cong S^\bullet_{F,w}$ as H_F-modules. Using Lemma 3.3(vi) and Remark 2.14 we get that $S^\bullet_{F,w}$ is a composition factor of $C'_w H_F$ and by Lemma 3.3(v) and Remark 2.12 we get that $C'_w H_F$ is a homomorphic image of $x_{\lambda'} H_F$. We can deduce that $S^\bullet_{F,w}$ and therefore $S_{F,w_{J(\lambda)}}$ is a composition factor of $x_{\lambda'} H_F$. Moreover, by the way it is defined, $S_{F,w_{J(\lambda)}}$ is a composition factor of $M_{F,w_{J(\lambda)}} = y_\lambda H_F$ (see Lemma 2.11). Now from [6, Lemma 4.1], in the case that λ is a partition, we get that $x_{\lambda'} H_F$ and $y_\lambda H_F$ have a unique common composition factor so the first statement of the Lemma follows. To establish the second statement in the Lemma, note that $S_F^{\lambda'} = x_{\lambda'} T_{w_{\lambda'}} y_\lambda H_F$ is a homomorphic image of $y_\lambda H_F$ and a submodule of $x_{\lambda'} H_F$. ■

We are now able to establish the isomorphism of the H-modules $S_{w_{J(\lambda)}}$ and $S^{\lambda'}$, over the scalars A, from which we can deduce a corresponding isomorphism over any commutative ring of scalars with 1.

Theorem 3.5 *If λ is a partition of $n+1$, then $S_{w_{J(\lambda)}} \cong S^{\lambda'}$ as H-modules.*

Proof. Define $\theta : M_{w_{J(\lambda)}} \to S^{\lambda'}$ by $m\theta = x_{\lambda'} T_{w_{\lambda'}} m$. Clearly, $\operatorname{im}\theta = S^{\lambda'}$. Next, we show that $\hat{M}_{w_{J(\lambda)}} \subseteq \ker\theta$. For this purpose, we extend scalars from A to F. If we had $\hat{M}_{w_{J(\lambda)}} \not\subseteq \ker\theta$, then we would also have $\hat{M}_{F,w_{J(\lambda)}} \not\subseteq \ker\theta_F$. This, in turn, implies that the restriction of θ_F to $\hat{M}_{F,w_{J(\lambda)}}$ is non-zero. Since the image of θ_F is the simple H_F-module $S_F^{\lambda'}$, we conclude that $S_F^{\lambda'}$ occurs as a composition factor of $\hat{M}_{F,w_{J(\lambda)}}$, contrary to Lemma 3.4. Thus, $\hat{M}_{w_{J(\lambda)}} \subseteq \ker\theta$. It then follows that the elements $x_{\lambda'} T_{w_{\lambda'}} C_w$, where $w \sim_R w_{J(\lambda)}$, form an F-spanning set of $S_F^{\lambda'}$. Since $\dim_F S_F^{\lambda'} = \dim_F S_{F,w_{J(\lambda)}}$ by Lemma 3.4, the set $\{x_{\lambda'} T_{w_{\lambda'}} C_w : w \sim_R w_{J(\lambda)}\}$ is linearly independent over F and hence over A. Now, let $r \in \ker\theta$. Then $r = m + \sum_{w \sim_R w_{J(\lambda)}} \alpha_w C_w$ for some $m \in \hat{M}_{w_{J(\lambda)}}$ and $\alpha_w \in A$. Hence, $0 = r\theta = m\theta + \sum_{w \sim_R w_{J(\lambda)}} \alpha_w C_w \theta = \sum_{w \sim_R w_{J(\lambda)}} \alpha_w (C_w \theta)$. But we have already shown that $\{C_w\theta : w \sim_R w_{J(\lambda)}\}$ is F-linearly independent, so $\alpha_w = 0$ for all such w. Hence, $r \in \hat{M}_{w_{J(\lambda)}}$, as desired. So, $\ker\theta = \hat{M}_{w_{J(\lambda)}}$ The isomorphism in the theorem is now immediate. ■

This theorem provides us with a straightforward proof that $S^{\lambda'}$ is a free A-module, a fact already known in [6], since $S_{w_{J(\lambda)}}$ is a free A-module by construction. If we had already established this fact, the second part of the proof could have been shortened as follows. The homomorphism θ induces a surjective H-module homomorphism $\bar{\theta} : S_{w_{J(\lambda)}} \to S^{\lambda'}$, since $S_{w_{J(\lambda)}} = M_{w_{J(\lambda)}}/\hat{M}_{w_{J(\lambda)}}$. Since both $S_{w_{J(\lambda)}}$ and $S^{\lambda'}$ are free A-modules of the same rank, $\bar{\theta}$ is injective. Hence, $\bar{\theta}$ is the required isomorphism.

Corollary 3.6 *Let R be any commutative ring with 1 and let $A \to R$ be any ring homomorphism. If λ is a partition of $n+1$, then $S_{R,w_{J(\lambda)}} \cong S_R^{\lambda'}$ as H_R-modules.*

Proof. We need only observe that the isomorphism $\bar{\theta}$ in the proof of Theorem 3.5 extends naturally to an isomorphism $\bar{\theta}_R : S_{R,w_{J(\lambda)}} \to S_R^{\lambda'}$. ■

Remark 3.7 By considering the induced isomorphism $\bar{\theta}$ in Theorem 3.5, it is immediate that the set $\{x_{\lambda'}T_{w_{\lambda'}}C_w : w \sim_R w_{J(\lambda)}\}$ is an A-basis of $S^{\lambda'}$ – we call it the C-basis of $S^{\lambda'}$. We note that by using different methods, Du in [7, 2.3(i)] proves that this is an A-basis of the Specht module and calls it the canonical basis.

We also note that in [16] Mathas relates the Kazhdan-Lusztig cell module and the corresponding Specht module in the special case of the group algebra.

4 The transition from the T-basis to the C-basis of the Specht module

In this section, we again suppose that W is the Weyl group of type A_n and address in the case of the Hecke algebra (see Theorem 4.1) the problem that Garsia and McLarnan address in [9] in the case of the symmetric group and thus obtain a generalization of [9, Theorem 5.3]. To achieve this we consider two different bases for the Specht module.

Following the discussion in the proof of Proposition 2.13 it will also be useful to have the associated functions $\bar{f}^{\lambda}_{y,x} = \bar{g}^{\lambda}_{w_{J(\lambda)}y,w_{J(\lambda)}x}$ and $f^{\lambda}_{y,x} = g^{\lambda}_{w_{J(\lambda)}y,w_{J(\lambda)}x}$ defined for all x and y with $x \le_R w_{J(\lambda)}$ and $y \le_R w_{J(\lambda)}$. Hence, for each $e \in X_J$,

$$C_{w_{J(\lambda)}}T_e = \sum_{d \in X_{J(\lambda)},\ d \text{ a prefix of } w_\lambda} g^{\lambda}_{e,d}C_{w_{J(\lambda)}d} + \sum_{d \in X_{J(\lambda)},\ d \text{ not a prefix of } w_\lambda} g^{\lambda}_{e,d}C_{w_{J(\lambda)}d}$$ So, letting $y = w_{J(\lambda)}e$, $C_{w_{J(\lambda)}}T_{w_{J(\lambda)}y} = \sum_{x \sim_R w_{J(\lambda)}} f^{\lambda}_{y,x}C_x + \sum_{x <_R w_{J(\lambda)}} f^{\lambda}_{y,x}C_x$. But $x_{\lambda'}T_{w_{\lambda'}}C_x = 0$ for all $x <_R w_{J(\lambda)}$ by Theorem 3.5. So, $x_{\lambda'}T_{w_{\lambda'}}C_{w_{J(\lambda)}}T_{w_{J(\lambda)}y} = \sum_{x \sim_R w_{J(\lambda)}} f^{\lambda}_{y,x}x_{\lambda'}T_{w_{\lambda'}}C_x$.

From this last equation we get a system of equations by letting y run through the elements of the right cell containing $w_{J(\lambda)}$. Now $f^{\lambda}_{y,y} \neq 0$ for all $y \sim_R w_{J(\lambda)}$ from Proposition 2.13 so we can invert this system of equations and deduce that the set $\{x_{\lambda'}T_{w_{\lambda'}}C_{w_{J(\lambda)}}T_{w_{J(\lambda)}y} : y \sim_R w_{J(\lambda)}\}$ which is the same as the set $\{x_{\lambda'}T_{w_{\lambda'}}C_{w_{J(\lambda)}}T_d :$ d a prefix of $w_\lambda\}$ is also a basis of $S^{\lambda'}$ – we call it the T-basis of this module. We point out that this last set is already proved to be a basis of $S^{\lambda'}$ in the Standard Basis Theorem, [6, Theorem 5.6].

It is clear from the above that the elements of the T-basis of $S^{\lambda'}$ are indexed by the elements in the right cell containing $w_{J(\lambda)}$. We order the elements of this right cell with respect to the Bruhat order and we let ρ be the matrix representation corresponding to $S^{\lambda'}$ with respect to this ordering of its T-basis. Keeping to the same ordering of the elements of the cell containing $w_{J(\lambda)}$, we also let σ be the matrix representation corresponding to this cell and Z be the change-of-basis matrix from the T-basis to the C-basis of $S^{\lambda'}$ It is also clear from the discussion above and the proof of Proposition 2.13 that Z is an invertible triangular matrix over A.

We also note that it is an immediate consequence of Theorem 3.5 that the representing matrices of $S^{\lambda'}$, with respect to the C-basis of this module, are exactly the same as the representing matrices of the cell representation corresponding to the cell containing $w_{J(\lambda)}$ and hence are given by the representation σ.

We sum up the above observations in the following theorem, using ρ, σ, Z and $f^{\lambda}_{y,x}$ as defined above:

Theorem 4.1 *The matrix representation ρ of the Specht module $S^{\lambda'}$ with respect to its standard basis and the matrix representation σ of the cell corresponding to this Specht module are related by $h\sigma = Z^{-1}h\rho Z$ for all $h \in H$, where Z is a unitriangular matrix in A with entries given by the $f^{\lambda}_{y,x}$.*

In Table 1, we give as an example the transition matrix for the representations corresponding to the right cell {121, 1213, 12132, 12134, 121324, 1213243}.

Table 1: The transition matrix from the T-basis of $S^{(3,1,1)}$ to the C-basis

	C_{121}	C_{1213}	C_{12132}	C_{12134}	C_{121324}	$C_{1213243}$
$C_{121}T_{\emptyset}$	1					
$C_{121}T_3$	q	$q^{1/2}$				
$C_{121}T_{32}$		$q^{3/2}$	q			
$C_{121}T_{34}$	q^2	$q^{3/2}$		q		
$C_{121}T_{324}$		$q^{5/2}$	q^2	q^2	$q^{3/2}$	
$C_{121}T_{3243}$				q^3	$q^{5/2}$	q^2

Finally we explain how Theorem 4.1 relates to [9, Theorem 5.3] so for the discussion that follows, we consider the specialization $q^{1/2} \mapsto 1$. We refer to James [13] for the notation and for the basic results. The only change we make is that we write $\bar{S}^{\lambda}$ and $\bar{M}^{\lambda}$ instead of S^{λ} and M^{λ} for the corresponding modules as defined in James [13], since we are already using the symbols S^{λ} and M^{λ} for the submodules of the Hecke algebra in the specialization under consideration.

Let λ be a partition of $n+1$ and let t be a λ-tableau. We define the elements ρ_t, κ_t and e_t of the group algebra FS_{n+1} as follows. Let $\rho_t = \sum_{\sigma \in R_t} \sigma$, and $\kappa_t = \sum_{\sigma \in C_t} \mathrm{sgn}(\sigma)\sigma$, where R_t, C_t denote the row group and the column group of t, respectively, and let e_t be the polytabloid $\{t\}\kappa_t$, where $\{t\}$ denotes the tabloid defined by t. Recall that $\bar{M}^{\lambda}$ is the F-space spanned by the tabloids and that $\bar{S}^{\lambda}$ is the F-space spanned by the polytabloids.

Following [13, pp.16,17], let $\theta : \rho_t FS_{n+1} \to \bar{M}^{\lambda}$ be the mapping defined by $\rho_t\pi \mapsto \{t\}\pi$, for $\pi \in S_{n+1}$. This is clearly a well-defined FS_{n+1}-isomorphism from the right ideal $\rho_t FS_{n+1}$ onto $\bar{M}^{\lambda}$. Restricting θ to the right ideal $\rho_t\kappa_t FS_{n+1}$ gives an isomorphism onto $\bar{S}^{\lambda}$.

Consider now the special case $t = t^{\lambda}$. Then $\rho_{t^{\lambda}} = x_{\lambda}$ and $\kappa_{t^{\lambda}w_{\lambda}} = y_{\lambda'}$. Hence, $w_{\lambda}\kappa_{t^{\lambda}w_{\lambda}}w_{\lambda}^{-1} = \kappa_{t^{\lambda}}$ since $\kappa_{t\pi} = \pi^{-1}\kappa_t\pi$ for all $\pi \in S_{n+1}$. So, $\rho_{t^{\lambda}}\kappa_{t^{\lambda}} = x_{\lambda}w_{\lambda}y_{\lambda'}w_{\lambda}^{-1}$

and $\rho_{t^\lambda}\kappa_{t^\lambda}FS_{n+1} = x_\lambda w_\lambda y_{\lambda'} w_\lambda^{-1} FS_{n+1} = x_\lambda w_\lambda y_{\lambda'} FS_{n+1} = S^\lambda$. That is, $S^\lambda\theta = \bar{S}^\lambda$. Now, $S^\lambda = x_\lambda w_\lambda y_{\lambda'} FS_{n+1}$ and the set $\{x_\lambda w_\lambda y_{\lambda'} d : d \text{ a prefix of } w_{\lambda'}\}$ is the T-basis of S^λ as defined above.

But $\{d : d \text{ a prefix of } w_{\lambda'}\} = \{w_\lambda^{-1} d : d \text{ a prefix of } w_\lambda\}$ since $w_{\lambda'} = w_\lambda^{-1}$. Hence, the T-basis of S^λ can be written as $\{x_\lambda w_\lambda y_{\lambda'} w_\lambda^{-1} d : d \text{ a prefix of } w_\lambda\} = \{\rho_{t^\lambda}\kappa_{t^\lambda} d : d \text{ a prefix of } w_\lambda\}$. Now, for d a prefix of w_λ, $(\rho_{t^\lambda}\kappa_{t^\lambda} d)\theta = \{t\}\kappa_{t^\lambda} d = e_{t^\lambda} d = e_{t^\lambda d}$ and $\{e_{t^\lambda d} : d \text{ a prefix of } w_\lambda\} = \{e_t : t \text{ a standard } \lambda\text{-tableau}\}$. This last basis is the basis of $\bar{S}^\lambda$ which gives Young's natural representation (see [13, p.114 and Lemma 8.4]).

It is clear from the above argument that the representing matrices of S^λ with respect to the T-basis, suitably ordered, are the same as the matrices for Young's natural representation for $\bar{S}^\lambda = S^\lambda\theta$. Combining this with the fact that the representing matrices with respect to the C-basis of the Specht module are exactly the representing matrices of the corresponding cell representation we can conclude that Theorem 4.1 gives a generalization to [9, Theorem 5.3].

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