On embedding certain Kazhdan–Lusztig cells of S_n into cells of S_{n+1}

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Abstract

In this paper, we consider a particular class of Kazhdan-Lusztig cells in the symmetric group S_n, the cells containing involutions associated with compositions λ of n. For certain families of compositions we are able to give an explicit description of the corresponding cells by obtaining reduced forms for all their elements. This is achieved by first finding a particular class of diagrams $\mathcal{E}^{(\lambda)}$ which lead to a subset of the cell from which the remaining elements of the cell are easily obtained. Moreover, we show that for certain cases of related compositions λ and $\hat{\lambda}$ of n and $n+1$ respectively, the members of $\mathcal{E}^{(\lambda)}$ and $\mathcal{E}^{(\hat{\lambda})}$ are also related in an analogous way. This allows us to associate certain cells in S_n with cells in S_{n+1} in a well-defined way, which is connected to the induction and restriction of cells.

Key words: symmetric group; Kazhdan-Lusztig cell; induction and restriction
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1 Introduction

In [13], when investigating the representations of a Coxeter group and its associated Hecke algebra, Kazhdan and Lusztig introduced the *left cells*, *right cells* and *two-sided cells* of a Coxeter group. The cell to which an element of the symmetric group S_n belongs can be determined by examining the tableaux resulting from an application of the *Robinson-Schensted process* to that element. Also, the elements of a cell can be computed by applying the reverse of the Robinson-Schensted process to a suitable selection of tableaux pairs. This, however, does not provide a straightforward way of obtaining reduced expressions for the elements of these cells.

In this paper, we provide an alternative process for determining the elements of a selection of cells. These are cells which are associated in a natural way with compositions of n, as the unique involution they contain is an element of longest length in the corresponding standard parabolic subgroup of S_n. We show how to construct a certain set $\mathcal{E}^{(\lambda)}$ of

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diagrams associated with the composition λ. A diagram D, which is a subset of $\mathbb{Z}^2$, is a generalization of a Young diagram, in particular, we do not insist that $\mu''_D = \lambda'_D$ (where λ_D and μ_D are respectively the row- and column-compositions of D). Each diagram D in $\mathcal{E}^{(\lambda)}$, where $\lambda = \lambda_D$, uniquely determines an element w_D of S_n and all the elements $w_{J(\lambda)} w_D$ belong to the same cell. We say the elements w_D, with $D \in \mathcal{E}^{(\lambda)}$, form the *rim* of the cell. The remaining elements of the cell are the products of $w_{J(\lambda)}$ with the prefixes of the elements in the rim.

In this way we are able to give reduced expressions for all the elements in cells corresponding to certain families of compositions λ (examples of these are given at the end of Section 3). The techniques introduced earlier on in Section 3 (these are based on the work in Schensted [20] and Greene [9] on increasing and decreasing subsequences) are crucial in achieving this and in fact they provide an extension to some of the techniques used in [17]. The main contribution of the paper is to show, by the use of the above techniques, that for certain related compositions λ and $\hat{\lambda}$ of n and $n+1$ respectively, the members of $\mathcal{E}^{(\lambda)}$ and $\mathcal{E}^{(\hat{\lambda})}$ are also related in an analogous way. This allows us to associate certain cells in S_n with cells in S_{n+1} in a well-defined way which is in fact connected with the induction and restriction of Kazhdan-Lusztig cells (see Barbasch and Vogan [2]).

Before discussing briefly the main results of the paper (Theorems 4.3 and 4.8 in Section 4) we need to introduce some more notation. For $\lambda = (\lambda_1, \ldots, \lambda_r)$ a composition of n, let $W_{J(\lambda)}$ be the standard parabolic (Young) subgroup of S_n corresponding to λ and let $w_{J(\lambda)}$ be the longest element of $W_{J(\lambda)}$. Also let $\mathfrak{X}_{J(\lambda)}$ be a complete set of distinguished right coset representatives of $W_{J(\lambda)}$ in S_n. The right cell, $\mathfrak{C}(\lambda)$, containing $w_{J(\lambda)}$ has form $w_{J(\lambda)} Z(\lambda)$ for some subset $Z(\lambda)$ of $\mathfrak{X}_{J(\lambda)}$. Also set $Y(\lambda) = \{x \in Z(\lambda) : x \text{ is not a prefix of any other } y \in Z(\lambda)\}$. Then $Y(\lambda) = \{w_D : D \in \mathcal{E}^{(\lambda)}\}$ with $\mathcal{E}^{(\lambda)}$ and w_D as above. (Note that for a diagram D, w_D is the unique element of minimum length in the double coset $W_{J(\lambda_D)} w_D W_{J(\mu_D)}$, and this double coset has the trivial intersection property.)

We will associate to the composition λ of n the compositions λ_* and $\lambda^{(k)}$ of $n+1$, where $\lambda_* = (\lambda_1, \ldots, \lambda_r, 1)$ and $\lambda^{(k)} = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + 1, \lambda_{k+1}, \ldots, \lambda_r)$ for $1 \le k \le r$. Moreover, given a diagram D with row-composition $\lambda_D = \lambda$ we form diagrams D' (with $\lambda_{D'} = \lambda_*$) and $D^{(k)}$ (with $\lambda_{D^{(k)}} = \lambda^{(k)}$) by inserting a new node to D in a well-defined way.

In Theorem 4.3 we define, via diagram pairs (D, D'), an injection θ_* from $Z(\lambda)$ to $Z(\lambda_*)$ (this induces an injection from $\mathcal{E}^{(\lambda)}$ to $\mathcal{E}^{(\lambda_*)}$) which satisfies $Y(\lambda)\theta_* \subseteq Y(\lambda_*) \subseteq Z(\lambda)\theta_*$. So, θ_* not only relates the rims of the cells $\mathfrak{C}(\lambda)$ and $\mathfrak{C}(\lambda_*)$ but also gives a way of determining (and obtaining reduced forms for) all the elements of cell $\mathfrak{C}(\lambda_*)$ once we have the corresponding information about cell $\mathfrak{C}(\lambda)$. In addition, we give the connection of the process described in this theorem with the induction of cells.

Finally, in Theorem 4.8 (under the assumption that $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a composition of n with $\lambda_k \ge \lambda_i$ for $1 \le i \le r$) we define, now via diagram pairs $(D, D^{(k)})$ an injection θ from $Z(\lambda)$ to $Z(\lambda_*)$ and give some sufficient conditions for θ to satisfy $Y(\lambda)\theta \subseteq Y(\lambda^{(k)})$. In particular, when $k = 1$, we have equality $Y(\lambda)\theta = Y(\lambda_*)$ (thus, in this case, θ induces a bijection from $\mathcal{E}^{(\lambda)}$ to $\mathcal{E}^{(\lambda^{(k)})}$) and this allows us to determine the rim of $\mathfrak{C}(\lambda^{(k)})$, and hence obtain reduced forms for all the elements of this cell, just from knowledge of the rim of $\mathfrak{C}(\lambda)$. We also show the precise connection of this process with the restriction of cells.

2 Induction and restriction of Kazhdan-Lusztig cells of S_n

We begin this section by recalling some of the basic concepts and results of the Kazhdan-Lusztig representation theory of Coxeter groups and Hecke algebras. For any Coxeter system (W, S), Kazhdan and Lusztig [13] introduced three preorders $\leqslant_L$, $\leqslant_R$ and $\leqslant_{LR}$, with corresponding equivalence relations $\sim_L$, $\sim_R$ and $\sim_{LR}$, whose equivalence classes are called *left cells*, *right cells* and *two-sided cells*, respectively.

For basic concepts relating to Coxeter groups and Hecke algebras, see Geck and Pfeiffer [8] and Humphreys [10]. In particular, for a Coxeter system (W, S), $W_J = \langle J \rangle$ denotes the parabolic subgroup determined by a subset J of S, w_J denotes the longest element of W_J, $\mathfrak{X}_J$ denotes the set of minimum length elements in the right cosets of W_J in W (the distinguished right coset representatives), $\leqslant$ denotes the strong Bruhat order on W, and $w < w'$ means $w \leqslant w'$ and $w \neq w'$ if $w, w' \in W$. The pair (W_J, J) is a Coxeter system whose length function is the same as the restriction of the length function of (W, S) to it; consequently, w_J is determined entirely by J. Also recall the *prefix relation* on the elements of W: if $x, y \in W$ we say that x is a *prefix* of y if y has a *reduced form* beginning with a reduced form for x.

The Hecke algebra $\mathcal{H}$ corresponding to (W, S) and defined over the ring $A = \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, where q is an indeterminate, has a free A-basis $\{T_w : w \in W\}$ and multiplication defined by the rules

(i) $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$ and
(ii) $(T_s + 1)(T_s - q) = 0$ if $s \in S$.

The basis $\{T_w : w \in W\}$ is called the *T-basis* of $\mathcal{H}$. (See [13]).

Result 1 ([13, Theorem 1.1]). $\mathcal{H}$ has a basis $\{C_w : w \in W\}$, the *C-basis*, whose terms have the form $C_y = \sum_{x \leqslant y} (-1)^{l(y)-l(x)} q^{\frac{1}{2}l(y)-l(x)} P_{x,y}(q^{-1}) T_x$, where $P_{x,y}(q)$ is a polynomial in q with integer coefficients of degree $\leq \frac{1}{2}(l(y) - l(x) - 1)$ if $x < y$ and $P_{y,y} = 1$.

The following result collects some useful propositions concerning cells. For proofs of (i), (ii) and (iii), see [13, 2.3ac], [14, 5.26.1], and [15, Corollary 1.9(c)], respectively. See also [7, Lemma 5.3] for a more elementary algebraic proof of (iii) in the case that W is the symmetric group.

Result 2 ([13, 14, 15]).
(i)   If x, y, z are elements of W such that x is a prefix of y, y is a prefix of z and $x \sim_R z$ then $x \sim_R y$.
(ii)   If $J \subseteq S$, then the right cell containing w_J is contained in $w_J \mathfrak{X}_J$.
(iii) If W is a crystallographic group and $x, y \in W$ are such that $x \sim_{LR} y$ and $x \leqslant_R y$ then $x \sim_R y$.

Each cell of W provides an integral representation of W, with the C-basis of $\mathcal{H}$ playing an important role in the construction of this representation; see Kazhdan and Lusztig [13, § 1]. Barbasch and Vogan [2] have addressed the question of induction and restriction of such representations in relation to parabolic subgroups, where W is a Weyl group. They obtained the following results, which have natural analogues for right cells. We quote these results of Barbasch and Vogan as they are formulated in [18].

Result 3 ([2, Proposition 3.15]). Let H be a parabolic subgroup of a Weyl group W. Let $\mathfrak{C}$ be a Kazhdan-Lusztig left cell of H and let $\rho^{\mathfrak{C}}$ be its Kazhdan-Lusztig representation of H. Let L be the set of all representatives of minimal length of the left cosets of H in W. Then $L\mathfrak{C} = \{rv : r \in L, v \in \mathfrak{C}\}$ is a union of Kazhdan-Lusztig left cells. The associated Kazhdan-Lusztig representation $\rho^{L\mathfrak{C}}$ is isomorphic to the induced representation $\rho^{\mathfrak{C}}\!\uparrow_H^W$.

Result 4 ([2, Proposition 3.11]). Let H be a parabolic subgroup of a Weyl group W. Let R be the set of representatives of minimal length of the right cosets of H in W. Let $\mathfrak{C}$ be a Kazhdan-Lusztig left cell of W, and let $\rho^{\mathfrak{C}}$ be the Kazhdan-Lusztig representation of W associated with $\mathfrak{C}$. Then $\mathfrak{C}$ is a disjoint union of sets of the form $\mathcal{D}r$ where $r \in R$ and $\mathcal{D}$ is a Kazhdan-Lusztig cell of H. The direct sum of the associated Kazhdan-Lusztig left representations of H, $\bigoplus_{\mathcal{D}} \rho^{\mathcal{D}}$, corresponds to the decomposition of the restricted representation $\rho^{\mathfrak{C}}\!\downarrow_H^W$.

These results have been generalized to all Coxeter groups by Roichman [18] and Geck [6].

We will focus our attention on the symmetric group. For the basic definitions and background concerning partitions, compositions, Young diagrams and Young tableaux we refer to James [12], Fulton [5] or Sagan [19].

All our partitions and compositions will be assumed to be *proper* (that is, with no zero parts). We use the notation $\lambda \vDash n$ (respectively, $\lambda \vdash n$) to say that λ is a composition (respectively, partition) of n. For $\lambda \vDash n$ having r' as its maximum part, recall that the *conjugate* composition $\lambda' = (\lambda'_1, \ldots, \lambda'_{r'})$ of λ is defined by $\lambda'_i = |\{j : 1 \leq j \leq r \text{ and } i \leq \lambda_j\}|$ for $1 \leq i \leq r'$. It is immediate that λ' is a partition of n with r' parts.

If λ and μ are partitions of n, write $\lambda \trianglelefteq \mu$ if, for all k, $\sum_{1 \leq i \leq k} \lambda_i \leq \sum_{1 \leq i \leq k} \mu_i$. This is the dominance order of partitions (see [12, p.8]). If $\lambda \trianglelefteq \mu$ and $\lambda \neq \mu$, we write $\lambda \lhd \mu$.

In the case of the symmetric group S_n, the Robinson-Schensted correspondence gives a combinatorial method of identifying the Kazhdan-Lusztig cells. The Robinson-Schensted correspondence is a bijection of S_n to the set of pairs of standard Young tableaux $(\mathcal{P}, \mathcal{Q})$ of the same shape and with n entries, where the shape of a tableau is the partition counting the number of entries on each row. See [5] or [19] for a good description of this correspondence. Denote this correspondence by $w \mapsto (\mathcal{P}(w), \mathcal{Q}(w))$. Then $\mathcal{Q}(w) = \mathcal{P}(w^{-1})$. The *shape* of w, denoted by $\text{sh}\, w$, is defined to be the common shape of the Young tableaux $\mathcal{P}(w)$ and $\mathcal{Q}(w)$. The tableaux $\mathcal{P}(w)$ and $\mathcal{Q}(w)$ are called the *insertion tableau* and the *recording tableau*, respectively, for w.

The following result in [13] characterises the cells in S_n.

Result 5 ([13], see also [1, Theorem A] or [7, Corollary 5.6]). If $\mathcal{P}$ is a fixed standard Young tableau then the set $\{w \in W : \mathcal{P}(w) = \mathcal{P}\}$ is a left cell of W and the set $\{w \in W : \mathcal{Q}(w) = \mathcal{P}\}$ is a right cell of S_n. Conversely, every left cell and every right cell arises in this way. Moreover, the two-sided cells are the subsets of W of the form $\{w \in W : \text{sh}\, \mathcal{P}(w) \text{ is a fixed partition}\}$.

The *shape* $\text{sh}\, \mathfrak{C}$ of a cell $\mathfrak{C}$ is $\text{sh}\, w$ for any $w \in \mathfrak{C}$.

For the rest of the paper, unless explicitly mentioned otherwise, W and W' will be the symmetric groups S_n on $\{1, \ldots, n\}$ and S_{n+1} on $\{1, \ldots, n+1\}$, respectively, with the natural embedding. Let $s_i = (i, i+1)$ for $1 \leq i \leq n$, let $S = \{s_1, \ldots, s_{n-1}\}$ and

4

$S' = \{s_1, \ldots, s_n\}$, so that (W, S) and (W', S') are Coxeter systems. Let $\mathfrak{X}'_K$ be the set of distinguished right coset representatives of W'_K in W', for any subset $K \subseteq S'$. Note that W is embedded naturally in W' as W'_S. We write $\mathfrak{X}'$ for $\mathfrak{X}'_S$. Then $\mathfrak{X}' = \{x_i \colon 1 \leq i \leq n+1\}$ where, for $1 \leq i \leq n+1$, $x_i = (i, i+1, \ldots, n, n+1) = s_n \cdots s_i$ (the empty product is 1, by convention).

We will describe an element w of S_n in different forms: as a word in the generators s_1, $\ldots$, s_{n-1}, as products of disjoint cycles on $1, \ldots, n$, and in *row-form* $[w_1, \ldots, w_n]$ where $w_i = iw$ for $i = 1, \ldots, n$. Also if $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a *composition* of n with r *parts*, we define the subset $J(\lambda)$ of S to be $S \backslash \{s_{\lambda_1}, s_{\lambda_1 + \lambda_2}, \ldots, s_{\lambda_1 + \ldots + \lambda_{r-1}}\}$. We make similar definitions for S_{n+1} and compositions of $n+1$.

For a Young diagram D corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, let $\mathrm{ic}(D)$ and $\mathrm{oc}(D)$ be the sets of *inner corners* and *outer corners*, respectively, of D; that is,

$$\begin{aligned}
\mathrm{ic}(D) &= \{(i, \lambda_i)) \colon 1 \leq i \leq r-1 \text{ where } \lambda_i > \lambda_{i+1}\} \cup \{(r, \lambda_r)\}, \\
\mathrm{oc}(D) &= \{(1, \lambda_1 + 1)\} \cup \{(i, \lambda_i + 1)) \colon 2 \leq i \leq r \text{ where } \lambda_{i-1} > \lambda_i\} \cup \{(r+1, 1)\}.
\end{aligned}$$

We denote by $<$ the total order on the nodes of a diagram given by $(i, j) < (i', j')$ if, and only if, $i < i'$ or $i = i'$ and $j < j'$.

Proposition 2.1. *Let $\mathfrak{C}$ be a right cell of W, let A be the recording tableau of elements of $\mathfrak{C}$ and let D be its underlying diagram. For each $k \in \mathrm{oc}(D)$, let A_k be the tableau obtained from A by adding the entry $n+1$ at node k and let $\mathfrak{C}_k$ be the right cell of W' corresponding to the recording tableau A_k. Then $\mathfrak{C}\mathfrak{X}' = \bigcup_{k \in \mathrm{oc}(D)} \mathfrak{C}_k$.*

Furthermore, if $k, k' \in \mathrm{oc}(D)$ and $k < k'$ then $\mathrm{sh}\,\mathfrak{C}_{k'} \lhd \mathrm{sh}\,\mathfrak{C}_k$.

Proof. Let $w = [w_1, \ldots, w_n] \in \mathfrak{C}$. Then $\mathcal{Q}(w) = A$. Let $\mathcal{P}(w) = B$. For each $k \in \mathrm{oc}(D)$, we let D_k be the diagram obtained from D by adding node k. If $1 \leq i \leq n+1$, then $wx_i = [\bar{w}_1, \ldots, \bar{w}_n, i]$ where $\bar{w}_j = w_j$ if $1 \leq w_j < i$ and $\bar{w}_j = w_j + 1$ if $i \leq w_j \leq n$. In determining the tableaux associated with wx_i, the tableaux which arise after the first n insertions are $\bar{B} = Bx_i$ and A. Hence, $\mathcal{Q}(wx_i) = A_k$ for some $k \in \mathrm{oc}(D)$. So, $wx_i \in \mathfrak{C}_k$.

On the other hand, suppose that $w' \in \mathfrak{C}_k$ where $k \in \mathrm{oc}(D)$. Write $w' = [w'_1, \ldots, w'_{n+1}]$ and let $i = w'_{n+1}$. Following the first n insertions of w' the recording tableau is A. Hence, $w = w'x_i^{-1} \in S_n$ and $\mathcal{Q}(w) = A$, so that $w \in \mathfrak{C}$ and $w' \in \mathfrak{C}x_i$.

It follows that $\mathfrak{C}\mathfrak{X}' = \bigcup_{k \in \mathrm{oc}(D)} \mathfrak{C}_k$. The final sentence in the proposition is immediate. $\square$

Proposition 2.2. *Let $\mathfrak{C}$ be a right cell of W' and let A be the recording tableau of elements of $\mathfrak{C}$ and let D be its underlying diagram. For each $k \in \mathrm{ic}(D)$, if $i(k)$ is the entry on the first row of A removed by reverse inserting from node k and A' is the resulting tableau, let $d_k = x_{i(k)}^{-1}$ and let $A_k = A'd_k$ (so that A_k is a standard Young tableau on $1, \ldots, n$). Let $\mathfrak{C}_k$ be the right cell of W corresponding to the recording tableau A_k. Then $\mathfrak{C} = \bigcup_{k \in \mathrm{ic}(D)} d_k \mathfrak{C}_k$.*

Furthermore, if $k, k' \in \mathrm{ic}(D)$ and $k < k'$ then $\mathrm{sh}\,\mathfrak{C}_k \lhd \mathrm{sh}\,\mathfrak{C}_{k'}$ and $d_k \leqslant d_{k'}$.

Proof. Let $w \in \mathfrak{C}$. Then $\mathcal{Q}(w) = A$. Let $\mathcal{P}(w) = B$. Then $\mathcal{P}(w^{-1}) = A$ and $\mathcal{Q}(w^{-1}) = B$. Let $w^{-1} = [v_1, \ldots, v_{n+1}]$ and let $k \in \mathrm{ic}(D)$ be the node at which B has entry $n+1$. Let B' be obtained from B by removing the entry $n+1$. Let A' be obtained from A by reverse-insertion from node k. Then $i(k)$ $(= v_{n+1})$ is the entry removed from the first row of A by this process. Also, A' and B' are the insertion and recording tableaux,

respectively, arising from the insertion of the first n entries of the row-form of w^{-1}. Hence, $w^{-1}d_k = w^{-1}x_{i(k)}^{-1} \in W$ and $\mathcal{P}(w^{-1}d_k) = A'd_k = A_k$. So, $d_k^{-1}w \in W$ and $\mathcal{Q}(d_k^{-1}w) = A_k$. That is, $w \in d_k\mathfrak{C}_k$.

The preceding argument is easily seen to be reversible. Hence, $\mathfrak{C} = \bigcup_{k \in \mathrm{ic}(D)} d_k\mathfrak{C}_k$. The first part of the final sentence in the proposition is immediate. Since the reverse insertion path from node k' must pass the row of k weakly left of that node, it must remain weakly left of the reverse insertion path from node k. Hence, $i(k') \leq i(k)$. So, $d_k = s_{i(k)} \cdots s_{n-1}s_n \leqslant s_{i(k')} \cdots s_{n-1}s_n = d_{k'}$. $\qquad\square$

3 Paths and admissible diagrams: determining the rim of certain cells

We recall the generalizations of the notions of diagram and tableau, commonly used in the basic theory, which we described in [17]. A *diagram D* is a finite subset of $\mathbb{Z}^2$. We will assume, where possible, that D has no empty rows or columns. These are the principal diagrams of [17]. We will also assume that both rows and columns of D are indexed consecutively from 1; a node in D will be given coordinates (a, b) where a and b are the indices respectively of the row and column which the node belongs to (rows are indexed from top to bottom and columns from left to right). For a principal diagram D we denote by c_D and r_D the number of columns and rows of D respectively. The *row-composition λ_D* (respectively, *column-composition μ_D*) of D is defined by setting $\lambda_{D,k}$ (respectively, $\mu_{D,k}$) to be the number of nodes on the k-th row (respectively, column) of D for $1 \leq k \leq r_D$ (respectively, $1 \leq k \leq c_D$). If λ and μ are compositions, we will write $\mathcal{D}^{(\lambda,\mu)}$ for the set of (principal) diagrams D with $\lambda_D = \lambda$ and $\mu_D = \mu$. We also define $\mathcal{D}^{(\lambda)}$ to be the set $\bigcup_{\mu \vDash n} \mathcal{D}^{(\lambda,\mu)}$ of (principal) diagrams with $\lambda_D = \lambda$. A *special diagram* is a diagram obtained from a Young diagram by permuting the rows and columns. Special diagrams are characterised in the following proposition.

Result 6 ([17, Proposition 3.1], Compare [3, Lemma 5.2]). Let D be a diagram. The following statements are equivalent. (i) D is special; (ii) $\lambda_D'' = \mu_D'$; (iii) for every pair of nodes $(i, j), (i', j')$ of D with $i \neq i'$ and $j \neq j'$, at least one of (i', j) and (i, j') is also a node of D.

If D is a diagram with n nodes, a *D-tableau* is a bijection $t : D \to \{1, \ldots, n\}$ and we refer to $(i, j)t$, where $(i, j) \in D$, as the *(i, j)-entry* of t. The group W acts on the set of D-tableaux in the obvious way—if $w \in W$, an entry i is replaced by iw and tw denotes the tableau resulting from the action of w on the tableau t. We denote by t^D and t_D the two D-tableaux obtained by filling the nodes of D with $1, \ldots, n$ by rows and by columns, respectively, and we write w_D for the element of W defined by $t^D w_D = t_D$.

Now let D be a diagram and let t be a D-tableau. We say t is *row-standard* if it is increasing on rows. Similarly, we say t is *column-standard* if it is increasing on columns. We say that t is *standard* if $(i', j')t \leq (i'', j'')t$ for any $(i', j'), (i'', j'') \in D$ with $i' \leq i''$ and $j' \leq j''$. Note that a standard D-tableau is row-standard and column-standard, but the converse is not true, in general.

Result 7 ([17, Proposition 3.5]. Compare [4, Lemma 1.5]). *Let D be a diagram. Then the mapping $u \mapsto t^D u$ is a bijection of the set of prefixes of w_D to the set of standard D-tableaux.*

In general, an element of W will have an expression of the form w_D for many different diagrams D of size n. The following result shows how to locate suitable diagrams.

Result 8 ([17, Proposition 3.7]). *Let $\lambda \vDash n$ and let $d \in \mathfrak{X}_{J(\lambda)}$. Then $d = w_D$ for some diagram $D \in \mathcal{D}^{(\lambda)}$.*

The proof involves the construction of a principal diagram $D(d, \lambda)$ with the desired properties. This is formed by partitioning the row-form of d in parts of sizes corresponding to λ, placing these parts on consecutive rows and moving the entries on the rows minimally to make a tableau of the form t_D.

Now let $d \in \mathfrak{X}_{J(\lambda)}$ and denote by $\mathcal{D}_d^{(\lambda)}$ the set of principal diagrams $D \in \mathcal{D}^{(\lambda)}$ for which $w_D = d$. The following result, which will turn out to be useful later on, tells us in what way two elements of $\mathcal{D}_d^{(\lambda)}$ can differ from one another.

Result 9 ([17, Proposition 3.8]). *Let $\lambda \vDash n$, let $d \in \mathfrak{X}_{J(\lambda)}$, let $D = D(d, \lambda)$ and let $E \in \mathcal{D}_d^{(\lambda)}$. Then the set of columns of E may be partitioned into sets of consecutive columns so that, for $j \geq 1$,*

(i) *for any two columns in the j-th set, the nodes in the column with lesser column index have row indices which are less than all the indices of the nodes in the column with greater column index;*

(ii) *the row indices of the nodes occurring in columns of the j-th set are precisely the row indices of the nodes in the j-th column of D.*

In particular, D is the unique diagram in $\mathcal{D}_d^{(\lambda)}$ with the minimum number of columns.

We illustrate some of the above concepts with an example.

Example. Let $\lambda = (3^2, 2, 1)$ and $d = [3, 4, 7, 2, 6, 8, 1, 9, 5] \in \mathfrak{X}_{J(\lambda)}$. From the 'corresponding' row-standard tableau, $\begin{smallmatrix} 3 & 4 & 7 \\ 2 & 6 & 8 \\ 1 & 9 \\ 5 \end{smallmatrix}$, we produce the tableau $\begin{smallmatrix} 3 & 4 & & 7 \\ 2 & & 6 & 8 \\ 1 & & & 9 \\ & 5 \end{smallmatrix}$ using the procedure in [17, Proposition 3.7] (see comments after Result 8). Now let

$$D = \begin{smallmatrix} & & \times & \times & & \times & \\ & \times & & & \times & \times & \\ \times & & & & & \times & \\ & & \times & & & \times & \end{smallmatrix} \quad \text{and} \quad E = \begin{smallmatrix} & & \times & \times & & & \times & \\ & \times & & & & \times & & \times \\ \times & & & & & & & \times \\ & & & \times & & & & \end{smallmatrix} .$$

Then $D = D(d, (3^2, 2, 1))$ and $w_D = w_E = d$ (compare with Result 9). Note that the 4-th and 6-th sets of columns in E referred to in Result 9 are $\{4, 5\}$ and $\{7, 8\}$. We also have $D = \{(1, 3), (1, 4), (1, 6), (2, 2), (2, 5), (2, 6), (3, 1), (3, 6), (4, 4)\}$, $c_D = 6$ and $r_D = 4$. Now let $e_1 = [2, 3, 4, 1, 6, 7, 5, 8, 9]$ and $e_2 = [2, 5, 6, 1, 4, 7, 3, 8, 9]$. Then,

$$t^D e_1 = \begin{smallmatrix} & 2 & 3 & & 4 \\ 1 & & & 6 & 7 \\ 5 & & & & 8 \\ & & 9 \end{smallmatrix} \quad \text{and} \quad t^D e_2 = \begin{smallmatrix} & 2 & 5 & & 6 \\ 1 & & & 4 & 7 \\ 3 & & & & 8 \\ & & 9 \end{smallmatrix} .$$

As $t^D e_1$ is standard but $t^D e_2$ is not standard, we conclude from Result 7 that e_1 is a prefix of d while e_2 is not a prefix of d.

For a composition λ of n, we define the following subsets of $\mathfrak{X}_{J(\lambda)}$ and $\mathcal{D}^{(\lambda)}$:

$$
\begin{aligned}
Z(\lambda) &= \{e \in \mathfrak{X}_{J(\lambda)} \colon w_{J(\lambda)}e \sim_R w_{J(\lambda)}\}, \\
Z_s(\lambda) &= \{e \in Z(\lambda) \colon e = w_D \text{ for some special diagram } D \in \mathcal{D}^{(\lambda)}\}, \\
Y(\lambda) &= \{x \in Z(\lambda) \colon x \text{ is not a prefix of any other } y \in Z(\lambda)\}, \\
Y_s(\lambda) &= Y(\lambda) \cap Z_s(\lambda), \\
\mathcal{E}^{(\lambda)} &= \{D(y, \lambda) \colon y \in Y(\lambda)\}.
\end{aligned}
$$

In view of Result 2(i) and (ii), $Z(\lambda)$ is closed under the taking of prefixes and $w_{J(\lambda)}Z(\lambda)$ is the right cell of W containing $w_{J(\lambda)}$. We denote this right cell by $\mathfrak{C}(\lambda)$. Note that $e \in Z(\lambda)$ if, and only if, $e \in \mathfrak{X}_{J(\lambda)}$ and $\mathcal{Q}(w_{J(\lambda)}e) = \mathcal{Q}(w_{J(\lambda)})$.

A knowledge of $Y(\lambda)$ leads directly to $Z(\lambda)$ by determining all prefixes. We call $Y(\lambda)$ the *rim* of the cell $\mathfrak{C}(\lambda)$.

Remark 3.1. In the case that λ is a partition of n, it follows from [16, Lemma 3.3] that $Y(\lambda) = Y_s(\lambda) = \{w_D\}$, where D is the Young diagram with $\lambda_D = \lambda$, and $Z(\lambda)$ is the set of prefixes of w_D.

We now show how a knowledge of the increasing subsequences of the row-form of $w_{J(\lambda_D)}e$, where D is any diagram and e is a prefix of w_D, helps to determine whether $w_{J(\lambda_D)}e$ is in the same right cell as $w_{J(\lambda_D)}$.

Lemma 3.2. *Let D be a diagram of size n and let e be a prefix of w_D.*
  (i)   *If (a, b) and (c, d) are nodes of D satisfying $a < c$ and $b \le d$, then $(a, b)t^D e < (c, d)t^D e$.*
  (ii) *If (a, b) and (c, d) are nodes of D satisfying $a < c$ and $b \le d$ and $k = (a, b)t^D w_{J(\lambda_D)}$ and $l = (c, d)t^D w_{J(\lambda_D)}$, then $k < l$ and $kw_{J(\lambda_D)}w_D < lw_{J(\lambda_D)}w_D$.*
  (iii) *If k and l are integers satisfying $1 \le k < l \le n$ and (a, b) and (c, d) are the nodes of D at which the tableau t^D has the entries $kw_{J(\lambda_D)}$ and $lw_{J(\lambda_D)}$, respectively, and if $kw_{J(\lambda_D)}w_D < lw_{J(\lambda_D)}w_D$, then $a < c$ and $b \le d$.*

Proof. (i) Since e is a prefix of w_D, $t = t^D e$ is a standard D-tableau. Hence, $(a, b)t < (c, d)t$. That is, $(a, b)t^D e < (c, d)t^D e$.

(ii) Since t^D is standard, $kw_{J(\lambda_D)} = (a, b)t^D < (c, d)t^D = lw_{J(\lambda_D)}$. As $kw_{J(\lambda_D)}$ is on a row of t^D of lower index than $lw_{J(\lambda_D)}$ and $w_{J(\lambda_D)}$ just permutes the entries on each row of t^D, $k < l$. Since $t_D = t^D w_D$ is standard and $(a, b)t_D = kw_{J(\lambda_D)}w_D$ and $(c, d)t_D = lw_{J(\lambda_D)}w_D$, $kw_{J(\lambda_D)}w_D < lw_{J(\lambda_D)}w_D$.

(iii) By hypothesis, $(a, b)t^D = kw_{J(\lambda_D)}$ and $(c, d)t^D = lw_{J(\lambda_D)}$. If $a = c$, then since $kw_{J(\lambda_D)}$ and $lw_{J(\lambda_D)}$ are on the same row of t^D and $w_{J(\lambda_D)}$ rearranges the entries on each row of t^D in decreasing order, $lw_{J(\lambda_D)} < kw_{J(\lambda_D)}$. As $t_D = t^D w_D$ is a standard D-tableau, $lw_{J(\lambda_D)}w_D < kw_{J(\lambda_D)}w_D$, contrary to hypothesis. Hence, $a \ne c$.

If $c < a$ then $lw_{J(\lambda_D)} < kw_{J(\lambda_D)}$. Since the entries of t^D are increasing by rows and $w_{J(\lambda_D)}$ only rearranges the entries on each row, $l < k$ contrary to hypothesis. Hence, $a < c$.

If $b > d$ then $kw_{J(\lambda_D)}$ appears in a later column of t^D than $lw_{J(\lambda_D)}$. Hence, $kw_{J(\lambda_D)}w_D > lw_{J(\lambda_D)}w_D$, contrary to hypothesis. So, $b \le d$. $\qquad\square$

8

In view of the work in Schensted [20] and Greene [9], the preceding lemma motivates the following definition of a path in a diagram.

Definition. Let D be a diagram of size n.

(i) A *path* of *length* m in D is a sequence of nodes $((a_i, b_i))_{i=1}^m$ of D such that $a_i < a_{i+1}$ and $b_i \leq b_{i+1}$ for $i = 1, \ldots, m-1$. For $k \in \mathbb{N}$, a *k-path* in D is a sequence of k mutually disjoint paths in D; in particular, a 1-path is a path. The *length* of a k-path is the sum of the lengths of its constituent paths; this is the total number of nodes in the k-path.

(ii) A k-path and a k'-path in D are said to be *equivalent* to one another if they have the same set of nodes.

(iii) We say that D is of *subsequence type* ν, where $\nu = (\nu_1, \ldots, \nu_r) \vdash n$, if the maximum length of a k-path in D is $\nu_1 + \cdots + \nu_k$ whenever $1 \leq k \leq r$. (In particular, D has an r-path containing all its nodes.)

Remark 3.3. Using the notion of *k-increasing subsequence* of the row form of a permutation (see, for example, [19, Definition 3.5.1]), we see from Lemma 3.2 that there is a bijection between the set of k-paths in a diagram D and the set of k-increasing subsequences in $w_{J(\lambda_D)} w_D$, for any positive integer k.

In particular, the increasing subsequences occurring in the row-form of $w_{J(\lambda_D)} w_D$ are precisely the ones which have form $((a_i, b_i) t_D)_{i=1}^m$ for some path $((a_i, b_i))_{i=1}^m$ inside D, where $(a_i, b_i) t_D$ denotes the entry of the tableau t_D at the node (a_i, b_i).

Also note that if $d \in \mathfrak{X}_{J(\lambda)}$ and the tableau $t^D d$ is standard, then the entries of $t^D d$ appearing along paths of D give rise to increasing subsequences in the row-form of $w_{J(\lambda_D)} d$ but in general, if $d \neq w_D$, there are other increasing subsequences in the row-form of $w_{J(\lambda_D)} d$.

Example. The diagram $D = \begin{smallmatrix} \cdot & \cdot & \times & \times & \cdot & \times \\ \cdot & \times & \cdot & \cdot & \times & \times \\ \times & \cdot & \cdot & \cdot & \cdot & \times \\ \cdot & \cdot & \cdot & \times & \cdot & \cdot \end{smallmatrix}$ has maximal paths $((3,1), (4,4))$, $((2,2), (4,4))$, $((2,2), (3,6))$, $((1,3), (4,4))$, $((1,3), (2,5), (3,6))$, $((1,3), (2,6), (3,6))$, $((1,4), (2,5), (3,6))$, $((1,4), (2,6), (3,6))$, $((1,4), (4,4))$, and $((1,6), (2,6), (3,6))$.

In this case, $w_D = [3, 4, 7, 2, 6, 8, 1, 9, 5]$, $\lambda = (3, 3, 2, 1)$ and $w_{J(\lambda)} w_D = [7, 4, 3, 8, 6, 2, 9, 1, 5]$. The corresponding increasing subsequences in $w_{J(\lambda)} w_D$ are $(1, 5)$, $(2, 5)$, $(2, 9)$, $(3, 5)$, $(3, 6, 9)$, $(3, 8, 9)$, $(4, 6, 9)$, $(4, 8, 9)$, $(4, 5)$, and $(7, 8, 9)$, and may be read directly from the tableau $t^D w_D$, as $w_{J(\lambda)} w_D$ maps $t^D w_{J(\lambda)}$ to $t^D w_D$. Thus,

$$
\begin{smallmatrix}
 & 3 & 2 & & 1 & \\
 6 & & & 5 & 4 & \\
8 & & & & 7 & \\
 & 9 & & & &
\end{smallmatrix}
\quad \xrightarrow{\; w_{J(\lambda)} w_D \;} \quad
\begin{smallmatrix}
 & 3 & 4 & & 7 & \\
 2 & & & 6 & 8 & \\
1 & & & & 9 & \\
 & 5 & & & &
\end{smallmatrix}
$$

The tableau $\begin{smallmatrix} & 2 & 3 & & 4 & \\ 1 & & & 6 & 7 & \\ 5 & & & & 8 & \\ & 9 & & & & \end{smallmatrix}$ corresponds to $d = [2, 3, 4, 1, 6, 7, 5, 8, 9]$ and $w_{J(\lambda)} d = [4, 3, 2, 7, 6, 1, 8, 5, 9]$. In addition to the increasing subsequences corresponding to the paths in D, $w_{J(\lambda)} d$ also has increasing subsequences such as $(1, 5)$ and $(2, 7, 8, 9)$ which do not correspond to paths in D.

We will be particularly interested in diagrams D with subsequence type λ'_D, which we call *admissible* diagrams. We make the following observation about paths in such diagrams.

Proposition 3.4. *Let D be a diagram and write $\lambda'_D = (\lambda'_1, \ldots, \lambda'_{r'})$. If $1 \leq u \leq r'$, then a u-path in D of length $\sum_{1 \leq j \leq u} \lambda'_j$ contains all λ_i nodes on the i-th row of D if $\lambda_i \leq u$ and exactly u nodes on all remaining rows.*

If for each u, $1 \leq u \leq r'$, there is a u-path Π_u such that, for all i, Π_u has exactly $\min\{u, \lambda_i\}$ nodes on the i-th row, then D is an admissible diagram.

Proof. Write $\lambda_D = (\lambda_1, \ldots, \lambda_r)$ and let E be a (λ_D, λ'_D)-diagram. Let $T_1 = \{i \colon \lambda_i \geq u\}$ and $T_2 = \{i \colon \lambda_i < u\}$. By counting the nodes in the first u columns of E, we get $u|T_1| + \sum_{i \in T_2} \lambda_i = \sum_{1 \leq j \leq u} \lambda'_j$. Clearly, a u-path in D has at most $\min\{\lambda_i, u\}$ nodes on the i-th row of D, for all i. If the u-path has exactly $\sum_{1 \leq j \leq u} \lambda'_j$ nodes, these inequalities must be exact.

From the first part, we see that each Π_u has length $\sum_{1 \leq j \leq u} \lambda'_j$. Hence, D is an admissible diagram. $\square$

We have the following simple bounds on the subsequence type of a diagram.

Proposition 3.5. *If D is of subsequence type ν then $\mu''_D \trianglelefteq \nu \trianglelefteq \lambda'_D$. In particular, if D is special then $\nu = \lambda'_D$.*

Proof. Let $k \geq 1$. Since each column of D gives a path in D, the k columns in D with the greatest number of nodes give a k-path of length $\mu''_{D,1} + \cdots + \mu''_{D,k}$. Hence, $\mu''_{D,1} + \cdots + \mu''_{D,k} \leq \nu_1 + \cdots + \nu_k$. So, $\mu''_D \trianglelefteq \nu$.

Since the nodes of a path in D are in different rows, there are no more than k nodes of a k-path in any row of D. For a k-path of greatest length we get $\nu_1 + \cdots + \nu_k \leq \lambda'_{D,1} + \cdots + \lambda'_{D,k}$. So, $\nu \trianglelefteq \lambda'_D$.

If D is special then $\lambda'_D = \mu''_D$. Hence, $\nu = \lambda'_D$. $\square$

We now relate the subsequence type of a diagram D to the shape of the Robinson-Schensted tableau of the element $w_{J(\lambda_D)} w_D$, and establish a criterion for this element to be in the right cell of $w_{J(\lambda_D)}$.

Theorem 3.6. *Let D be a diagram of size n and let ν be a partition of n.*
  (i) $\operatorname{sh}(w_{J(\lambda_D)} w_D) = \nu$ *if, and only if, D is of subsequence type ν.*
  (ii) $w_{J(\lambda_D)} w_D \sim_R w_{J(\lambda_D)}$ *if, and only if, D is of subsequence type λ'_D.*

Proof. (i) Let D be of subsequence type ν and let $\operatorname{sh}(w_{J(\lambda_D)} w_D) = \tilde{\nu}$. By Remark 3.3, the maximum total length of a set of k disjoint increasing subsequences in the row-form of $w_{J(\lambda_D)} w_D$ is $\nu_1 + \cdots + \nu_k$, $k \geq 1$. From [19, Theorem 3.5.3], $\operatorname{sh}(w_{J(\lambda_D)} w_D) = \nu$. Hence, $\tilde{\nu} = \nu$.

(ii) From (i), D is of subsequence type λ'_D if, and only if, $\operatorname{sh}(w_{J(\lambda_D)} w_D) = \lambda'_D$. Since $\operatorname{sh}(w_{J(\lambda_D)}) = \lambda'_D$, we get $\operatorname{sh}(w_{J(\lambda_D)} w_D) = \lambda'_D$ if, and only if, $w_{J(\lambda_D)} w_D \sim_{LR} w_{J(\lambda_D)}$. Since $w_{J(\lambda_D)} w_D \leqslant_R w_{J(\lambda_D)}$, we get $w_{J(\lambda_D)} w_D \sim_{LR} w_{J(\lambda_D)}$ if, and only if, $w_{J(\lambda_D)} w_D \sim_R w_{J(\lambda_D)}$, by Lemma 2(iii). $\square$

Corollary 3.7. *If D is a special diagram then $w_{J(\lambda_D)} w_D \sim_R w_{J(\lambda_D)}$.*

Proof. This follows from Proposition 3.5 and Theorem 3.6. $\square$

We can also deduce from Theorem 3.6 that $\nu \trianglelefteq \lambda'_D$, using the fact that $w_{J(\lambda_D)}w_D \leqslant_R w_{J(\lambda_D)}$ and that $\mathrm{sh}\,(x) \trianglelefteq \mathrm{sh}\,(y)$ if $x, y \in S_n$ and $x \leqslant_R y$ by [7, Theorem 5.1] from which it follows that $\nu = \mathrm{sh}\,(w_{J(\lambda_D)}w_D) \trianglelefteq \mathrm{sh}\,(w_{J(\lambda_D)}) = \lambda'_D$.

Note that Theorem 3.6 (ii) is trivially equivalent to "$w_D \in Z(\lambda_D)$ if, and only if, D is of subsequence type λ'_D". In particular, $w_D \in Z(\lambda_D)$ whenever D is special.

Example. Consider $D = $
$$\begin{array}{cccc} & & \times & \times \ \times \\ \times & \times & & \\ & & \times & \end{array}$$
. Then $\mu''_D = (2, 2, 1, 1)$, $\lambda'_D = (3, 2, 1)$ and D is of subsequence type $(3, 1, 1, 1)$. Consequently, $w_D \notin Z(\lambda_D)$ since D is not admissible.

Proposition 3.8. *Let $\lambda \vDash n$ and let $D \in \mathcal{D}^{(\lambda)}$ be an admissible diagram. Then $w_D \notin Y(\lambda)$ if, and only if, there exists an admissible diagram $E \in \mathcal{D}^{(\lambda)}$ such that $w_D \neq w_E$ and $t^E w_D$ is a standard E-tableau.*

Proof. By hypothesis and Theorem 3.6 (ii), $w_D \in Z(\lambda)$.

For the 'if' part: In this case, since $t^E w_D$ is a standard E-tableau, w_D is a prefix of w_E. Moreover, $w_E \in Z(\lambda)$ since E is admissible. Since $w_D \neq w_E$, $w_D \notin Y(\lambda)$.

For the 'only if' part: Since $w_D \notin Y(\lambda)$, there is an $s \in S$ such that $l(w_D s) = l(w_D) + 1$ and $w_D s \in Z(\lambda)$. Let $E = D(w_D s, \lambda)$. Clearly, E is admissible and $w_E = w_D s \neq w_D$. Since w_D is a prefix of w_E, $t^E w_D$ is a standard E-tableau. This completes the proof. $\square$

Example. Consider the diagrams D and E and the standard E-tableau t where
$$D = \begin{array}{ccc} & \times & \\ \times & \times & \\ \times & \times & \times \\ & \times & \end{array} \text{ and } E = \begin{array}{ccc} & \times & \\ \times & \times & \\ \times & & \times \ \times \\ & \times & \end{array} \text{ and } t = \begin{array}{ccc} & & 3 \\ & 1 & 4 \\ 2 & 5 & 7 \\ & 6 & \end{array}.$$
We then get that $w_D \in Z(\lambda_D) \setminus Y(\lambda_D)$ in view of Proposition 3.8 (note that $\lambda_D = \lambda_E$ and both D and E are of subsequence type $(4, 2, 1) = \lambda'_D$).

The *reverse* composition $\dot{\lambda}$ of a composition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the composition $(\lambda_r, \ldots, \lambda_1)$ obtained by reversing the order of the entries. For a principal diagram $D \in \mathcal{D}^{(\lambda)}$, the diagram $\dot{D} \in \mathcal{D}^{(\dot{\lambda})}$ is the diagram obtained by rotating D through $180°$. If $D \in \mathcal{D}^{(\lambda,\mu)}$, then $\dot{D} \in \mathcal{D}^{(\dot{\lambda},\dot{\mu})}$. Since rotating D through $180°$ maps k-paths into k-paths, for any k, D and $\dot{D}$ have the same subsequence type.

Proposition 3.9. *Let $\lambda \vDash n$. Then there is a natural bijection $Y(\lambda) \to Y(\dot{\lambda})$ given by $y \mapsto w_0 y w_0$, where w_0 is the longest element in W, which extends to a bijection $Z(\lambda) \to Z(\dot{\lambda})$ and restricts to a bijection $Y_s(\lambda) \to Y_s(\dot{\lambda})$. Moreover, $\mathfrak{C}(\dot{\lambda}) = w_0 \mathfrak{C}(\lambda) w_0$.*

Proof. Let $y \in Y(\lambda)$ and write $y = w_D$ where $D \in \mathcal{D}^{(\lambda)}$. Then D is admissible by Theorem 3.6(ii). Hence, $\dot{D} \in \mathcal{D}^{(\dot{\lambda})}$ is admissible. Let $\dot{y} = w_{\dot{D}}$. Then $\dot{y} \in Z(\dot{\lambda})$ by Theorem 3.6(ii). Also, $\dot{y} = w_0 y w_0$. If $\dot{y} \notin Y(\dot{\lambda})$, then $\dot{y}$ is a proper prefix of some $z \in Y(\dot{\lambda})$. So, y is a proper prefix of $\dot{z}$, where $\dot{z} = w_0 z w_0$. By the preceding argument, $\dot{z} \in Z(\lambda)$. This contradicts the fact that $y \in Y(\lambda)$. Hence, $\dot{y} \in Y(\dot{\lambda})$.

If w is an arbitrary prefix of y, it is immediate that $w_0 w w_0$ is a prefix of $\dot{y}$. Since $Z(\lambda)$ is the set of all prefixes of elements of $Y(\lambda)$, the second bijection is established. The third bijection comes from the fact that the diagram D is special if, and only if, $\dot{D}$ is special.

Finally, $\mathfrak{C}(\dot{\lambda}) = w_{J(\dot{\lambda})} Z(\dot{\lambda}) = w_0 w_{J(\lambda)} w_0 w_0 Z(\lambda) w_0 = w_0 \mathfrak{C}(\lambda) w_0$. $\square$

Note that if ρ is the representation of S_n corresponding to the cell $\mathfrak{C}(\lambda)$, then the representation corresponding to $\mathfrak{C}(\dot\lambda)$ is given by $s_i \mapsto s_{n-i}\rho$ for all i.

When λ is a partition we have seen in Remark 3.1 that $Y(\lambda) = \{w_D\}$ where D is the Young diagram corresponding to λ. It then follows from Proposition 3.9 that in the case of a composition μ such that $\dot\mu$ is a partition, we have $Y(\mu) = \{w_0 w_E w_0\}$, where E is the Young diagram corresponding to $\dot\mu$.

However, determining $Y(\lambda)$ for an arbitrary composition λ turns out to be much more complicated in general. We conclude this section with two propositions in which $Y(\lambda)$ is determined for two families of compositions. We will first deal with compositions which are rearrangements of hook partitions and in this case special diagrams will turn out to play an important part. Recall that if λ, μ are compositions of n with $\mu'' = \lambda'$, then there is a unique diagram D with $\lambda_D = \lambda$ and $\mu_D = \mu$ (such a diagram D is special).

Proposition 3.10. *Let $\lambda \vDash n$ and suppose λ is a rearrangement of the hook partition $m^1 1^{r-1}$ where $m > 1$ and $r \geq 3$. Suppose further that none of λ or $\dot\lambda$ is a partition. Then $Y(\lambda) = \{w_D : D \in \mathcal{D}^{(\lambda)} \text{ and } \mu_D'' = \lambda'\}$. In particular, $Y(\lambda) = Y_s(\lambda)$ and $|Y(\lambda)| = m$.*

Proof. Assume the hypothesis. Then $\lambda = (\lambda_1, \ldots, \lambda_r)$ where $\lambda_k = m$ for some k with $1 < k < r$ and $\lambda_i = 1$ for all $i \in \{1, \ldots, r\} \setminus \{k\}$. Let $y \in Y(\lambda)$ and let $D = D(y, \lambda)$. It follows that D has a path Π of length r which necessarily has a node on each row of D. Let (i, b_i), $1 \leq i \leq r$, be the nodes of Π. From the definition of path, $b_i \leq b_{i+1}$ for $1 \leq i \leq r - 1$. Now suppose that $b_l < b_{l+1}$ for some $l \in \{1, \ldots, r-1\}$. If $l \geq k$ (resp. $l < k$) consider the diagram E obtained from D by removing the node in position $(l+1, b_{l+1})$, (resp. (l, b_l)) and introducing a node in position $(l+1, b_l)$, (resp. (l, b_{l+1})). Clearly, $\lambda_E = \lambda$. Since all nodes of D which are not on the k-th row are on the path Π, $t^E w_D$ is standard. The subsequence type ν of E has r as its first entry and satisfies $\nu \trianglelefteq \lambda'$ by Proposition 3.5. Hence, $\nu = \lambda'$ (that is, E is admissible). Clearly, $w_E \neq w_D$ so by Proposition 3.8, $w_D \notin Y(\lambda)$, contrary to hypothesis. Hence, $b_i = b_{i+1}$ for $1 \leq i \leq r - 1$. We conclude that D is a special diagram.

For each λ there are m $(= n - r + 1)$ principal special diagrams. Moreover, our hypothesis that $1 < k < r$ ensures that the corresponding w_D are not prefixes of one another by Result 7. The required result now follows easily. $\square$

We remark here that Proposition 3.10 follows from [17, Proposition 5.2]. However, its proof is done in the spirit of the techniques developed in this paper.

Next we deal with a family of compositions $\lambda = \lambda(r)$ for which, as it turns out, $Y(\lambda) \neq Y_s(\lambda)$ for $r \geq 4$.

Proposition 3.11. *Let $r \geq 3$ and let $\lambda = (2, 1^{r-2}, 2) \vDash n = r + 2$. Then $Y(\lambda) = \{w_{D(a)} : 2 \leq a \leq r\}$ where, for $2 \leq a \leq r$, $D(a)$ is the diagram defined by $D(a) = \{(i, 1) : 1 \leq i \leq a-1\} \cup \{(i, 2) : a \leq i \leq r\} \cup \{(r, 1), (1, 2)\}$.*

Proof. Let λ, r, a and $D(a)$ be as in the statement of the proposition.

$$D(a) \;\; = \;\; \begin{matrix} \times & \times \\ \vdots & \\ \times & \\ & \times \\ & \vdots \\ \times & \times \end{matrix}$$

It is easy to observe from Result 7 that for i, j with $2 \leq i, j \leq r$ and $i \neq j$, we have that $w_{D(i)}$ cannot be a prefix of $w_{D(j)}$. It is thus sufficient to show that any element of $Z(\lambda)$ is a prefix of $w_{D(a)}$ for some a with $2 \leq a \leq r$.

Let $z \in Z(\lambda)$ be given and set $D = D(z, \lambda)$. Then D is admissible by Theorem 3.6. Suppose $D = \{(i, j_i) : 1 \leq i \leq r\} \cup \{(i, j_1'), (r, j_r')\}$ where $j_1 < j_1'$ and $j_r < j_r'$. Since D is of subsequence type $\lambda' = (r, 2)$, it has a path Π of length r and two disjoint paths Π_1 and Π_2 which contain all nodes of D. Necessarily, Π contains the nodes (i, j_i) with $2 \leq i \leq r - 1$ and precisely one node from each of the remaining rows. It follows that $j_1 \leq j_2 \leq \ldots \leq j_{r-1} \leq j_r'$. Moreover, we must have that Π_1 contains precisely one node from each of the first and last rows of D while Π_2 contains the remaining two nodes from these two rows. This forces $j_1 \leq j_r$ and $j_1' \leq j_r'$.

If $j_1' \leq j_2$, it is then an easy consequence of Result 7 that $z \, (= w_D)$ is a prefix of $w_{D(2)}$. Similarly, if $j_{r-1} \leq j_r$ we can see that z is a prefix of $w_{D(r)}$.

It remains to consider the case when $j_1' > j_2$ and $j_{r-1} > j_r$. First, we show that if i satisfies $2 \leq i \leq r - 1$, then either $j_i \leq j_r$ or $j_1' \leq j_i$. Suppose, on the contrary, that $j_i > j_r$ and $j_1' > j_i$ for some i with $2 \leq i \leq r - 1$. Then $j_1' > j_i > j_r$ for some i with $2 \leq i \leq r - 1$. Hence the nodes $(1, j_1')$ and (r, j_r) cannot both belong to the same one of the paths Π_1 or Π_2 described above. It follows that $(1, j_1)$ and (r, j_r) belong to the same path, say Π_1, and $(1, j_1'), (r, j_r')$ belong to Π_2. But $(i, j_i) \notin \Pi_1$ since $j_i > j_r$ and $(i, j_i) \notin \Pi_2$ since $j_i < j_1'$. This contradicts the fact that $D = \Pi_1 \cup \Pi_2$ and shows that either $j_i \leq j_r$ or $j_1' \leq j_i$ if $2 \leq i \leq r - 1$. In particular, taking $i = r - 1$, we get $j_1' \leq j_{r-1}$.

Now let a be the smallest integer with $2 \leq a \leq r$ such that $j_a \geq j_1'$. Note that $2 < a < r$ since $j_1' > j_2$ by our assumption, and $j_1' \leq j_{r-1}$, as we have seen already. It follows that $j_{a-1} < j_1'$ and hence $j_{a-1} \leq j_r$ in view of our observation above. This ensures that $\Pi_1' = \{(1, j_1), (2, j_2), \ldots, (a - 1, j_{a-1}), (r, j_r)\}$ and $\Pi_2' = \{(1, j_1'), (a, j_a), \ldots, (r, j_r')\}$ are both paths in D and hence the tableau $t^{D(a)}z$ is standard. Invoking Result 7, we conclude that z is a prefix of $w_{D(a)}$. $\qquad\square$

4 Lifting cells

In this section, we examine how the cell associated with a composition λ of n may be related to the cells associated with certain related compositions of $n + 1$.

4.1 A process relating to the induction of cells

The *lower star* composition λ_* of a composition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n is the composition $(\lambda_1, \ldots, \lambda_r, 1)$ of $n + 1$.

If $\lambda = (1, 2, 1, 2)$, then $Y(\lambda) = \{y_1, y_2\}$, where $y_1 = [3, 1, 4, 5, 2, 6]$ and $y_2 = [1, 2, 5, 3, 4, 6]$, and $Y(\lambda_*) = \{y_1', y_2', y_3'\}$, where $y_1' = [1, 2, 6, 3, 4, 7, 5]$, $y_2' = [3, 1, 4, 5, 2, 6, 7]$, and $y_3' = [2, 1, 3, 4, 5, 7, 6]$.

Extending the diagram underlying y_1 to an admissible diagram in $\mathcal{D}^{(1,2,1,2,1)}$ by placing a node in the fifth row requires the node to be in the second column (or later), whereas the diagram underlying y_2 can be so extended by placing the node anywhere in the fifth row.

$$
\begin{array}{ccccc}
\begin{array}{ll} & 3 \\ 1 & 4 \\ & 5 \\ 2 & 6 \end{array}
&
\begin{array}{ll} 1 & \\ 2 & 5 \\ 3 & \\ 4 & 6 \end{array}
&
\begin{array}{ll} 1 & \\ 2 & 6 \\ 3 & \\ 4 & 7 \\ 5 & \end{array}
&
\begin{array}{ll} & 3 \\ 1 & 4 \\ & 5 \\ 2 & 6 \\ & 7 \end{array}
&
\begin{array}{ll} & 2 \\ 1 & 3 \\ & 4 \\ & 5 \ \ 7 \\ & 6 \end{array}
\\[2em]
y_1 & y_2 & y_1' & y_2' & y_3'
\end{array}
$$

Table 1: The tableaux of the elements of $Y(\lambda)$ and $Y(\lambda_*)$, $\lambda = (1,2,1,2)$.

Definition. Let $\lambda \vDash n$ and let $D \in \mathcal{D}^{(\lambda)}$. For $1 \leq j \leq n$, let u_j be the node u of D for which $ut_D = j$ (so we have $(u_j)t_D = j$). Suppose further that $u_j = (a_j, b_j)$ for $1 \leq j \leq n$.

(i) Define $D_0' \in \mathcal{D}^{(\lambda_*)}$ by $D_0' = \{(i, j+1): \ (i,j) \in D\} \cup \{(r_D + 1, 1)\}$.

(ii) For $1 \leq i \leq c_D$, define $D_i' \in \mathcal{D}^{(\lambda_*)}$ by $D_i' = \{(a,b): \ (a,b) \in D\} \cup \{(r_D + 1, i)\}$.

(iii) For each i, with $1 \leq i \leq n$, define $D'(u_i) \in \mathcal{D}^{(\lambda_*)}$ as follows:
(a) If $b_{i+1} > b_i$, then $D'(u_i) = D \cup \{(r_D + 1, b_i)\}$.
(b) If $b_{i+1} = b_i$, then $D'(u_i) = \{u_j \in D: \ j \leq i\} \cup \{(r_D + 1, b_i)\} \cup \{(a_j, b_j + 1): (a_j, b_j) = u_j \in D \text{ and } j > i\}$.

Informally, for item (iii), if $u_i = (a_i, b_i)$ is the last node in its column, $D'(u_i)$ is obtained from D by introducing a node in the same column as u_i (column b_i) in a row immediately after the last row of D. Otherwise, $D'(u_i)$ is obtained by inserting a new column to D immediately after column b_i and moving into this new column all nodes in column b_i below node u_i while introducing an extra node in column b_i in a row immediately after the last row of D.

Now let $D \in \mathcal{D}^{(\lambda)}$ where $\lambda \vDash n$. It is clear from the above definition that D_0' is not admissible even in the case D is assumed to be admissible. On the other hand, if D is admissible then $D'(u_n)$ is also admissible. This is because any k-path of length l in D can be converted to a k-path of length $l+1$ in $D'(u_n)$ by just adding to it the single node in the last row of $D'(u_n)$. It is also easy to observe that for j, with $1 \leq j \leq c_D$, we have that $D_j' = D'(u_t)$ where u_t is the last node in column j of D; in particular if $s = c_D$ then $D_s' = D'(u_n)$.

An immediate consequence of the above is that whenever D is an admissible (principal) diagram of size n then both sets $\{i: \ 1 \leq i \leq n \text{ and } D'(u_i) \text{ is admissible}\}$ and $\{j: \ 1 \leq j \leq c_D \text{ and } D_j' \text{ is admissible}\}$ are nonempty. This leads to the following definition.

Definition. Let D be an admissible (principal) diagram of size n. We set $p(D) = \min\{i: \ 1 \leq i \leq n \text{ and } D'(u_i) \text{ is admissible}\}$ and $q(D) = \min\{j: \ 1 \leq j \leq c_D \text{ and } D_j' \text{ is admissible}\}$.

Remark 4.1. Let $\lambda \vDash n$ and let $D \in \mathcal{D}^{(\lambda)}$ be admissible.

(i) It then follows that $D'(u_i)$ is admissible for $i \geq p(D)$ and $D'(u_i)$ is not admissible for $i < p(D)$. This is because for $j < l$, with $1 \leq j, l \leq n$, we have that for any k-path in $D'(u_j)$ there is a 'corresponding' k-path of the same length in $D'(u_l)$.

(ii) Fix i with $1 \leq i \leq n$ and set $D' = D'(u_i)$. An easy computation shows that $w_{D'} = w_D(i+1, i+2, \ldots, n+1) = w_D s_n s_{n-1} \cdots s_{i+1}$. Hence, for $j > i$ we have, on setting $E' = D'(u_j)$, that $w_{E'}$ is a prefix of $w_{D'}$ which is in agreement with the above observation that $w_{E'} \in Z(\lambda_*)$ whenever $w_{D'} \in Z(\lambda_*)$.

14

On the other hand, if we set $D' = D'_0$ it is also easy to observe that $w_{D'} = w_D s_n s_{n-1} \ldots s_2 s_1$. We can deduce that $\mathfrak{X}' = \{w_D^{-1} w_{D'} : D' = D'_0 \text{ or } D' = D'(u_i) \text{ for some } i \text{ with } 1 \leq i \leq n\}$.

Lemma 4.2. *Let $\lambda \vDash n$ and let $D \in \mathcal{D}^{(\lambda)}$ be an admissible (principal) diagram. Also let $p = p(D)$ and $q = q(D)$. We then have that u_p is the last node in column q of D (and hence $D'(u_p) = D'_q$).*

Proof. We assume the hypothesis and suppose that $\lambda' = (\lambda'_1, \ldots, \lambda'_{r'})$. For each k with $1 \leq k \leq r'$, let $\mathfrak{P}_k$ be the set of k-paths in D of length $\lambda'_1 + \ldots + \lambda'_k$ (recall that $\mathfrak{P}_k$ is nonempty for $1 \leq k \leq r'$ since D is admissible). For each $\Pi \in \mathfrak{P}_k$, let $m_\Pi = \min\{i :$ there exists a path in Π that terminates at $u_i\}$. Also let $m_k(D) = \min\{m_\Pi : \Pi \in \mathfrak{P}_k\}$.

For the moment, we fix k. We write $l = m_k(D)$. We claim that u_l is the last node in its column. This is immediate for $k = 1$, since u_l is at the end of a path of length λ'_1 in this case. So we need only consider $1 < k \leq r'$. Also observe that what is stated in the claim is trivially true when $l = n$.

Assume now that $l < n$ and that u_l and u_{l+1} are in the same column of D (that is, u_l is not the last node in its column) and let Π' be a k-path in D of length $\lambda'_1 + \ldots + \lambda'_k$ such that $m_{\Pi'} = l$. Let Π_1 be a path in Π' that terminates at u_l and let l_1 be the length of Π_1. Then $\Pi'_1 = \Pi_1 \cup \{u_{l+1}\}$ is a path in D of length $l_1 + 1$. If $u_{l+1} \notin \Pi'$ then $(\Pi' \setminus \Pi_1) \cup \Pi'_1$ forms a k-path in D of length $\lambda'_1 + \ldots + \lambda'_k + 1$. This is impossible since D is of subsequence type $(\lambda'_1, \ldots, \lambda'_{r'})$. Hence, $u_{l+1} \in \Pi'$.

Since u_l is the last node in path Π_1 of Π', u_{l+1} is in a different path Π_2 ($\neq \Pi_1$) of Π'. Let $\overline{\Pi}_2 = \{u_i \in \Pi_2 : i \geq l + 1\}$. Then $\Pi_1 \cup \overline{\Pi}_2$ forms a path in D and its nodes are in Π'. If $\Pi_2 \subseteq \overline{\Pi}_2$, it would then follow that Π' is equivalent to a $(k-1)$-path of length $\lambda'_1 + \ldots + \lambda'_k$ contrary to the admissibility of D since $\lambda'_k \geq 1$. Hence $\Pi_2 \setminus \overline{\Pi}_2 \neq \varnothing$ and $\Pi_2 \setminus \overline{\Pi}_2$ is a path terminating at node $u_{l'}$ where $l' < l$. Considering the decomposition $(\Pi' \setminus (\Pi_1 \cup \Pi_2)) \cup (\Pi_1 \cup \overline{\Pi}_2) \cup (\Pi_2 \setminus \overline{\Pi}_2)$ we see that Π' is equivalent to a k-path Π'' in D where $m_{\Pi''} \leq l' < l$. This contradicts the choice of l and establishes that u_l and u_{l+1} are in different columns of D. In particular, u_l is the last node in its column.

Now we define $m(D) = \max\{m_k(D) : 1 \leq k \leq r'\}$ and let $m = m(D)$. It is clear from this definition that u_m is the last node in its column. Moreover, the above construction ensures that $D'(u_i)$ is admissible for $i \geq m$ and $D'(u_i)$ is not admissible for $i < m$. We conclude that $m(D) = p(D)$. The required result now follows easily. $\qquad \square$

We are now ready to state the main result of this subsection.

Theorem 4.3. *Let $\lambda \vDash n$. For $z \in Z(\lambda)$ set $D = D(z, \lambda)$ and $D' = D'_q$, where $q = q(D)$. Then there is a well-defined injective map θ_* from $Z(\lambda)$ to $Z(\lambda_*)$ in which $z \mapsto z'$, where $z' = w_{D'}$. Moreover, θ_* has the following properties.*

(i) For $z \in Z(\lambda)$ we have $z^{-1}(z\theta_*) \in \mathfrak{X}'$ and whenever $zx \in Z(\lambda_*)$ with $x \in \mathfrak{X}'$ we have that x is a prefix of $z^{-1}(z\theta_*)$.

(ii) $Y(\lambda)\theta_* \subseteq Y(\lambda_*) \subseteq Z(\lambda)\theta_*$.

Proof. It is clear that the map θ_* from $Z(\lambda)$ to $Z(\lambda_*)$ described in the statement of the theorem is well-defined. Moreover, with z, D, D' and z' as above, it is easy to observe using Result 9 that diagram D' is in fact exactly the same as diagram $D(z', \lambda_*)$. Suppose now that $z_1, z_2 \in Z(\lambda)$ satisfy $z_1\theta_* = z' = z_2\theta_*$ for some z' in $Z(\lambda_*)$. Now z'

together with λ_* determine uniquely diagram D' (this is because $D' = D(z', \lambda_*)$). But the construction via which θ_* is defined ensures that, by removing the single node in its last row, D' determines diagrams $D(z_1, \lambda)$ and $D(z_2, \lambda)$ uniquely, forcing them to be equal. The equality $D(z_1, \lambda) = D(z_2, \lambda)$ ensures that $z_1 = z_2$ and this is enough to establish that θ_* is injective.

For item (i), let $z \in Z(\lambda)$ and let $D = D(z, \lambda)$. Then $z\theta_* = w_{D'}$ where $D' = D'_q$ ($q = q(D)$) by the definition of θ_*. We know from Lemma 4.2 that $D'_q = D'(u_p)$ where $p = p(D)$. It is now immediate from Remark 4.1 that any $x \in \mathfrak{X}'$ which satisfies $zx \in Z(\lambda_*)$ must be of the form $z^{-1}w_E$ where $E = D'(u_j)$ for some j with $j \geq p$ and, in addition, that such an x must be a prefix of $z^{-1}(z\theta_*)$.

In order to establish the second inclusion of sets in item (ii), suppose $\overline{y} \in Y(\lambda_*)$ and consider diagram $C = D(\overline{y}, \lambda_*)$. Then $C \in \mathcal{D}^{(\lambda_*)}$ and C is an admissible diagram. Now let $B \in \mathcal{D}^{(\lambda)}$ be the diagram obtained from C by removing the single node in its last row. Let ν be the subsequence type of B. Then $\nu \trianglelefteq \lambda' = (\lambda'_1, \ldots, \lambda'_{r'})$ by Proposition 3.5. As C is of subsequence type $(\lambda'_1 + 1, \lambda'_2 \ldots, \lambda'_{r'}) = \lambda'_*$, if $1 \leq k \leq r'$ then any k-path of length $\lambda'_1 + \cdots + \lambda'_k + 1$ in C gives a k-path of length $\geq \lambda'_1 + \cdots + \lambda'_k$ in B. So $\lambda' \trianglelefteq \nu$. Thus B is admissible.

Now let $F = D(w_B, \lambda)$. Then $w_F = w_B \in Z(\lambda)$ as B is admissible. From the construction of B, either $F = B$ or F is obtained from B by merging two adjacent columns (see Result 9). Since C is admissible and $C = D(\overline{y}, \lambda_*)$, we have $C = F'(u_i)$ for some i with $p(F) \leq i \leq n$ (see Remark 4.1). Since $\overline{y} \in Y(\lambda_*)$, $i = p(F)$ again by Remark 4.1. Hence $C = F'_{q(F)}$ by Lemma 4.2. We deduce that $\overline{y} (= w_C) = w_F\theta_* \in Z(\lambda)\theta_*$ as required.

Finally we want to establish that $Y(\lambda)\theta_* \subseteq Y(\lambda_*)$. For this, let $y \in Y(\lambda)$ and suppose that $y\theta_* \notin Y(\lambda_*)$. Let $y' = y\theta_*$. Also let $D = D(y, \lambda)$ and $D' = D(y', \lambda_*)$. Then $y' = w_{D'}$ and $D' = D'(u_p) = D'_q$ where $p = p(D)$ and $q = q(D)$. It follows that $(n+1)y' = p+1$. Since $y' \notin Y(\lambda_*)$, by Proposition 3.8 there exists an admissible (principal) diagram $E \in \mathcal{D}^{(\lambda_*)}$ such that $t^E y'$ is standard and $w_E \neq w_{D'}$. We can assume without loss of generality that $E = D(w_E, \lambda_*)$ since, if $E' = D(w_E, \lambda_*)$, then $w_E = w_{E'}$ and $t^{E'} y'$ is standard from Result 9. Now let F be the diagram obtained from E by removing the single node on its last row. Then F is admissible by similar argument as above. Next, we would like to compare the tableaux $t^E y'$ and $t^F y$. For this, define subsets T_1 and T_2 of $\{1, \ldots, n\}$ as follows : $T_1 = \{i : 1 \leq i \leq n$ and $iy' < p+1\}$ and $T_2 = \{i : 1 \leq i \leq n$ and $iy' > p+1\}$. We then have $iy = iy'$ for $i \in T_1$ and $iy = (iy') - 1$ for $i \in T_2$. Thus, tableau $t^F y$ is obtained from $t^E y'$ by removing the single entry in the last row of $t^E y'$ and replacing each number $l > p+1$ by $l-1$. It follows that $t^F y$ is standard since $t^E y'$ is standard. By Result 7, y is a prefix of w_F. Since $w_F \in Z(\lambda)$ as F is admissible and $y \in Y(\lambda)$, we get $w_F = y (= w_D)$. At this point it is also useful to observe that the single node in the last row of $E (= D(w_E, \lambda_*))$ cannot be the sole node of E in its column. For this, let the single node in row $r_E + 1$ of E be in column j. That this node cannot be the sole node in its column follows in the case $j = 1$ from Remark 4.1 since E is admissible, and in the case $j > 1$ from the fact that $E = D(w_E, \lambda_*)$.

Combining the above observations with Result 9 we see that either $F = D$ or F can be obtained from D by splitting a column of D into two successive columns so that the nodes in the column with lesser column index have row indices which are less than the row indices of all the nodes in the column with greater column index.

We conclude that $E = D'(u_a)$ for some a. Since E is admissible, $a \geq p$. Since $D' = D'(u_p)$

and $E \neq D'$, $a \neq p$. It follows from this that the entry $p + 1$ which appears in the last row of $t^E y'$ must lie in a column which includes entries greater than $p + 1$ since $p + 1$ is not the sole entry in its column. This contradicts the fact that $t^E y'$ is standard. Hence $y' \in Y(\lambda_*)$. $\qquad \square$

We illustrate Theorem 4.3 with the following example. Consider $\lambda = (2, 1, 5)$. Then $\lambda_* = (2, 1, 5, 1)$. $Y(\lambda) = \{y_1, y_2\}$, where $y_1 = (1, 4)(2, 6, 3, 7, 5)$ and $y_2 = (1, 4)(2, 7, 6, 3, 5)$, and $Y(\lambda_*) = \{y_1', y_2', y_3', y_4', y_5'\}$, where $y_1' = (1, 4)(2, 6, 3, 7, 5)$, $y_2' = (1, 4)(2, 8, 9, 7, 6, 3, 5)$, $y_3' = (1, 3, 4)(2, 8, 9, 6, 5)$, $y_4' = (1, 2, 8, 9, 5, 4)$, and $y_5' = (2, 8, 9, 4, 3)$. Putting $x_1 = 1$ and $x_2 = (7, 8, 9) = s_8 s_7$, we see that $y_1' = y_1 x_1$, $y_2' = y_2 x_2$, and $x_1, x_2 \in \mathfrak{X}'$.

Remark 4.4. (i) If in the proof for the second inclusion of sets in item (ii) of Theorem 4.3 we assume in place of "$\overline{y} \in Y(\lambda_*)$" that "$\overline{y} \in Z(\lambda_*)$" instead, the proof goes through without any change up to the point where we deduce that there exists i with $p(F) \leq i \leq n$ such that $C = F'(u_i)$. It follows from this that $\overline{y} = w_C = w_F s_n \ldots s_{i+1} = z s_n \ldots s_{i+1}$ where $z \in Z(\lambda)$ (see Remark 4.1). Since $w_{J(\lambda_*)} = w_{J(\lambda)}$, we deduce that $\mathfrak{C}(\lambda_*) \subseteq \mathfrak{C}(\lambda)\mathfrak{X}'$ (in fact this last observation follows easily from Result 3 or Proposition 2.1). Thus the map θ_* of the preceding theorem is compatible with the induction of cells. Moreover, the map θ_* enables us to obtain some additional information about the induction process for the particular cells involved ($\mathfrak{C}(\lambda)$ and $\mathfrak{C}(\lambda_*)$) by relating their rims. In fact, knowledge of $Z(\lambda)$ enables us to determine the rim of $\mathfrak{C}(\lambda_*)$ and hence obtain reduced forms for all the elements of $\mathfrak{C}(\lambda_*)$. The disadvantage of the above process is that the map θ_* is difficult to construct.

(ii) In view of Result 3, a consequence of the above is that the cell module corresponding to $\mathfrak{C}(\lambda_*)$ occurs as a constituent of the cell module corresponding to $\mathfrak{C}(\lambda)$ induced up to S_{n+1}. This also agrees with the Branching Theorem (see [12, Theorem 9.2(i)]); this last observation follows once we combine the above with [16, Theorem 3.5] or [17, Theorem 4.6].

Finally for this subsection, we consider a dual construction to that of the composition λ_*. The *upper star* composition λ^* is the composition of $n + 1$ formed by prepending a new part 1 to the composition λ. Thus, $\lambda^* = \dot{\mu}$ where $\mu = (\dot{\lambda})_*$. It is easy to see that, combining Theorem 4.3 with Proposition 3.9, we get the following corollary.

Corollary 4.5. *Let λ be a composition of n. Then there is a natural injective mapping $\theta^*: Z(\lambda) \to Z(\lambda^*)$ such that $z\theta^* \in (w_{S'} w_S z w_S w_{S'}) w_{S'} \mathfrak{X}' w_{S'}$ for every $z \in Z(\lambda)$. Moreover, $Y(\lambda)\theta^* \subseteq Y(\lambda^*)$.*

If μ is the composition defined before the statement of the theorem, then θ^* is obtained by composing the bijection of Proposition 3.9, the injection of Theorem 4.3 for $\dot{\lambda}$ and the bijection of Proposition 3.9 for μ.

4.2   A process relating to the restriction of cells

We now consider a different type of 'extension' of a composition.

Definition. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be an r-part composition of n.

Table 2: $\mathcal{E}^{(1,2,2,1)}$.

```
  ×              ×              ×
× ×            × ×            × ×
× ×            × ×              × ×
×              ×              ×
```

Table 3: $\mathcal{E}^{(1,3,2,1)}$.

```
×          ×          ×            ×          ×
× × ×      × × ×      × ×   ×      × × ×      × × ×
× ×        × ×        × ×          ×   ×      × ×
×          ×          ×            ×          ×
```

Table 4: $\mathcal{E}^{(1,2,3,1)}$.

```
  ×          ×            ×          ×            ×
× ×        × ×          × ×        ×   ×        × ×
× × ×      × × ×      ×   × ×      × × ×        × × ×
  ×          ×            ×          ×            ×
```

(i) Define $\max \lambda = \max\{\lambda_i \colon 1 \le i \le r\}$ and $M(\lambda) = \{j \colon 1 \le j \le r$ and $\lambda_j = \max \lambda\}$. For $k \in M(\lambda)$, define the r-part composition $\lambda^{(k)}$ of $n+1$ by $\lambda_k^{(k)} = \lambda_k + 1$ and $\lambda_i^{(k)} = \lambda_i$ if $i \ne k$.

(ii) For D a (principal) diagram with $\lambda_D = \lambda$ and $k \in M(\lambda)$, define a new diagram $D^{(k)}$ by $D^{(k)} = D \cup \{(k, c_D + 1)\}$. Then $D^{(k)}$ is a principal diagram with $\lambda_{D^{(k)}} = \lambda^{(k)}$. Informally, to construct $D^{(k)}$ a new column is inserted in D immediately after its last column and a single node is inserted in this new column on the k-th row.

Example. If $\lambda = (1,2,2,1)$, then $Y(\lambda) = \{y_1, y_2, y_3\}$, where $y_1 = [1,\,2,\,5,\,3,\,6,\,4]$, $y_2 = [3,\,1,\,4,\,2,\,5,\,6]$, and $y_3 = [2,\,1,\,3,\,4,\,6,\,5]$. Also, $Y_s(\lambda) = \{y_1, y_2\}$. We display $\mathcal{E}^{(\lambda)}$ in Table 2.

In this case, $M(\lambda) = \{2,3\}$. $\lambda^{(2)} = (1,3,2,1)$ and $\lambda^{(3)} = (1,2,3,1)$. Moreover, $Y(\lambda^{(2)}) = \{y_1^{(2)}, y_2^{(2)}, y_3^{(2)}, y_4^{(2)}, y_5^{(2)}\}$ and $Y(\lambda^{(3)}) = \{y_1^{(3)}, y_2^{(3)}, y_3^{(3)}, y_4^{(3)}, y_5^{(3)}\}$, where $y_1^{(2)} = [1,\,2,\,5,\,7,\,3,\,6,\,4]$, $y_2^{(2)} = [3,\,1,\,4,\,7,\,2,\,5,\,6]$, $y_3^{(2)} = [2,\,1,\,3,\,7,\,4,\,6,\,5]$, $y_4^{(2)} = [4,\,1,\,3,\,5,\,2,\,6,\,7]$, and $y_5^{(2)} = [3,\,1,\,2,\,4,\,5,\,7,\,6]$, and $y_1^{(3)} = [4,\,2,\,5,\,1,\,3,\,6,\,7]$, $y_2^{(3)} = [2,\,3,\,6,\,1,\,4,\,7,\,5]$, $y_3^{(3)} = [3,\,2,\,4,\,1,\,5,\,7,\,6]$, $y_4^{(3)} = [1,\,2,\,6,\,3,\,5,\,7,\,4]$, $y_5^{(3)} = [2,\,1,\,3,\,4,\,6,\,7,\,5]$, Also, $Y_s(\lambda^{(2)}) = \{y_1^{(2)}, y_2^{(2)}, y_4^{(2)}\}$ and $Y_s(\lambda^{(3)}) = \{y_1^{(3)}, y_2^{(3)}, y_4^{(3)}\}$. We display $\mathcal{E}^{(\lambda^{(2)})}$ and $\mathcal{E}^{(\lambda^{(3)})}$ in Table 3 and Table 4, respectively.

Observe that $\{D^{(2)} : D \in \mathcal{E}^{(1,2,2,1)}\} \subseteq \mathcal{E}^{(1,3,2,1)}$. Note also that there are various injections $Y(\lambda) \to Y(\lambda^{(2)})$ given by $y_1 \mapsto y_1^{(2)}$, $y_2 \mapsto y_2^{(2)}$ or $y_4^{(2)}$, and $y_3 \mapsto y_3^{(2)}$ or $y_5^{(2)}$. These injections induce injections $Y_s(\lambda) \to Y_s(\lambda^{(2)})$. There is a similar situation for $\lambda^{(3)}$.

While the aim is to show that the situation in the preceding example generalizes, the following lemmas establish a complementary result.

Lemma 4.6. *Let $\lambda \vDash n$ and $k \in M(\lambda)$. Let $B, C \in \mathcal{D}^{(\lambda)}$ and let B be an admissible diagram. Let $\overline{B} = B^{(k)}$ and $\overline{C} = C^{(k)}$. Then,*
(i) $\overline{B}$ is an admissible diagram.
(ii) If $w_C = w_B$ then $w_{\overline{B}} = w_{\overline{C}}$.

Proof. (i) Let ν be the subsequence type of $\overline{B}$ and let $\overline{\lambda} = \lambda^{(k)}$. Since $\lambda_{\overline{B}} = \overline{\lambda}$ we have $\nu \trianglelefteq \overline{\lambda}'$ by Proposition 3.5. Write $\lambda' = (\lambda_1', \ldots, \lambda_{r'}')$. Since B is admissible, we see from the construction of $\overline{B}$ that $(\lambda_1', \ldots, \lambda_{r'}', 1) \trianglelefteq \nu$. The assumption $k \in M(\lambda)$ ensures that $\overline{\lambda}' = (\lambda_1', \ldots, \lambda_{r'}', 1)$, hence $\overline{\lambda}' \trianglelefteq \nu$. We conclude that $\nu = \overline{\lambda}'$ and $\overline{B}$ is admissible.

(ii) is immediate. $\qquad\square$

```
× ×          × ×          × ×
×            ×            ×
×            ×              ×
× ×          × ×          × ×
```

```
× × ×        × × ×        × × ×
×            ×            ×
×            ×              ×
× ×          × ×          × ×
```

```
× ×          × ×          × ×
×            ×            ×
×              ×            ×
× × ×        × × ×        × × ×
```

Lemma 4.7. *Let $\lambda \vDash n$, $z \in Z(\lambda)$, $D = D(z,\lambda)$ and $k \in M(\lambda)$. Also let $\overline{\lambda} = \lambda^{(k)}$, $\overline{D} = D^{(k)}$ and $\overline{z} = w_{\overline{D}}$. We then have*

(i) $\overline{D}$ is an admissible diagram and $\overline{z} \in Z(\overline{\lambda})$. (Hence, there is a well-defined map θ from $Z(\lambda)$ to $Z(\overline{\lambda})$ in which $z \mapsto \overline{z}$.)

(ii) If $z \notin Y(\lambda)$ then $\overline{z} \notin Y(\overline{\lambda})$.

Proof. Assume the hypothesis.

(i) We have $z = w_D$ and $z \in Z(\lambda)$ so D is admissible by Theorem 3.6. Invoking Lemma 4.6(i) we see that $\overline{D}$ is also admissible and hence $\overline{z} = w_{\overline{D}} \in Z(\overline{\lambda})$.

(ii) Suppose now that $z \notin Y(\lambda)$. Since $z \in Z(\lambda)$, $z = w_D$, $\lambda_D = \lambda$ and D is admissible, we get from Proposition 3.8 that there is an admissible diagram $E \in \mathcal{D}^{(\lambda)}$ such that $w_E \neq w_D$ and $t^E w_D$ is a standard E-tableau. The fact that $w_E \neq w_D$ ensures that $t = t^E w_D \neq t_E$. Let $F = E^{(k)}$. Then $\lambda_F = \overline{\lambda}$ and F is admissible by Lemma 4.6(i). Now let $t' = t^F w_{\overline{D}}$. Then $(u)t' = (u)t$ for all $u \in E$ and $(k, c_E + 1)t' = n + 1$ so t' is a standard F-tableau. It is also easy to see from this construction that $(u)t_F = (u)t_E$ for all $u \in E$. The fact that $t \neq t_E$ ensures that $t^F w_{\overline{D}} = t' \neq t_F = t^F w_F$. We conclude that $w_{\overline{D}} \neq w_F$ and this is enough to complete the proof in view of Proposition 3.8. $\qquad\square$

As motivation for our next result, we now consider the case $\lambda = (2,1,1,2)$, in which $1 \in M(\lambda)$. Here, $Y(\lambda) = \{y_1, y_2, y_3\}$, where $y_1 = [1, 5, 2, 3, 4, 6]$, $y_2 = [1, 3, 4, 5, 2, 6]$, and $y_3 = [1, 4, 2, 5, 3, 6]$. Also, $Y_s(\lambda) = \{y_1, y_2\}$. We display $\mathcal{E}^{(\lambda)}$ in Table 5.

In this case, $\lambda^{(1)} = (3,1,1,2)$. Moreover, $Y(\lambda^{(1)}) = \{y_1^{(1)}, y_2^{(1)}, y_3^{(1)}\}$ where $y_1^{(1)} = [1, 5, 7, 2, 3, 4, 6]$, $y_2^{(1)} = [1, 3, 7, 4, 5, 2, 6]$, and $y_3^{(1)} = [1, 4, 7, 2, 5, 3, 6]$, and $Y_s(\lambda^{(1)}) = \{y_1^{(1)}, y_2^{(1)}\}$. We find in this case that there is a bijection $Y(\lambda) \to Y(\lambda^{(1)})$, which induces a bijection $Y_s(\lambda) \to Y_s(\lambda^{(1)})$. We display $\mathcal{E}^{(\lambda^{(1)})}$ in Table 6.

Since $4 \in M(\lambda)$, we get a similar analysis and result. Here, $\lambda^{(4)} = (2,1,1,3)$ and $Y(\lambda^{(4)}) = \{y_1^{(4)}, y_2^{(4)}, y_3^{(4)}\}$, where $y_1^{(4)} = [2, 4, 5, 6, 1, 3, 7]$, $y_2^{(4)} = [2, 5, 3, 6, 1, 4, 7]$, and $y_3^{(4)} = [2, 6, 3, 4, 1, 5, 7]$, and $Y_s(\lambda^{(4)}) = \{y_1^{(4)}, y_3^{(4)}\}$. There is, however, a very close connection between $Y(\lambda^{(1)})$ and $Y(\lambda^{(4)})$, since every diagram $D \in \mathcal{D}^{(\lambda^{(1)})}$ corresponds, on rotation through $180°$, to a diagram $\dot{D}$ with $\dot{D} \in \mathcal{D}^{(\lambda^{(4)})}$, and this correspondence is bijective. We display $\mathcal{E}^{(\lambda^{(4)})}$ in Table 7.

Theorem 4.8. *Let $\lambda \vDash n$, let $k \in M(\lambda)$ and let $\overline{\lambda} = \lambda^{(k)}$. Also let $d = s_{p+1} \ldots s_n$ and $\overline{d} = s_{q+1} \ldots s_n$, where $p = \sum_{1 \le i \le k} \lambda_i$ and $q = \sum_{1 \le i \le k-1} \lambda_i$ (so that $q = 0$ when $k = 1$). Finally, let $\theta : Z(\lambda) \to Z(\overline{\lambda}) : z \mapsto \overline{z}$ be the map defined in Lemma 4.7. Then,*

(i) For each $z \in Z(\lambda)$ we have $z\theta \, (= \overline{z}) = dz$ (hence θ is injective),

(ii) $\overline{d}\mathfrak{C}(\lambda) \subseteq \mathfrak{C}(\overline{\lambda})$,

(iii) Any one of conditions (a), (b) or (c) below is sufficient to ensure that $Y(\lambda)\theta \subseteq Y(\overline{\lambda})$.

(a) $k = 1$,

(b) $k \ne \max M(\lambda)$,

(c) $M(\lambda) = \{k\}$ and $\max\{i : \lambda_i = m\} > k$ where $m = \max\{\lambda_j : 1 \le j \le r \text{ and } j \ne k\}$.

Moreover, in case $k = 1$, we have equality $Y(\lambda)\theta = Y(\overline{\lambda})$ and θ also induces a bijective mapping $\mathcal{E}^{(\lambda)} \to \mathcal{E}^{(\overline{\lambda})}$ in which $D \mapsto D^{(k)}$.

Proof. Let λ, k, $\overline{\lambda}$, d, $\overline{d}$ and θ be as in the statement of the theorem. For each $z \in Z(\lambda)$, we define $\overline{z} = w_{\overline{D}}$ where $D = D(z, \lambda)$ and $\overline{D} = D^{(k)}$. From the construction of $\overline{D}$ we have $i\overline{z} = iz$ if $1 \le i \le p$, $(p+1)\overline{z} = n+1$, and $i\overline{z} = (i-1)z$ if $p+2 \le i \le n+1$. Thus, $z\theta = \overline{z} = dz$, where $d = (n+1, n, \ldots, p+1) = s_{p+1} \ldots s_n$. It follows that if $z_1, z_2 \in Z(\lambda)$ and $z_1 \ne z_2$ then $\overline{z_1} \ne \overline{z_2}$. Hence the mapping $z \mapsto \overline{z}$ is injective. By a straightforward calculation, it can be shown that $\overline{d}w_{J(\lambda)} = w_{J(\overline{\lambda})}d$. Hence, $\overline{d}\mathfrak{C}(\lambda) = \overline{d}w_{J(\lambda)}Z(\lambda) = w_{J(\overline{\lambda})}dZ(\lambda) \subseteq w_{J(\overline{\lambda})}Z(\overline{\lambda}) = \mathfrak{C}(\overline{\lambda})$. This completes the proof of items (i) and (ii) of the theorem.

In order to prove item (iii) suppose, in addition, that any one of conditions (a), (b) or (c) holds. Also let $y \in Y(\lambda)$, and set $\overline{y} = w_{\overline{D}}$ where $D = D(y, \lambda)$ and $\overline{D} = D^{(k)}$. We suppose that $\overline{y} \notin Y(\overline{\lambda})$ and aim to obtain a contradiction. By Proposition 3.8, there exists a diagram F of subsequence type $\overline{\lambda}' = (\lambda'_1, \ldots, \lambda'_{r'}, 1)$, where $\lambda' = (\lambda'_1, \ldots, \lambda'_{r'})$, satisfying $\lambda_F = \overline{\lambda}$ and $w_F \ne w_{\overline{D}}$ and a standard F-tableau $t = t^F w_{\overline{D}} \ne t^F w_F = t_F$. We may suppose that F has no empty columns, that is F is principal. The last entry of the k-th row of t is $(p+1)\overline{y} = n+1$. Let (k, j) be the node of F for which $(k, j)t = n+1$. Since t is standard, the only node (i', j') of F with $i' \ge k$ and $j' \ge j$ is the node (k, j). Hence, any path in F containing the node (k, j) terminates in this node. We construct a diagram E from F by removing the node (k, j) and use E to obtain a contradiction. The tableau $\hat{t}$ obtained from t by removing the entry $(k, j)t$ is a standard E-tableau and $\hat{t} = t^E w_D$. Clearly, $\lambda_E = \lambda$. Let ν be the subsequence type of E. By Proposition 3.5, $\nu \lhdeq \lambda'$. We deal separately with the cases (a)–(c).

(a) Here, $k = 1 \in M(\lambda)$. Then $n+1$ is the only entry in the last column of t. Hence, $(1, c_F)$ is a node of F and is the only node in the last column of F. Since the only path in F containing $(1, c_F)$ has length 1, $\nu = \lambda'$ and E is admissible.

(b) Now let $l = \max M(\lambda)$ so that $k < l$ and $\lambda_k = \lambda_l = r'$. Let Π_u be a u-path in F of length $\sum_{1 \le j \le u} \lambda'_j$ where $1 \le u \le r'$. Then, by Proposition 3.4, Π_u has u nodes on each of the k-th and l-th rows. Thus, all u paths of Π_u contain nodes on the l-th row. Since a path containing node (k, j) must terminate at it, (k, j) cannot be a node of Π_u. Hence, Π_u is a u-path of E. This proves that E is admissible. By considering $\Pi_{r'}$, we see that all nodes of E on the l'-th row, where $l' < l$, are in columns containing nodes of E on the l-th row or to the left of such columns. Hence, node (k, j) is the only node of F on the j-th column and there are no nodes of E on or to the right of the j-th column.

(c) In this case, we first let $u = \lambda_l$ where $l = \max\{i : \lambda_i = m\}$ and let Π be a u-path of F of length $\sum_{1 \leq j \leq u} \lambda'_j$. By Proposition 3.4, Π contains all nodes on the l-th row of F and by the argument in (b), Π cannot contain node (k, j). Hence, Π is a u-path in E.

Since $\lambda'_j = 1$ for $u < j \leq r'$, we can form a u'-path in E of length $\sum_{1 \leq j \leq u'} \lambda'_j$ for any u' such that $u < u' \leq r'$ by adding $u' - u$ paths of length 1 whose nodes are chosen arbitrarily from the $r' - u$ nodes on the k-th row of E which are not in Π.

Finally, if $1 \leq u' < u$, let Π be a u'-path of F of length $\sum_{1 \leq j \leq u'} \lambda'_j$. By Proposition 3.4, Π contains exactly u' nodes on the l-th row of F and again by the argument in (b), Π cannot contain node (k, j). Hence, Π is a u'-path in E. So, in this case, E is again admissible. By an argument similar to that in case (b), we again find that node (k, j) is the only node of F on the j-th column and there are no nodes of E on or to the right of the j-th column.

We complete cases (a)–(c) by noting that, since $w_{\overline{D}} \neq w_F$ and $n + 1$ appears as the sole entry in the last columns of $t_{\overline{D}}$ and t_F, we get $w_D \neq w_E$. By Proposition 3.8, $y \notin Y(\lambda)$ contrary to hypothesis.

Next we establish that $Y(\lambda)\theta = Y(\overline{\lambda})$ in the case $k = 1$. Let $x \in Y(\overline{\lambda})$. We must show that $x = y\theta$ for some $y \in Y(\lambda)$. Let $C = D(x, \overline{\lambda})$. Then $\lambda_C = \overline{\lambda}$ and, since $x \in Z(\overline{\lambda})$, C is of subsequence type $\overline{\lambda}'$ by Theorem 3.6(ii). Since $\overline{\lambda}_1 > \overline{\lambda}_i$ for $i > 1$, $\overline{\lambda}_1$ is the number of parts of the partition $\overline{\lambda}'$ and the last part of $\overline{\lambda}'$ is 1. Let B be the diagram obtained from C by removing the last node of the first row, and let ν be its subsequence type. Then $\lambda_B = \lambda$. So, $\nu \trianglelefteq \lambda'$ by Proposition 3.5.

Let Π be a u-path in C where $1 \leq u < \overline{\lambda}_1$, and let $(1, j_1), \ldots, (1, j_u)$ be the first u nodes on the first row of C. By Proposition 3.4, Π has exactly u nodes on the first row of C; call these nodes $(1, j'_1), \ldots, (1, j'_u)$. Then $\Pi' = (\Pi \backslash \{(1, j'_1), \ldots, (1, j'_u)\}) \cup \{(1, j_1), \ldots, (1, j_u)\}$ forms a u-path in C of the same length as Π. Since Π' does not contain the final node of the first row of C, Π' is a u-path of B. Since we can choose Π to have length $\sum_{1 \leq j \leq u} \overline{\lambda}'_j = \sum_{1 \leq j \leq u} \lambda'_j$, we find a u-path of this length in B. Hence, B is admissible, $\nu = \lambda'$ and $w_B \in Z(\lambda)$ by Theorem 3.6(ii).

Let $\overline{B} = B^{(1)}$. Then $\overline{B}$ is admissible by Lemma 4.6(i), since B is admissible. We want to show that $C = \overline{B}$. Suppose on the contrary that $C \neq \overline{B}$. In view of the way $C = D(x, \overline{\lambda})$ is constructed from x and $\overline{\lambda}$ (and also the way B is constructed from C) we would then have that the last node on the first row of C is not in the last column of C or is not the sole node in the last column of C. It follows from this that $w_C \neq w_{\overline{B}}$. On the other hand $t^{\overline{B}}w_C$ is a standard $\overline{B}$-tableau since $t^{\overline{B}}w_C$ is constructed from $t_C = t^C w_C$ by moving the last entry on the first row of t_C to the position of the last node of the first row of $\overline{B}$ and keeping all other entries of t_C fixed. This contradicts the fact that $x = w_C \in Y(\overline{\lambda})$. Hence $C = \overline{B} = B^{(1)}$.

Let $y = w_B$. Then $w_{\overline{B}} = y\theta$ from Lemma 4.6(ii) (or by noting that $B = D(w_B, \lambda)$). As $w_{\overline{B}} = w_C = x$, we get $x = y\theta$. Since B is admissible, $y \in Z(\lambda)$. Since $y\theta = x \in Y(\overline{\lambda})$, it now follows from Lemma 4.7(ii) that $y \in Y(\lambda)$. This completes the proof that in the case $k = 1$ we have equality $Y(\lambda)\theta = Y(\overline{\lambda})$. Finally, it is clear that θ induces a bijective mapping $\mathcal{E}^{(\lambda)} \to \mathcal{E}^{(\overline{\lambda})}$, given by $D \mapsto D^{(1)}$. $\qquad\square$

We illustrate Theorem 4.8 with the following example. Consider $\lambda = (2, 1, 2, 2)$. Then $M(\lambda) = \{1, 3, 4\}$ and $Y(\lambda) = \{(2, 4)(3, 5, 6), (2, 5, 6, 4, 3)\}$. If $k = 1$, $\overline{\lambda} = (3, 1, 2, 2)$, $Y(\overline{\lambda})$

$= \{(2,4,5)(3,8,7),(2,5,3,8,7,4)\}$, and $d = (8,7,6,5,4,3)$ is a prefix of both elements of $Y(\overline{\lambda})$ and $d^{-1}Y(\overline{\lambda}) = Y(\lambda)$. If $k = 3$, $\overline{\lambda} = (2,1,3,2)$, and $Y(\overline{\lambda}) = \{(1,2,5,3,6,7,4), (2,5,4)(3,6,7), (1,2,6,7,5,4), (2,4)(3,5,6,8,7), (2,5,6,8,7,4,3), (2,6,7,4,3)\}$. The element $d = (8,7,6)$ is a prefix of the fourth and fifth elements of $Y(\overline{\lambda})$ and $dY(\lambda)$ consists of these two elements. If $k = 4$, $\overline{\lambda} = (2,1,2,3)$, and $Y(\overline{\lambda}) = \{(1,2,5,7,4,3,6),(1,2,6)(5,7)\}$. Neither of the elements of $Y(\lambda)$ is a suffix of either of these elements.

Next we comment on the link Theorem 4.8 provides with the restriction of cells.

Remark 4.9. (i) Comparing Theorem 4.8 (and keeping the notation introduced there) with the 'right analogue' of Result 4 we see, under the assumption $k \in M(\lambda)$, that the Kazhdan-Lusztig cell $\mathfrak{C}(\lambda)$ is one of the cells appearing in the decomposition of $\mathfrak{C}(\overline{\lambda})$ into a disjoint union of sets of the form $e\mathcal{D}$ where $\mathcal{D}$ is a right cell of S_n and $e \in \mathfrak{X}'^{-1}$. To see this, note first that $\mathfrak{X}'^{-1}$ is the set of distinguished left coset representatives of S_n into S_{n+1}. Now $\overline{d} \in \mathfrak{X}'^{-1}$, so the above is an immediate consequence of the uniqueness of decomposition of any $x' \in S_{n+1}$ in the form $e'x$ where $e' \in \mathfrak{X}'^{-1}$ and $x \in S_n$.
The theorem also provides information about which particular element of $\mathfrak{X}'^{-1}$ we need to premultiply the elements of $\mathfrak{C}(\lambda)$ for the particular cases of cells we are considering. Note that this information is provided by Proposition 2.2 in the more general case, however this involves applying the reverse Robinson-Schensted process. See also [11, Theorem 3.3] for a different result related to the above discussion.

(ii) In addition to providing a link with the restriction of cells, the theorem also relates the rims of the cells involved. If we know the rim of $\mathfrak{C}(\lambda)$ we can immediately deduce information about the rim of $\mathfrak{C}(\overline{\lambda})$. In particular, when $k = 1$, mere knowledge of the rim of $\mathfrak{C}(\lambda)$ allows us to determine completely the rim of $\mathfrak{C}(\overline{\lambda})$ and, as a consequence, obtain reduced forms for all the elements in this cell (as the remaining elements of $\mathfrak{C}(\overline{\lambda})$ are the products of $w_{J(\overline{\lambda})}$ with the prefixes of the elements in its rim).

(iii) In view of Result 4, another consequence of the above (when $k \in M(\lambda)$) is that the cell module corresponding to $\mathfrak{C}(\lambda)$ is a constituent of the cell module corresponding to $\mathfrak{C}(\overline{\lambda})$ when restricted to S_n. This can also be seen to agree with the Branching Theorem (see [12, Theorem 9.2(ii)]) by using the same results as for the induction case earlier on.

Finally, by applying Proposition 3.9, we can obtain a dual result to Theorem 4.8 in a similar manner to how Corollary 4.5 is obtained from Theorem 4.3. In particular, if λ is a composition of n with r parts and $r \in M(\lambda)$, there is a bijection between the elements of $Y(\lambda)$ and the elements of $Y(\lambda^{(r)})$.

References

[1] S. Ariki. *Robinson-Schensted correspondence and left cells*, pages 1–20. Number 28 in Advanced studies in Pure Mathematics. Math. Soc. of Japan, Kinokuniya, Tokyo, 2000. conference held in Kyoto, 1998. 4

[2] D. Barbasch and D. Vogan. Primitive ideals and orbital integrals in complex exceptional groups. *J. Algebra*, 80:350–382, 1983. 2, 3, 4

[3] D. Deriziotis, T. P. McDonough, and C. A. Pallikaros. On root subsystems and involutions in S_n. *Glasgow Mathematical Journal*, 52:357–369, 2010. 6

[4] Richard Dipper and Gordon James. Representations of Hecke algebras of general linear groups. *Proc. London Math. Soc.*, 52:20–52, 1986. 7

[5] W. Fulton. *Young Tableaux*. Cambridge University Press, Cambridge, 1997. 4

[6] M. Geck. On the induction of Kazhdan-Lusztig cells. *Bull. London Math. Soc.*, 35:608–614, 2003. 4

[7] M. Geck. Kazhdan-Lusztig cells and the Murphy basis. *Proc. London Math. Soc.*, 93:635–665, 2006. 3, 4, 11

[8] M. Geck and G. Pfeiffer. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*. Clarendon Press, Oxford, 2000. 3

[9] C. Greene. An extension of Schensted's theorem. *Advances in Math.*, 14:254–265, 1974. 2, 9

[10] James E. Humphreys. *Reflaction Groups and Coxeter Groups*. Number 29 in Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge, 1990. 3

[11] N. Jacon. Calcul explicite de certaines cellules de kazhdan-lusztig pour le type a_{n-1}. *Publications Mathématiques de Besançon*, 2006, 2006. 22

[12] G. D. James. *The representation theory of the symmetric groups*. Number 682 in Lecture Notes in Mathematics. Springer, Berlin, 1978. 4, 17, 22

[13] D. A. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math*, 53:165–184, 1979. 1, 3, 4

[14] G. Lusztig. *Characters of reductive groups over a finite field*, volume 107 of *Ann. of Math. Stud.* Princeton University Press, 1984. 3

[15] George Lusztig. Cells in affine Weyl groups II. *J. Algebra*, 109:536–548, 1987. 3

[16] T. P. McDonough and C. A. Pallikaros. On relations between the classical and the Kazhdan-Lusztig representations of symmetric groups and associated Hecke algebras. *Journal of Pure and Applied Algebra*, 203:133–144, 2005. 8, 17

[17] Thomas P. McDonough and Christos A. Pallikaros. On double cosets with the trivial intersection property and kazhdan-lusztig cells in S_n. *International Journal of Group Theory*, 4(2):25–48, 2015. 2, 6, 7, 12, 17

[18] Yuval Roichman. Induction and restriction of Kazhdan-Lusztig cells. *Adv. in Math.*, 134:384–398, 1998. 3, 4

[19] B. Sagan. *The Symmetric group, Representations, Combinatorial Algorithms and Symmetric Functions*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. 4, 9, 10

[20] C. Schensted. Longest increasing and decreasing subsequences. *Canadian J. Math.* 2, 9