The Stokes experiment in a foam

S.J. Cox, M.D. Alonso,* S. Hutzler and D. Weaire
Department of Physics, Trinity College, Dublin 2, Ireland.

Abstract

The well-known experiment of Stokes, where the viscosity of a liquid can be determined by measuring the velocity of a sphere falling through it, was performed for the case of an aqueous foam. The liquid fraction of the foam was kept uniform throughout the sample using forced drainage and the velocity of a falling ball was then determined as a function of the liquid fraction. In a similar experiment, the position of a light sphere placed in a dry foam was recorded; the rate of descent of the sphere is then governed by the coarsening of the foam. Results are compared with a simulation of the foam as a continuous medium using the model of a Herschel-Bulkley fluid. This includes a finite yield stress, above which the foam flows as a liquid and below which it remains solid.

1 Introduction

In this paper we give the first detailed analysis (as far as we are aware) of the Stokes experiment for a sphere falling through a foam and arrive at a tentative power law relation between force and velocity.

While considerable progress has been made in the physics of foams within recent decades [1, 2] reliable knowledge of rheological properties remains limited. Quasi-static properties (shear modulus, yield stress) are well understood, although their precise dependence on liquid fraction remains debatable [3].

The flow of foam when the yield stress is exceeded has presented a number of interesting questions, still under investigation. These topics include avalanches [4, 5, 6], convective instability [7, 8] and hysteresis [9].

Even apparently elementary experiments, including Couette viscometry, have proved difficult to analyse in the past [10, 11, 12]. We are not aware of any previous discussion of the Stokes experiment in this context. Our analysis suggests that it can provide significant information on the relationship between stress and strain rate close to the yield stress, with a minimum of instrumentation.

1.1 Phenomenology

In the familiar application of this experiment to find the viscosity, η, of a Newtonian liquid the relationship between the drag force F and velocity v for a sphere of radius a is

$$F = 6\pi\eta v a. \tag{1}$$

However, this formula is not applicable to a foam, since it is highly non-Newtonian.

Instead we suggest the following: since foam has a finite yield stress (at least in a practical sense, regardless of whether theoretical arguments to this effect are satisfactory), we might expect to find

$$\begin{aligned} F - F_0 &= \kappa v^n, \quad F \geq F_0 \quad \text{and} \\ v &= 0, \quad F \leq F_0 \end{aligned} \tag{2}$$

where κ is a constant dependent on the radius a, the liquid fraction Φ_l of the foam and its rheological properties. A minimum force F_0 is required to move the sphere through this medium. At all points we neglect inertia and it is therefore assumed that the forces acting on the sphere (weight and drag) are in equilibrium.

Our goal is this work is to analyse a range of data in this way and relate these findings to the Herschel-Bulkley relation

$$S = S_y + K\dot{\epsilon}^m, \tag{3}$$

which is often assumed to relate stress, S, and strain rate, S_y, in a foam undergoing continuous shear ($\dot{\epsilon}$ is shear rate). Theoretical models [12] have supported such a relation, with m generally less than 1 [13, 14]. For $m = 1$, equation (3) defines a Bingham plastic and K represents the viscosity. The question is: can we relate m to the exponent n in equation (2)?

*Corresponding author. Email: lalonso@phy.tcd.ie

2 Experiments

We use a soap solution prepared with ordinary dishwasher detergent (Fairy Liquid) and push gas through a sparger immersed in the solution. The foam which is created is polydisperse, containing bubbles with an average diameter of less than 1mm. It is introduced into a glass cylinder. The addition of soap solution to the top of the foam (forced drainage) results in a foam with uniform liquid fraction. Changing the flow rate Q changes the liquid fraction, which allows us to analyse Stokes flow through foams with different, but constant, liquid fractions.

For a range of sphere sizes and flow rates Q, we placed a small plastic sphere at the top of the foam and timed its descent through the foam. We assume that the sphere reaches its terminal velocity immediately i.e. that inertia is negligible. In figure 1 we plot the sphere velocity as a function of Q, for spheres with varying radii. We find that for each sphere size, the velocity varies approximately linearly with flow rate and, since, for the detergent solution we used, the liquid fraction $\Phi_l \propto \sqrt{Q}$, it varies with the square of liquid fraction.

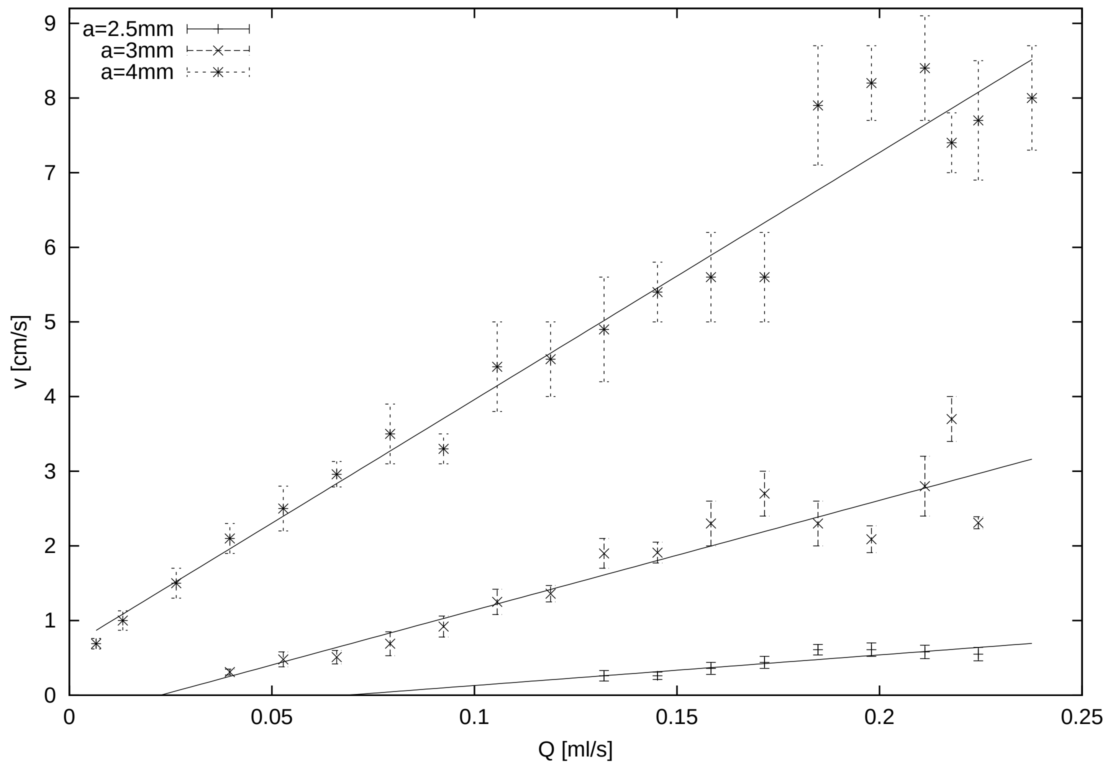

Fig. 1: The velocity of spheres of radii $a = 2.5\mathrm{mm}, 3\mathrm{mm}$ and $4\mathrm{mm}$ immersed in a liquid foam, along with their linear best fit lines. The $4\mathrm{mm}$ ball has a finite velocity for all flow rates, whereas at low flow rates the smaller balls don't move through the foam. A flow rate of $Q = 0.25\mathrm{ml/s}$ corresponds to a foam with a liquid fraction of roughly $\Phi_l \simeq 0.08$. The diameter of the tube was $40\mathrm{mm}$ and its length varied between 15 and $30\mathrm{cm}$.

However, since the actual data is somewhat scattered, we decided to concentrate on the case of zero flow rate, and we postpone any discussion of the effects of liquid fraction. In this experiment, since the foam is dry, we can monitor the position of the sphere in the foam visually, as was not possible at higher flow rates. We use a sphere that will stay on the surface initially (the results in figure 1 suggest that its radius must be less than $4\mathrm{mm}$). Figure 2 records the position of the sphere against time. Initially the sphere is found to move downwards very slowly (velocities of the order of $0.5\mathrm{mm/min}$), but eventually accelerates to achieve larger velocities, of the order of $1\mathrm{cm/s}$. Occasionally it is brought to a stand-still (or it reverts to movement with a slow velocity).

We interpret this behaviour as follows: topological rearrangements due to the applied load cause a slow yielding as is familiar in metallurgy. This *creep*, as we shall call it, is believed to cause the slow initial downward motion. Further, a foam sample of this kind coarsens continuously due to gas diffusion, which results in an increase of average bubble diameter in time. This temporal variation is eventually governed by the square root of time, but over the relatively short time scale of the experiment we will treat it as linear. The effect of coarsening is the gradual reduction of the yield stress, S_y, since [1]

$$S_y \propto d^{-1} \tag{4}$$

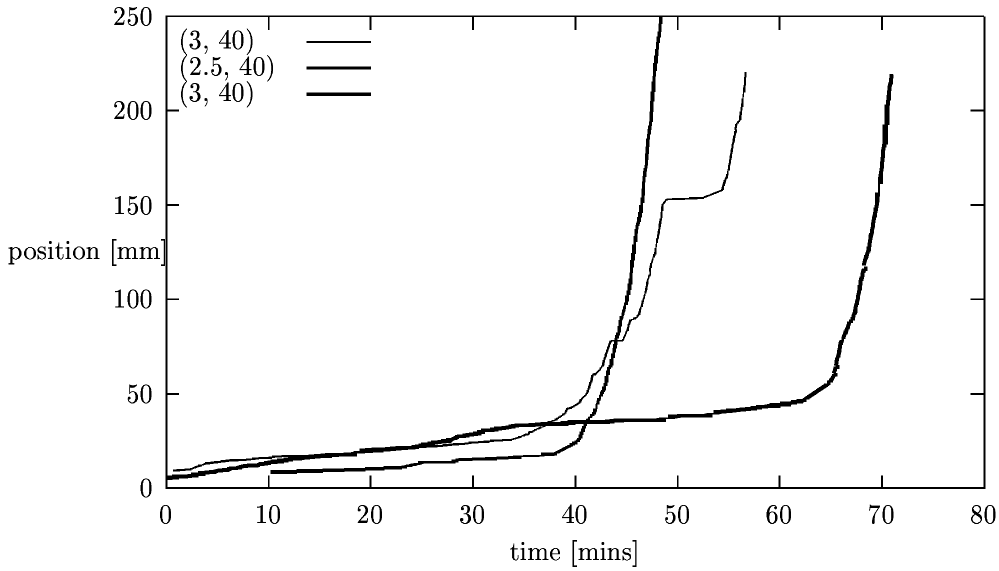

Fig. 2: The sphere is dropped into a dry foam at time zero and its position recorded. Local inhomogeneities in the foam structure cause the sphere to hesitate in its descent, giving the horizontal plateaux, which make curve fitting difficult. In the key (a, D) refers to a sphere of radius a in a cylindrical tube of diameter D (both measured in millimetres). All the lines show a transition from creep to accelerated motion; after correction for creep the latter is roughly characterised by a power law of position $\sim$ time$^{3\pm0.5}$.

where d is the mean bubble diameter. Hence there is a point at which the yield stress is reached close to the surface of the sphere, and a different regime is encountered. As the yield stress continues to decrease, the sphere accelerates since its velocity is always proportional to the net force acting on it (inertia being negligible). Local inhomogeneities in the polydisperse foam cause the sphere to slow and re-enter the creeping regime; once these regions have coarsened sufficiently, the local yield stress decreases and the sphere's velocity increases again.

Such a description suggests the following analysis, to eliminate the effect of creep in the data.

2.1 Data reduction

In the regime that we have characterised as creep, we typically observe a roughly linear variation in the vertical position with time. We therefore assume that this always occurs; we make a linear fit to the first part of the data and subtract this from the later data, to remove the contribution of creep. That which remains is fitted to a power law

$$x = c(t - t_0)^k, \quad t > t_0. \tag{5}$$

Results obtained in this way for the index k proved to be quite variable, because it is only possible to fit to the parts of the data between the hesitations. We suggest as a preliminary finding that

$$k \sim 3 \pm 0.5 \tag{6}$$

on the basis of three experiments.

3 Theoretical ideas

We now attempt to relate these experimental results to the Herschel-Bulkley model. In this scenario, the value of the yield stress is important: whenever the sphere is sufficiently light (and the foam bubbles sufficiently small) the net force acting (weight minus buoyancy, although the latter is a small correction here) is not sufficient for the yield stress to be exceeded and the sphere is initially supported on the surface.

Consider again the relations (2) and (3) and figure 3. We employ the following heuristical argument to rationalise the results of the experiment and the numerical simulations which follow.

The force F is constant while F_0 decreases with time; we assume this decrease to be linear over the time range of the data analysed. We take the derivative of (5) to see that the velocity of the ball varies as t^{k-1}. Comparison

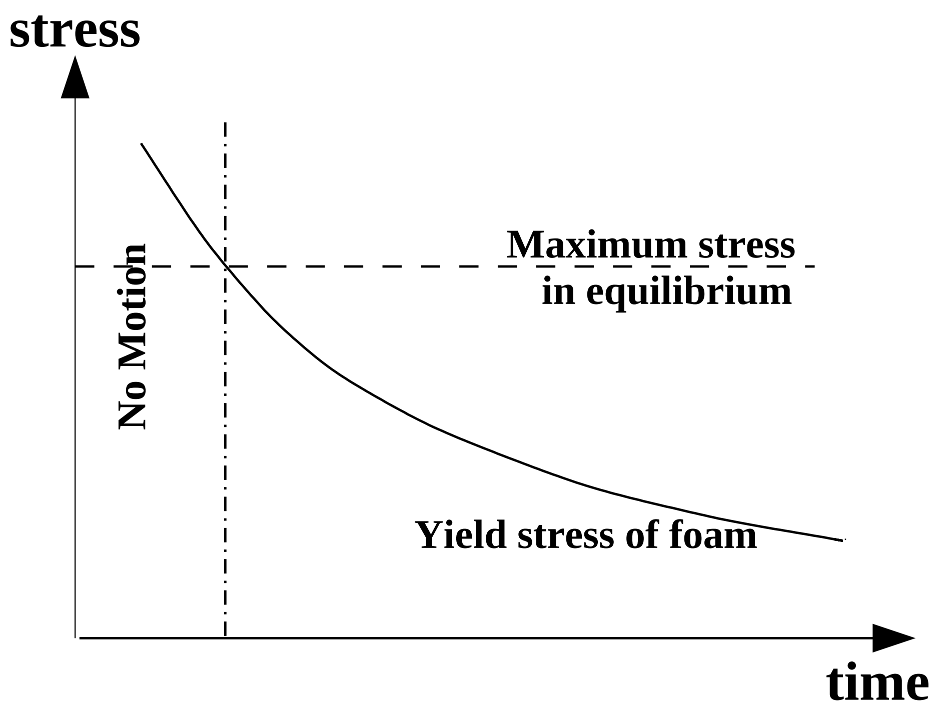

Fig. 3: While the ball is in equilibrium the stress remains constant. The yield stress decreases however, due to a coarsening induced increase in bubble size. Until the yield stress falls below the equilibrium stress there is no motion apart from that due to creep.

with (2), where v^n varies linearly in time, shows that the index k from the experimental data is trivially related to n by

$$k = 1 + 1/n. \tag{7}$$

When F_0 becomes equal to F, the sphere acquires a non-zero velocity v. It is then contained within a small, roughly spherical, yielding region of radius $a + \delta a$; see figure 4.

Equivalently, consider the stress on the sphere, as sketched in figure 3. In equilibrium the stress S is constant and less than the yield stress S_y which decreases with time. We assume the decrease in yield stress is also linear for the duration of the experiment. At a distance δa from the surface of the sphere the stress and yield stress are equal. Now if the stress decreases linearly with distance from the sphere, which may be questionable, then this distance δa is directly proportional to the 'excess' stress $S - S_y$. In any case, the resulting relation between n and m will be directly tested and validated in the next section.

Now, at the sphere's surface the rate of strain is approximately $v/\delta a$ so we find from equation (3) that

$$S - S_y \propto \left(\frac{v}{\delta a}\right)^m.$$

But since $\delta a \propto S - S_y$ we conclude that

$$S - S_y \propto v^{m/(m+1)} \quad \text{and} \tag{8}$$

$$\delta a \propto v^{m/(m+1)}. \tag{9}$$

When $m = 1$, the Bingham case, this gives $\delta a \propto \sqrt{v}$.

We wish to relate the stresses and forces around the ball, for which we need a length scale. The relevant length here is the radius of the sphere, so we can immediately equate the exponent of velocity in the stress equation (8) with its exponent n in the force equation (2):

$$\boxed{n = \frac{m}{m+1}.} \tag{10}$$

So the tentative answer to our earlier question is YES, we *can* relate n to m.

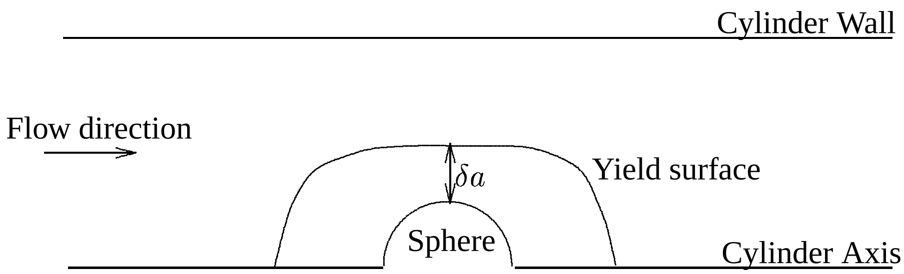

Fig. 4: The foam flows past a stationary sphere. We sketch a cross-section showing the position of the yield surface (dotted line) around the sphere. The size of the yielded region between the yield surface and the sphere increases with velocity v and decreases with yield stress S_y.

4 Numerical modelling

In §2.1 we found that the position of the ball in the foam varies as t^k with $k = 3 \pm 0.5$; Then according to (7) we get values of the exponent n of velocity in the force equation between 0.4 and 0.67. We use (10) to obtain $0.67 \leq m \leq 2$. This conclusion is consistent with previous theoretical and experimental values of m, and we now test it with a numerical model of the behaviour seen in the experiments.

We therefore simulate the flow of a yield stress fluid around a sphere confined within a cylinder, similar to the analysis of Mitsoulis and co-workers [15, 16]. Using an axisymmetric formulation, we hold the sphere fixed and allow the fluid to flow along the frictionless tube. The boundary condition on the sphere is that of no-slip. Using the Herschel-Bulkley relation (3) we define an effective viscosity

$$\eta_{\text{eff}} = \frac{S}{\dot{\epsilon}} = \frac{S_y}{\dot{\epsilon}} + K\dot{\epsilon}^{m-1} \tag{11}$$

where K is some ‘asymptotic viscosity’ (at high strain rate). We instigate a cut-off for values of $\eta_{\text{eff}} > \eta_s$ (a ‘solid’ viscosity), after which η_{eff} is constant (at low strain-rate where the foam moves as a solid plug). (We find that, provided it is sufficiently large, the value of η_s is not significant.) We retain the foam density as $\rho = \rho_{water}\Phi_l \approx 10\text{kg}/\text{m}^3$ throughout (corresponding to $\Phi_l = 0.01$).

Streamlines of the motion are readily obtained, but prove of little value as there is little difference seen between runs with different flow parameters. More interesting is the position of the yield surfaces (see the schematic in figure 4); this is taken as the contour at which the effective viscosity is no longer equal to the solid viscosity. As we show in figure 5, our computations in the Bingham case $m = 1$ are consistent with the theoretical estimate (9): that the width δa of the yielded region does indeed increase with $\sqrt{v}$ gives us greater confidence in the validity of the assumptions made in §3.

We calculate the total force on the sphere as a sum of the viscous and pressure forces. This is illustrated in figure 6 for varying free stream foam velocity and three different values of m, which encompasses the range dictated by the experiments. For $m = 1$, the Bingham case, we find $F \sim v^{0.4}$; that is, $n \approx 0.4$. This and the other values obtained numerically are shown in Table 1; the numerical result proves to be slightly lower than the theoretical result that $m = 1 \Rightarrow n = 0.5$.

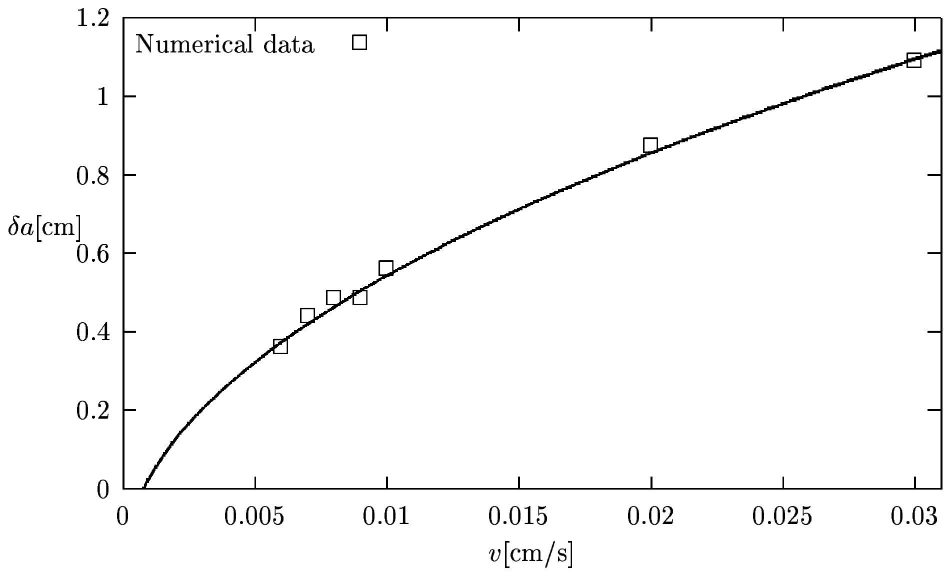

Fig. 5: The numerically calculated width (across the cylinder) of the yielded region as a function of free stream foam velocity. Also shown is a fit to $\delta a \sim \sqrt{v}$. Note that even at non-zero velocities, it may be possible that the fluid remains unyielded. Parameters are $m = 1, S_y = 0.01$ and $K = 0.01$ for a sphere of radius $a = 4$mm in a tube of diameter $D = 40$mm.

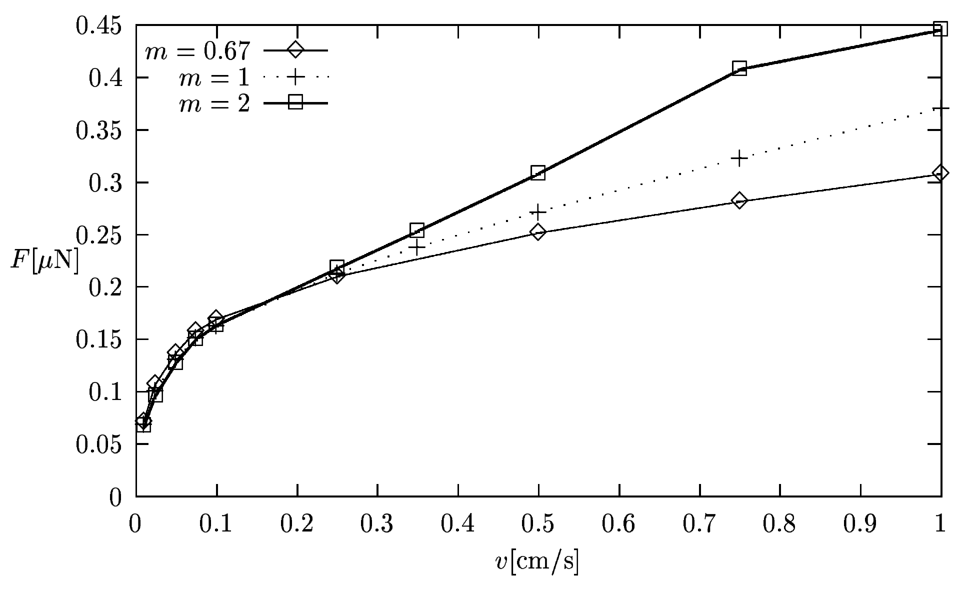

Fig. 6: In a numerical calculation, we vary the exponent m in (3) to find the corresponding variation of the force on the sphere with the velocity. For $F \sim v^n$, we see that n is an increasing function of m; the values are given in Table 1. Parameters are $K = 0.01$ and $S_y = 0.01$ Ns/m^2 for a sphere of radius $a = 4$mm in a tube of diameter $D = 40$mm.

m	n_{numeric}
2.0	0.6
1.0	0.4
0.67	0.17

$n_{\mathrm{expt}} \approx 0.53 \pm 0.13$

Table 1: The index m is that of the Herschel-Bulkley relation (3). For these values, numerical calculations give estimates of n, the exponent of velocity in the force equation (2), tabulated as n_{numeric}. These are consistent with the proposed relation (10), and are to be compared with the the value obtained from experiment, n_{expt}.

5 Concluding remarks

We have described a range of new experiments on spherical balls falling through foams. We find that the velocity of a ball varies linearly with flow rate, and therefore with the square of the liquid fraction of the foam. Further experimentation is now being undertaken to extract the exact dependence of the results on the bulk properties of the foam.

When a light ball descends through a coarsening foam, it initially creeps (with position varying linearly with time) and then accelerates after the yield stress of the foam has been reduced by the coarsening process. We find that in the latter part of the motion the velocity of a ball varies with the square of time.

Our heuristic theoretical model is qualitatively and semi-quantitatively consistent with expectations based on the Herschel-Bulkley model, and also shows agreement with our experiments, to within the wide limits of the present results. These suggest that the exponent m in the Herschel-Bulkley relation lies between 0.67 and 2. However, it is not yet possible to specify m more precisely at this stage

Our numerical calculations show the yield surfaces around a sphere and the variation of the force on a sphere with its velocity. The results support the conclusions of our theoretical model. Further analysis is now required for larger parameter ranges.

Acknowledgements

Research supported by the Irish Higher Education Authority under PRLT199 (MDA), Shell Amsterdam, Enterprise Ireland (SJC) and an EU Marie Curie fellowship (SJC). Thanks are due to the European Space Agency, for support of a Topical Team in this research area. The second experiment was originally suggested by D. Pardal. IITAC research studentship (MDA)

References

[1] D. Weaire and S. Hutzler. *The Physics of Foams*. Clarendon Press, Oxford (1999)

[2] D. Weaire and M.A. Fortes. *Advances Phys.* **43**, 685 (1994)

[3] A. Saint-Jalmes and D.J. Durian. *J. Rheol.* **43**, 1411 (1999)

[4] S. Hutzler, D. Weaire and F. Bolton. *Phil. Mag. B* **71**, 277 (1995)

[5] D.J. Durian. *Phys. Rev.* **55**, 1739 (1997)

[6] A. Abd el Kader and J.C. Earnshaw. *Phys. Rev. Lett.* **82**, 2610 (1999)

[7] S. Hutzler, D. Weaire and R. Crawford. *Europhysics Lett.* **41**, 461 (1998)

[8] M.U. Vera, A. Saint-Jalmes and D.J. Durian. *Phys. Rev. Lett.* **84**, 3001 (2000)

[9] R. Hoehler, S. Cohen-Addad and A. Asnacios. *Europhys. Lett.* **48**, 93 (1999)

[10] M.E. Rosa, A.C. Diodo and M.A. Fortes. *Langmuir* **11**, 2329 (1995)

[11] B.S. Gardiner. *PhD thesis, Univ. of Newcastle* (1999)

[12] H.M. Princen and A.D. Kiss. *J. Coll. Int. Sci.* **128**, 176 (1989)

[13] M.E. Cates and P. Sollich. In S.F. Sadoc and N. Rivier (eds), *Foams and Emulsions* Nato Science Series: E Applied Sciences. Kluwer, Dordrecht (1999)

[14] S.A. Khan, C.A. Schnepper and R.C. Armstrong. *J. Rheol.* **32**, 69 (1988)

[15] J. Blackery and E. Mitsoulis. *J. Non-Newtonian Fluid Mech.* **70**, 59 (1997)

[16] M. Beaulne and E. Mitsoulis. *J. Non-Newtonian Fluid Mech.* **72**, 55 (1997)