On the Irreducible Representations of the Specializations of the Generic Hecke Algebra of type F_4*

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24th June, 1998

Abstract

A complete determination of the irreducible modules of specialized Hecke algebras of type F_4, with respect to specializations with equal parameters, was obtained by Geck and Lux in [5]. A similar determination for specializations with $v = u^2$ and $v = u^4$ has been obtained by Bremke in [1]. In this paper, we determine the irreducible modules for all remaining specializations other than those into fields of characteristic 2 or 3, obtaining *en route* decompositions of the generic irreducible modules under such specializations. We find that the decomposition matrices may be expressed in lower uni-triangular form in all these cases.

*The authors wish to thank the University of Cyprus for supporting this research through exchange visits between the University of Cyprus and the University of Wales, Aberystwyth.

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1 Introduction

Let A be the ring $\mathbf{Z}[u, u^{-1}, v, v^{-1}]$, where u and v are algebraically independent indeterminates and let H be the generic Hecke algebra over A of type F_4. $H = H(u, v)$ is defined to be the associative A-algebra with generators T_i, $i = 1, 2, 3, 4$, and defining relations $T_i^2 = u1 + (u-1)T_i$ for $i = 1, 2$, $T_i^2 = v1 + (v-1)T_i$ for $i = 3, 4$, $T_iT_j = T_jT_i$ if $|i-j| > 1$, $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ for $i = 1, 3$ and $T_2T_3T_2T_3 = T_3T_2T_3T_2$.

With each homomorphism $f : A \to F$, where F is a field, there is an associated algebra H_F known as the specialized algebra with respect to f obtained by specializing u and v to $f(u)$ and $f(v)$, respectively, in the field F. The specialized algebra with respect to the homomorphism f is the F-algebra defined by a similar set of generators and relations which are obtained from those above by replacing u and v throughout by $\bar{u} = f(u)$ and $\bar{v} = f(v)$, respectively. It will be convenient to refer to this algebra either as $H_F(\bar{u}, \bar{v})$ or H_F. Where no confusion is likely to arise, we will use the same symbols T_i $(i = 1, 2, 3, 4)$ to refer to the generators of these algebras and refer to the specialization briefly by the pair $(\bar{u}, \bar{v})$.

It is the purpose of this paper to describe the irreducible H_F-representations for all specializations, where the field F has characteristic different from 2 and 3.

If K denotes the field of fractions of A, we will show that each irreducible H_K-representation has a realisation over A and describe how this module decomposes under such specializations.

We will then obtain the following theorem as a simple consequence of these results.

Theorem 1 *Let L be any algebraically closed field of characteristic $\neq 2, 3$ and let L_0 be the field of fractions of the image of A under the specialization $f : A \to L$. Then*

(a) *There is a well-defined decomposition map from the Grothendieck group of H_K to the Grothendieck group of H_L.*

(b) *The irreducible H_K-modules and the irreducible H_L-modules can be arranged so that the corresponding decomposition matrix has a lower uni-triangular shape. In particular, its elementary divisors are all 1.*

(c) *The field L_0 is a splitting field for H_{L_0}.*

Proof of Part (a): H_K is split semi-simple (see [4]). Furthermore, all irreducible H_K-representations can be realized over $\mathbf{Z}[u, v]$. All but those of degrees 12 and 16 and one of the degree 6 representations are so described in [7] and [8]. We are indebted to Geck for explicit descriptions of the generic irreducible representations of degrees 12 and 16 with respect to the parameters u^2 and v^2. It was not difficult to derive from them generic irreducible representations of degrees 12 and 16 of $H_K(u, v)$ realized over $\mathbf{Z}[u, v]$. A similar, but easier, task dealt with the remaining degree 6 representation which, as described in [7], was realized over $\mathbf{Z}[u, u^{-1}, v]$.

The images of each element of H under two H-representations, which are H_K-equivalent, have the same characteristic polynomial. Hence, by an adaptation of the argument of

[3, Theorem (82.1)], their specializations to H_L-representations have the same composition factors. Thus, the decomposition map is well-defined and Part (a) is established. We complete Parts (b) and (c) in Section 5.

Now, let F be an arbitrary field of characteristic $\neq 2, 3$ and let L be its algebraic closure. The regular representation of $H_F(\bar{u}, \bar{v})$ contains all its irreducible representations among its composition factors. Since this regular representation arises by specializing the regular representation of $H_K(u, v)$, every irreducible representation of $H_F(\bar{u}, \bar{v})$ occurs as a composition factor of the specialization of some generic irreducible representation. By Theorem 1(a), a complete determination of the $H_L(\bar{u}, \bar{v})$-irreducibles is achieved by determining the composition factors of the specializations of irreducible representations of H, one from each H_K-equivalence class. We find these by finding the composition factors of the specializations of these same generic irreducible representations over F and establishing that they are absolutely irreducible.

A complete determination of the $H_F(\bar{u}, \bar{v})$-irreducible modules and the decomposition of the specialized generic irreducible modules has been carried our by Geck and Lux [5] for the case $\bar{v} = \bar{u}$ and by Bremke [1] for the cases $\bar{v} = \bar{u}^2$ and $\bar{v} = \bar{u}^4$.

Geck has observed (see [4, Lemma 1.2]) that when $H_F(\bar{u}, \bar{v})$ is semisimple, all generic irreducible representations of $H_K(u, v)$ remain irreducible and pairwise inequivalent under the given specialization. The semisimplicity of $H_F(\bar{u}, \bar{v})$ is equivalent to the simultaneous nonvanishing of the elements $\overline{c_\chi(u, v)}$ of F, where χ ranges over the set of generic irreducibles of $H_K(u, v)$ (see [6, Proposition 4.3]). $c_\chi(u, v)$ are polynomials in $\mathbf{Z}[u, v]$ defined by $c_\chi(u, v) = P(u, v)/D_\chi(u, v)$ where $P(u, v)$ is the Poincare polynomial of $H_K(u, v)$ and $D_\chi(u, v)$ is the generic degree of the representation χ (see [2, p.450]). It is straightforward to check that each of the polynomials $c_\chi(u, v)$ is a product of factors taken from the following 18 polynomials:

$$\begin{array}{l} 2,\ 3,\ u+1,\ v+1,\ u+v,\ u^2+u+1,\ u^2-u+1,\ v^2+v+1, \\ v^2-v+1,\ u^2+v^2,\ u+v^2,\ u^2+v,\ uv+1,\ u^2v+1,\ uv^2+1, \\ u^2v^2+1,\ u^2v^2-uv+1,\ \text{and } u^2-uv+v^2. \end{array} \tag{1}$$

Also, each of these polynomials occurs at least once as a factor of some $c_\chi(u, v)$. So, semisimplicity of $H_F(\bar{u}, \bar{v})$ is equivalent to the simultaneous nonvanishing of the specializations of these 18 polynomials. It remains to consider the cases in which one or more of these polynomials vanishes under specialization.

For the present paper, we exclude consideration of fields of characteristics 2 and 3. These cases will be studied in a later paper. The remaining cases subdivide into two classes, the first contains the 16 cases in which exactly one of the 16 non-constant polynomials specializes to zero and the second contains the 45 cases in which more than one of these polynomials specialize to zero.

2 Equivalent Specializations

To simplify our exposition, we note that certain pairs of specializations give rise to isomorphic specialized algebras. Two such specializations will be said to be *equivalent*.

For example, if T_1, T_2, T_3 and T_4 is a generating system for $H_F(\bar{u},\bar{v})$ then T_4, T_3, T_2 and T_1 is a generating system for $H_F(\bar{v},\bar{u})$. Clearly, $H_F(\bar{u},\bar{v}) \cong H_F(\bar{v},\bar{u})$ as rings.

Also, $-\bar{u}T_1^{-1}$, $-\bar{u}T_2^{-1}$, $-\bar{u}T_3^{-1}$ and $-\bar{u}T_4^{-1}$ is a generating system for $H_F(\bar{u},\bar{v})$ and $-\bar{u}^{-1}T_1$, $-\bar{u}^{-1}T_2$, T_3 and T_4 is a generating system for $H_F(\bar{u}^{-1},\bar{v})$. Clearly, $H_F(\bar{u},\bar{v}) \cong H_F(\bar{u}^{-1},\bar{v})$ as rings.

From the preceding remarks, we see that the specializations $(\bar{u},\bar{v})$, $(\bar{u}^{-1},\bar{v})$, $(\bar{u},\bar{v}^{-1})$, $(\bar{u}^{-1},\bar{v}^{-1})$, $(\bar{v},\bar{u})$, $(\bar{v}^{-1},\bar{u})$, $(\bar{v},\bar{u}^{-1})$ and $(\bar{v}^{-1},\bar{u}^{-1})$ are equivalent to one another. Since the algebras corresponding to equivalent specializations are isomorphic, it will be sufficient to consider just one specialization from each equivalence class.

The 16 cases referred to in Section 1, in which exactly one of the non-constant polynomials in (1) specializes to zero, split into 7 equivalence classes, from which we choose the following representatives:

Case 1: $\bar{u}=u,\ \bar{v}=-1$. Case 2: $\bar{u}=u,\ \bar{v}=\zeta_3$. Case 3: $\bar{u}=u,\ \bar{v}=\zeta_6$.
Case 4: $\bar{u}=u,\ \bar{v}=-u$. Case 5: $\bar{u}=u,\ \bar{v}=-u^2$. Case 6: $\bar{u}=u,\ \bar{v}=\zeta_4 u$.
Case 7: $\bar{u}=u,\ \bar{v}=\zeta_6 u$.

Here, ζ_n denotes a primitive nth root of 1 in F. If F contains mnth roots of 1 where m and n are positive integers, we choose the notation so that $\zeta_{mn}^m=\zeta_n$.

The results we obtain for these 7 cases will apply to any further specialization of u which does not result in any more of the polynomials in (1) vanishing.

Of the remaining 45 cases referred to in Section 1, 26 are equivalent to those studied by Bremke, 5 are equivalent to those studied by Geck and Lux and the remainder split into 6 classes with representatives:

Case 8: $\bar{u}=\zeta_6,\ \bar{v}=-1$. Case 9: $\bar{u}=\zeta_8,\ \bar{v}=\zeta_8^3$. Case 10: $\bar{u}=\zeta_{10},\ \bar{v}=\zeta_{10}^3$.
Case 11: $\bar{u}=\zeta_{24},\ \bar{v}=\zeta_{24}^5$. Case 12: $\bar{u}=\zeta_6,\ \bar{v}=1$. Case 13: $\bar{u}=\zeta_3,\ \bar{v}=-1$.

In view of the detailed work of Bremke and Geck and Lux, we make no further mention of the cases equivalent to those with $\bar{v}=\bar{u}$, $\bar{u}^2$ or $\bar{u}^4$.

3 Classes of Representations

If ρ is a representation of $H_F(\bar{u},\bar{v})$, we can obtain representations of the algebras corresponding to equivalent specializations by combining the following constructions:

(C1) $\rho'(T_i)=-\bar{u}\rho(T_i^{-1})$, $i=1,2$; $\rho'(T_i)=-\bar{v}\rho(T_i^{-1})$, $i=3,4$.
(C2) $\rho'(T_i)=\rho(T_{5-i})$, $i=1,2,3,4$.
(C3) $\rho'(T_i)=-\bar{u}^{-1}\rho(T_i)$, $i=1,2$; $\rho'(T_i)=\rho(T_i)$, $i=3,4$.

In (C1), ρ' is also a representation of $H_F(\bar{u},\bar{v})$. In (C2), ρ' is a representation of $H_F(\bar{v},\bar{u})$. In (C3), ρ' is a representation of $H_F(\bar{u}^{-1},\bar{v})$.

It is clear that any representation ρ' arising in one of these ways has a decomposition similar to that of ρ, in which the composition factors are obtained from those of ρ by the same construction.

A representation ρ'' will be said to be an *associate* of ρ if it is equivalent to the representation ρ' arising from ρ by construction (C1). This relation is symmetric.

It will be useful to have a compact notation to describe the composition factors of a module. For example, $M = X : Y : Z$ will denote the fact that the module M has a series of submodules $0 \subseteq V_1 \subseteq V_1 \subseteq V_3 = M$ with $V_1 \cong X$, $V_2/V_1 \cong Y$ and $V_3/V_2 \cong Z$ If Y is a direct sum $U + V$, we will write $M = X : U + V : Z$.

Our listing of the irreducible modules of the generic Hecke algebra $H_K(u, v)$ follows the order used by Kondo (see [2, p.413]) and in GAP (see [10]). We label the modules $M_1, \ldots, M_{25}$. To avoid cumbersome notation, we will use these labels to describe the modules resulting from these under any specializations considered. The order of listing differs from that used by Geck in [4] and by the second author in [7] and [8]. The two orders of listing are related by an interchange of the parameters u and v.

For convenience, we provide the following table to help identify the representations. The second row lists the degrees, the third and fourth the character values on T_1 and T_4 respectively.

M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9
1	1	1	1	2	2	2	2	4
u	u	-1	-1	$2u$	-2	$u-1$	$u-1$	$2u-2$
v	-1	v	-1	$v-1$	$v-1$	$2v$	-2	$2v-2$

M_{10}	M_{11}	M_{12}	M_{13}	M_{14}	M_{15}	M_{16}	M_{17}	M_{18}
9	9	9	9	6	6	12	4	4
$6u-3$	$6u-3$	$3u-6$	$3u-6$	$3u-3$	$3u-3$	$6u-6$	$3u-1$	$3u-1$
$6v-3$	$3v-6$	$6v-3$	$3v-6$	$3v-3$	$3v-3$	$6v-6$	$3v-1$	$v-3$

M_{19}	M_{20}	M_{21}	M_{22}	M_{23}	M_{24}	M_{25}
4	4	8	8	8	8	16
$u-3$	$u-3$	$6u-2$	$2u-6$	$4u-4$	$4u-4$	$8u-8$
$3v-1$	$v-3$	$4v-4$	$4v-4$	$6v-2$	$2v-6$	$8v-8$

This table does not distinguish M_{14} from M_{15}. However, T_1T_4 has the character values $(2u-1)v - (u-2)$ and $(u-2)v - (2u-1)$ in M_{14} and M_{15}, respectively.

We note that the pairs of associated generic irreducible modules are: M_1, M_4; M_2, M_3; M_5, M_6; M_7, M_8; M_{10}, M_{13}; M_{11}, M_{12}; M_{17}, M_{20}; M_{18}, M_{19}; M_{21}, M_{22}; M_{23}, M_{24}. The remaining 5 modules are self-associates.

In the case of a representation ρ of the generic Hecke algebra $H_K(u, v)$, we can modify constructions (C2) and (C3) to give representations of the same generic Hecke algebra by interchanging the parameters u and v following (C2) and by substituting u^{-1} for u following (C3). Using combinations of these constructions, the full set of generic irreducible representations may be derived easily from M_1, M_5, M_9, M_{12}, M_{14}, M_{16}, M_{17}, M_{22} and M_{25}. We use the explicit descriptions of M_1, M_5, M_9, M_{14} and M_{17} given in [7, pp.295-497] and the explicit descriptions of M_{12} and M_{22} given in [8, pp.48-49] We give below the explicit descriptions of M_{16} and M_{25} referred to in the Introduction.

M_{16}:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u & u-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u & -u & 1 & u & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -3\,u & -3\,u \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & -3\,u & 0 & 0 & -3\,u \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -3\,u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & u-1 \end{bmatrix}$$

$$\begin{bmatrix} u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u & 1 & 0 & -u & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & u & -1 & -u^2 & u & u-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u & -1 & 3\,u & 0 & 3\,u^2 & 3\,u \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & u^2-u & 1-u \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & u-1 \end{bmatrix}$$

$$\begin{bmatrix} v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u+v & u+v & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u+v & -v-1 & 0 & u+v & u-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & u+v & 0 & 0 \\ v & v & 0 & 3\,uv & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v & 0 & 0 & 0 & -1 & 0 & 1-u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 & u-1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -3\,u & -3\,u \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 3\,u & 0 & -3\,u & -3\,u^2 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -u & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -u & -2\,u & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & u & 0 & 2\,u & u+1 \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v \end{bmatrix}$$

M_{25}:

$$\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
u & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & uy & 1 & u & y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & x
\end{bmatrix}$$

$$\begin{bmatrix}
u & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u & -u & 0 & u & -u & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & u & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u^2 & 0 & 0 & u & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & -u & 0 & -1 & -1
\end{bmatrix}$$

$$\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v & 0 & 0 & 0 & 0 & -uz & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v & -z & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & v & 0 & ux & 0 & 0 & 0 & 0 & -x & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & v & -x & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v & 0 & 0 & 0 & 0 & 0 & 0 & t & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -v & 0 & 0 & t & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-uv & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & z & 0 & v & 0 & 0 \\
0 & -v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -z & z & 0 & v & s \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}$$

$$\begin{bmatrix}
v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v & -v & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & v & 0 & v & -uv & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & uv & -uv & -uv & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 \\
0 & 0 & 0 & 0 & u^2v & 0 & -v & uv & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\
0 & 0 & 0 & 0 & -uv & -v & -xv & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v
\end{bmatrix}$$

where $x = u - 1$, $y = u + 1$, $z = u + v$, $s = 1 + uv$, and $t = -2uv$.

All the explicit descriptions above and those obtained from them by the modified constructions are realized over the ring $\mathbf{Z}[u, v]$ with the exception of M_{15}. But in the case of M_{15}, it is an elementary exercise to find an equivalent representation realized over $\mathbf{Z}[u, v]$.

4 Known Results and General Techniques

In [7], complete decompositions have been found of all generic irreducible representations of degrees ≤ 4 under all specializations other than $\bar{u} = \bar{v} = -1$. These results are contained in the details of the 13 cases referred to in the Introduction and listed in the next section.

For each case, we adopted the following approach. A generic irreducible representation was specialized over a finite field with a prime number of elements p. The GAP Meat-Axe package (see [9]) provided a complete decomposition of this representation. From the generators of the submodules provided by GAP we were able to determine elements which generated corresponding submodules of the representation in all specializations allowed by this case. In order to have a reasonable expectation that the decomposition over the initial finite field would be similar to the general case, we selected primes $p > 100$.

For example, in Case 4, consider M_{12}. We find that $[0, 0, 0, 0, 0, u, 1, u, 1]$ generates a 4-dimensional submodule $U \cong M_9$. U together with $[0, 0, 0, 0, 0, 0, 0, u, 1]$ generates an 8-dimensional submodule V with $V/U \cong M_{19}$. Also, U and $[2u^2, -u, 0, 0, 1, 0, 0, 1, 0]$ generates a 5-dimensional submodule W with $W/U \cong M_3$. Thus, $M_{12} = M_9 : M_{19} + M_3$. In this example, we can check that the factors given are indeed irreducible by consulting [7].

For each of the representations of degree 6 or more and for each new representation which appeared as a factor in some decomposition it was also necessary to verify its irreducibility. The principal technique used for this was a version of Norton's Criterion for Irreducibility (see [9]) as explained by Geck and Lux in [5].

The procedure required a matrix B in the algebra generated by the representation with the following three properties: (i) the nullspace of B has dimension 1; (ii) the space generated by a non-zero null vector of B under the action of the representation is the whole space; (iii) the space generated by a non-zero null vector of the transpose of B under the action of the transpose of the representation is the whole space. If such a matrix B is found, the representation is irreducible. Since extension of the base field in this case caused no change in the validity of (i), (ii) or (iii), this result guaranteed absolute irreducibility.

For each representation to which we applied this technique, we found a suitable matrix B which continued to have co-rank 1 for any further specialization consistent with the case being studied. Moreover, under such specializations, the spaces referred to in (ii) and (iii) above continued to have maximal dimension. So, the representations continued to be absolutely irreducible under all such specializations.

For example, M_{22} remains irreducible in Case 3, with $\bar{u} = u$ and $\bar{v} = \zeta_6$. We see this by taking B to be the matrix $\bar{u}\bar{v}^2 - \bar{v}T_1 + T_2 - \bar{u}T_4 + T_1T_4$—here, we use T_i to denote the matrix representing it. B is singular but has a minor with determinant $(2 - \bar{v})\ (\bar{u} + 1)^4\ (\bar{u} + \bar{v})\ (\bar{u} - \bar{v}^2)$. Since the characteristic is not 3 and $\bar{u} \neq -1$, $-\bar{v}$ or $\bar{v}^2$—these three possibilities give cases equivalent to Case 8 and two of Bremke's cases—B has a null space of dimension 1 for all values of $\bar{u}$ covered by this case. The null space of B is spanned by $v_1 = [0, 0, 1, 1, 0, 0, 0, 0]$ and that of its transpose B' by $v'_1 = [0, 0, -(\bar{v} + 1), -\bar{u}(\bar{v} + 1), 0, 0, \bar{u}, \bar{u}^2]$. The set of vectors $\{v_1,\ v_1T_1,\ v_1T_3,\ v_1T_1T_3,\ v_1T_3T_4,\ v_1T_1T_3T_2,\ v_1T_1T_3T_2T_3,\ v_1T_1T_3T_2T_3T_4\}$ is independent so long as $\bar{u} \neq -1$, $\bar{v}$ or $\bar{v}^2$. The set of vectors $\{v'_1,\ v'_1T'_1,\ v'_1T'_3,\ v'_1T'_1T'_3,\ v'_1T'_3T'_4,\ v'_1T'_1T'_3T'_2,\ v'_1T'_1T'_3T'_2T'_3,\ v'_1T'_1T'_3T'_2T'_3T'_4\}$ is independent so long as $\bar{u} \neq \bar{v}$ and the characteristic is not 2. Hence, Norton's Criterion has been satisfied for M_{22} for all values of $\bar{u}$ covered by Case 3.

In the following section, we list the decompositions which occur in each case and give generators for the proper submodules. However, we will omit details of the computations similar to those used in the preceding example. For such details, the reader should contact either of the authors by e-mail.

5 Details of the Thirteen Cases

In this section, we record the list of those generic irreducible modules which remain irreducible under the various specializations and any isomorphisms arising between these modules. We also record the decompositions of the remaining modules giving, in each case, vectors generating the submodules which arise.

Since we can use the same generating vectors for each of a pair of associated generic irreducible modules which decompose under specialization, we will record these vectors in just one of the two cases.

In our decompositions, we introduce 22 new irreducible modules which we label M_{26}, …, M_{47} so that M_{2i} and M_{2i+1} are associates for $i = 13, \ldots, 23$.

For the remainder of the paper, we will use ω, i, ε, δ, ζ and η for ζ_3, ζ_4, ζ_6, ζ_8, ζ_{10} and ζ_{24}, respectively.

Case 1: $\bar{u} = u$, $\bar{v} = -1$. All generic irreducible modules, with the exception of M_{23} and M_{24}, remain irreducible. We get the following isomorphisms: $M_1 \cong M_2$, $M_3 \cong M_4$, $M_7 \cong M_8$, $M_{10} \cong M_{11}$, $M_{12} \cong M_{13}$, $M_{14} \cong M_{15}$, $M_{17} \cong M_{18}$, $M_{19} \cong M_{20}$, and $M_{23} \cong M_{24}$. No other pair of generic irreducible modules are isomorphic to one another. We have the decomposition $M_{23} = M_{14} : M_7$. The vector $[1, 1, 1, 0, 0, 0, 0, 0]$ generates a 6-dimensional submodule isomorphic to M_{14}.

Case 2: $\bar{u} = u$, $\bar{v} = \omega$. All generic irreducible modules, with the exception of M_5, M_6, M_9, M_{21}, M_{22}, and M_{25}, remain irreducible and pairwise non-isomorphic. For the remaining modules, we get the decompositions $M_5 = M_2 : M_1$, $M_6 = M_4 : M_3$, $M_9 = M_8 : M_7$, $M_{21} = M_{17} : M_{18}$, $M_{22} = M_{20} : M_{19}$, and $M_{25} = M_{23} : M_{24}$. The proper submodules of M_5, M_9, M_{21}, and M_{25} are generated by $[1, -\omega^2]$, $[1, 0, -\omega^2, 0]$, $[0, 0, 0, \omega, \omega, -1, -1, 0]$, and $[0, 0, 0, 0, 0, -u, u, u, 0, 0, 0, 0, 0, \omega, 0, 0]$, respectively.

Case 3: $\bar{u} = u$, $\bar{v} = \varepsilon$. All generic irreducible modules, with the exception of M_{16}, M_{23}, and M_{24}, remain irreducible and pairwise non-isomorphic. For the remaining modules, we get the decompositions $M_{16} = M_{26} : M_{27}$, $M_{23} = M_{27} : M_7$, and $M_{24} = M_{26} : M_8$.

The following matrices give a realization of the new 6-dimensional representation M_{26}:

$$\begin{bmatrix} u & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 \\ \varepsilon & 0 & u-\varepsilon & -1 & -\varepsilon & -\varepsilon \\ 0 & 0 & u & 0 & u-\varepsilon & u-\varepsilon^2 \\ 0 & 0 & -u & 0 & \varepsilon & \varepsilon^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ u & u-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ \varepsilon-1 & \varepsilon^2 & 0 & -1 & -\varepsilon^2 & \varepsilon^2 \\ 0 & 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 0 & u \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon^2 & 0 & -\varepsilon^2 \\ 0 & 0 & \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon^2 & 1 & \varepsilon^2 & 0 & \varepsilon \\ 0 & 0 & u+\varepsilon & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & -1 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & -1 & 0 & 0 \\ \varepsilon & \varepsilon & 0 & 0 & -1 & 0 \\ -\varepsilon & -\varepsilon & 0 & 0 & 0 & -1 \end{bmatrix}$$

The proper submodules of M_{16} and M_{23} are generated by $[0, 0, -(1+\varepsilon)/3, 0, 0, 0, 0, 0, u, 0, -u, -u^2]$ and $[0, 0, 0, 0, 0, 0, 1, 1]$, respectively.

Case 4: $\bar{u} = u$, $\bar{v} = -u$. All generic irreducible modules, with the exception of M_{11}, M_{12}, M_{15}, M_{16}, M_{17}, and M_{20}, remain irreducible and pairwise non-isomorphic. For the remaining modules, we get the decompositions $M_{11} = M_9 : M_{18} + M_2$, $M_{12} = M_9 : M_{19} + M_3$, $M_{15} = M_2 + M_3 : M_9$, $M_{16} = M_{18} : M_9 : M_{19}$, $M_{17} = M_5 : M_7$, and $M_{20} = M_6 : M_8$.

As remarked in the preceding section, $[0, 0, 0, 0, 0, u, 1, u, 1]$, $[0, 0, 0, 0, 0, 0, 0, u, 1]$, and $[2u^2, -u, 0, 0, 1, 0, 0, 1, 0]$ generate the proper submodules of M_{12}. The proper submodules of M_{15} are generated by $[-u, 0, 1, 0, 0, 0]$ and $[0, u^2, 0, 1, -u, -u]$. Those of M_{16} are generated by $[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0]$ and $[1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$. The proper submodule of M_{17} is generated by $[0, 0, 0, 1]$.

Case 5: $\bar{u} = u$, $\bar{v} = -u^2$. All generic irreducible modules, with the exception of M_{10}, M_{13}, M_{23}, and M_{24}, remain irreducible and pairwise non-isomorphic. For the remaining modules, we get the decompositions $M_{10} = M_5 : M_{29}$, $M_{13} = M_6 : M_{28}$, $M_{23} = M_{29} : M_3$, and $M_{24} = M_{28} : M_2$.

The following matrices give a realization of the new 7-dimensional representation M_{28}:

$$\begin{bmatrix} u & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & u & u-1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ u & u-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ u & u-1 & 0 & 0 & 0 & 1 & 0 \\ -u & 0 & u & 0 & u-1 & 0 & 0 \\ 0 & -u & 0 & u & 0 & u-1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -u^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -u^2 & 0 & 0 & 0 \\ 0 & 0 & u-1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -u & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -u^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -u^2 & 0 & 0 & 0 & 0 & 0 \\ -u^2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -u^2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ -u & 1-u & 0 & 0 & 0 & -1 & 0 \\ u-1 & u-1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

The proper submodules of M_{10} and M_{23} are generated by $[u^2, -u, 1, 0, -u^2, 0, u, 0, -1]$ and $[0, 0, 0, 0, 0, 0, 1, 1]$, respectively.

Case 6: $\bar{u} = u$, $\bar{v} = iu$. All generic irreducible modules, with the exception of M_{11}, M_{12}, and M_{25}, remain irreducible and pairwise non-isomorphic. For the remaining modules, we get the decompositions $M_{11} = M_{31} : M_2$, $M_{12} = M_{30} : M_3$, and $M_{25} = M_{30} : M_{31}$.

The following matrices give a realization of the new 8-dimensional representation M_{30}:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u & u & 0 & 0 & u & 0 & 0 & u \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+i & u & i & i \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u & 1 & 0 & 0 & u+1 & 0 & 0 & u+i \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 1 & -1-i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & -ui & 0 \\ 0 & 0 & 0 & 0 & -1 & i & u-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} i\,(u+i) & 0 & 0 & 0 & -1 & 0 & u & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & ui & 0 & i & 0 & 0 & i \\ 0 & i & 0 & ui & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ui & 0 & -u & 0 \\ 0 & 1 & 0 & 0 & 0 & ui & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -ui & 0 & 0 & 0 & -ui & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} ui & 0 & u & 0 & 1 & 0 & 0 & 0 \\ 0 & ui & 0 & u & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & ui & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & ui & 0 \\ 0 & 0 & 0 & 0 & ui & 0 & 0 & ui \end{bmatrix}$$

The proper submodules of M_{11} and M_{25} are generated by $[1, 1, 0, 0, 0, 0, 0, 0, 0, 0]$ and $[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0]$, respectively.

Case 7: $\bar{u} = u$, $\bar{v} = \varepsilon u$. All generic irreducible modules, with the exception of M_{14}, M_{18}, and M_{19}, remain irreducible and pairwise non-isomorphic. For the remaining modules, we get the decompositions $M_{14} = M_{33} : M_{32}$, $M_{18} = M_{32} : M_2$, and $M_{19} = M_{33} : M_3$.

The new 3-dimensional representation M_{32} is the representation described in [7, Sect. 1.2.3].

The proper submodules of M_{14} and M_{18} are generated by $[0, -u, 1, -u, 1, 0]$ and $[1, 1, 0, 0]$, respectively.

Case 8: $\bar{u} = \varepsilon$, $\bar{v} = -1$. All generic irreducible modules, with the exception of M_{16}, M_{21}, M_{22}, M_{23}, and M_{24}, remain irreducible. We get the following isomorphisms: $M_1 \cong M_2$, $M_3 \cong M_4$, $M_7 \cong M_8$, $M_{10} \cong M_{11}$, $M_{12} \cong M_{13}$, $M_{14} \cong M_{15}$, $M_{17} \cong M_{18}$, $M_{19} \cong M_{20}$, and $M_{23} \cong M_{24}$. No other pair of generic irreducible modules are isomorphic to one another. For the remaining modules, we get the decompositions $M_{16} = M_{34} : M_{35}$, $M_{21} = M_{35} : M_5$, $M_{22} = M_{34} : M_6$, and $M_{23} = M_{14} : M_7$.

The representation M_{34} is obtained by specializing u to -1 in M_{26} and applying construction (C2).

The proper submodules of M_{16}, M_{21} and M_{23} are generated by $[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1 + \varepsilon, 0]$ $[0, 0, 0, 0, 0, 0, 1, 1]$, and $[0, 0, 0, 0, 0, 0, 1, 1]$, respectively.

Case 9: $\bar{u} = \delta$, $\bar{v} = \delta^3$. All generic irreducible modules, with the exception of M_{10}, M_{11}, M_{12}, M_{13}, M_{14}, M_{16}, M_{18}, M_{19}, and M_{25}, remain irreducible and pairwise non-isomorphic.

For the remaining modules, we get the decompositions $M_{10} = M_9 : M_{17} + M_1$, $M_{11} = M_{31} : M_2$, $M_{12} = M_{30} : M_3$, $M_{13} = M_9 : M_{20} + M_4$, $M_{14} = M_1 + M_4 : M_9$, $M_{16} = M_{20} : M_9 : M_{17}$, $M_{18} = M_8 : M_5$, $M_{19} = M_7 : M_6$, and $M_{25} = M_{30} : M_{31}$.

The vectors $[0, 0, 0, 0, 0, \delta, 1, \delta, 1]$, $[0, 0, 2\delta+1, 0, -2, -2, \delta^3-2, -\delta^3-2, 0]$, and $[0, 0, 0, 0, 0, 0, 0, \delta, 1]$ generate submodules of M_{10} of dimensions 4, 5, and 8 respectively, where the 4-dimensional module is a submodule of the other two submodules.

The vectors $[1, 1, 1, 0, 0, 0]$ and $[0, 1, 0, \delta, 0, \delta^2]$ generate two submodules of M_{14}, each of dimension 1, and isomorphic to M_1 and M_4, respectively.

The vectors $[-\delta, 2, 1/(1 + \delta), -(1 + \delta), -\delta, 1, 0, 0, 0, 0, 0, 0]$ and $[0, 0, 0, 1, 1, 0, 0, 0, \delta - 1, 0, 0, \delta - 1]$ generate the proper submodules of M_{16} of dimensions 4 and 8, respectively.

The proper submodule of M_{18} is generated by $[1, \delta, 0, \delta]$. The proper submodules of M_{11} and M_{25} can be obtained from the corresponding modules in Case 6 by specializing u to δ in that case.

Case 10: $\bar{u} = \zeta$, $\bar{v} = \zeta^3$. All generic irreducible modules, with the exception of M_{10}, M_{11}, M_{12}, M_{13}, M_{21}, M_{22}, M_{23}, and M_{24}, remain irreducible and pairwise non-isomorphic.

For the remaining modules, we get the decompositions $M_{10} = M_{37} : M_7$, $M_{11} = M_5 : M_{38}$, $M_{12} = M_6 : M_{39}$, $M_{13} = M_{36} : M_8$, $M_{21} = M_{37} : M_2$, $M_{22} = M_{36} : M_3$, $M_{23} = M_1 : M_{39}$, and $M_{24} = M_4 : M_{38}$.

The representation M_{36} is obtained by specializing u to ζ in M_{28} and applying construction (C2). For M_{38} we apply construction (C3) to M_{28} and then specialize u to ζ^{-1}. Note that M_{38} may also be obtained from M_{36} by applying the field automorphism $\zeta \to \zeta^{-3}$ followed by the constructions (C3) and (C2).

The proper submodules of M_{10}, M_{11}, M_{21}, and M_{23} are generated by $[1, 0, 0, 0, 0, 0, 0, 0, -\zeta^3]$, $[\zeta^2, -\zeta, 1, 0, -\zeta^2, 0, \zeta, 0, -1]$, $[0, 0, 0, 0, 0, 0, 1, 1]$, and $[-1 - \zeta^2, 1, -\zeta^2, 1 - \zeta + \zeta^2 -$

$\zeta^3, \zeta, 1+\zeta^2, \zeta^3+\zeta-1, \zeta^3-\zeta^2-1]$, respectively.

Case 11: $\bar{u}=\eta$, $\bar{v}=\eta^5$. All generic irreducible modules, with the exception of M_{10}, M_{13}, M_{14}, M_{18}, M_{19}, and M_{25}, remain irreducible and pairwise non-isomorphic.

For the remaining modules, we get the decompositions $M_{10}=M_{40}:M_1$, $M_{13}=M_{41}:M_4$, $M_{14}=M_{33}:M_{32}$, $M_{18}=M_{32}:M_2$, $M_{19}=M_{33}:M_3$, and $M_{25}=M_{40}:M_{41}$.

The representation M_{40} is obtained by specializing u to η^{-1} in M_{30} and applying construction (C3).

The proper submodules of M_{10} and M_{25} are generated by $[1,1,0,0,0,0,0,0,0,0]$ and $[0,0,0,0,0,0,0,0,0,0,1,-\eta,1+\eta,1/(1+\eta),1,0]$, respectively. The proper submodules of M_{14} and M_{18} are obtained from the corresponding modules in Case 7 by specializing u to η in that case.

Case 12: $\bar{u}=\varepsilon$, $\bar{v}=1$. All generic irreducible modules, with the exception of M_{14}, M_{15}, M_{16}, M_{17}, M_{18}, M_{19}, M_{20}, M_{21}, and M_{22}, remain irreducible and pairwise non-isomorphic.

For the remaining modules, we get the decompositions $M_{14}=M_{44}:M_{45}$, $M_{15}=M_{43}:M_{42}$, $M_{16}=M_{46}:M_{47}$, $M_{17}=M_1:M_{43}$, $M_{18}=M_{45}:M_2$, $M_{19}=M_{44}:M_3$, $M_{20}=M_4:M_{42}$, $M_{21}=M_{47}:M_5$, and $M_{22}=M_{46}:M_6$.

The representation M_{42} is obtained by applying construction (C3) to M_{32} and then specializing u to ε^{-1}. M_{44} is obtained from M_{42} by keeping the first two generators and replacing the last two by their negatives. M_{46} is obtained by specializing u to 1 in M_{26} and applying construction (C2). Note that M_{46} and M_{34} are both specializations of the same $H_F(\varepsilon, v)$-module.

The proper submodules of M_{14}, M_{15}, M_{16}, M_{17}, M_{18}, and M_{21} are generated by $[1,0,1,-1,0,0]$, $[0,1,1,-1,0,1]$, $[0,0,0,0,0,0,0,0,0,1,u+1,0]$, $[u-2,3,-2,1]$, $[1,1,0,0]$, and $[0,0,0,0,0,0,1,1]$, respectively.

Case 13: $\bar{u}=\omega$, $\bar{v}=-1$. Only the generic irreducible modules M_1, M_5, M_{10}, M_{16}, and M_{21} and their associates remain irreducible and pairwise non-isomorphic. M_2, M_{11}, and their associates are the only other generic irreducible modules which remain irreducible but we have the isomorphisms: $M_1 \cong M_2$, $M_3 \cong M_4$, $M_{10} \cong M_{11}$ and $M_{12} \cong M_{13}$. We also have $M_7 \cong M_8$.

For the remaining modules, we get the decompositions $M_7=M_4:M_1$, $M_9=M_6:M_5$, $M_{14}=M_{33}:M_{32}$, $M_{15}=M_{32}:M_{33}$, $M_{17}=M_1:M_{32}$, $M_{18}=M_{32}:M_1$, $M_{19}=M_{33}:M_4$, $M_{20}=M_4:M_{33}$, $M_{23}=M_{32}:M_1+M_{33}:M_4$, $M_{24}=M_{33}:M_4+M_{32}:M_1$, and $M_{25}=M_{21}:M_{22}$.

The proper submodules of M_7, M_9, M_{14}, M_{15}, M_{17}, M_{18}, and M_{25} are generated by $[\omega,-1]$, $[\omega,-1,0,0]$, $[1,2,3,1,2,0]$, $[1,0,0,0,0,\omega]$, $[\omega,1,0,-1]$, $[1,1,0,0]$, and $[1,-\omega,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$, respectively.

M_{23} has submodules X and Y of dimensions 4 and 6, respectively, and such that $X\cap Y$ has dimension 3. $X\cap Y$, X and Y are generated by $[\omega,\omega,0,0,\omega,0,-1,-1]$, $[0,\omega^2,0,0,0,1,1,1]$ and $[0,0,0,0,0,0,1,1]$ respectively.

Proof of Theorem 1 Concluded: For Part (b), it is sufficient to inspect the explicit decompositions obtained in this section and to rearrange the irreducible modules appropriately.

For Part (c), we note that each irreducible H_L-representation has either been given explicitly or described as arising from an explicit irreducible representation by means of certain operations—the actions (C1), (C2) and (C3) of Section 3. In all of these cases, the representations are realized over the field L_0, the subfield of L generated by $\bar{u}$ and $\bar{v}$. Thus, the irreducible H_L-modules all have the form $N \otimes_{L_0} L$ where N is an irreducible H_{L_0}-module.

Suppose that N' is an arbitrary irreducible H_{L_0}-module. Then $N' \otimes_{L_0} L$ has a quotient module isomorphic to $N \otimes_{L_0} L$ for some irreducible H_{L_0}-module N. Thus, $\mathrm{Hom}_{H_L}(N' \otimes_{L_0} L, N \otimes_{L_0} L) \neq 0$. By Curtis and Reiner [3, (29.5)]), $\mathrm{Hom}_{H_{L_0}}(N', N) \neq 0$. Since N and N' are both irreducible, $N \cong N'$ by Schur's lemma. We conclude that all irreducible H_{L_0}-modules are absolutely irreducible. Hence, L_0 is a splitting field for H_{L_0}.

6 Acknowledgements

The authors acknowledge the helpful suggestions, leading to an improved exposition, made by the referee. They also acknowledge the helpful suggestions made by M.Geck regarding earlier work on this topic by second author.

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