

# *The rheology of two-dimensional foams*

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## Abstract

We survey simulations of two-dimensional flowing foam inspired by recent illuminating experiments. We also describe the *viscous froth model*, an effective tool for such simulations, which accurately represents the detailed structure of the foam and includes a linear drag force.

**Key words.** foam – emulsion – two-dimensional – shear-banding – localization – viscous froth

## 1. Introduction

Why study the rheology of foams? The motivations are widespread and diverse (Weaire and Hutzler 1999). Foams are common in oil extraction and industrial cleaning. Closer to home, an understanding of flowing foams helps to extinguish fires more efficiently and to generate the perfect pint of beer. At low shear-rates a foam behaves as an elastic solid, while at high shear-rates it yields like a liquid, generating a rich range of behaviours. Foams are non-Newtonian, but have a specific structure which allows analysis of their non-Newtonian behaviour. Foams can thus serve as a prototype for other non-Newtonian fluids and even for granular materials.

In addition to rheology, we seek to understand the *drainage* (usually gravitational) of liquid through the foam structure. High flow-rate drainage can cause the structure itself to flow; these convective instabilities (Weaire *et al* 2003) further motivate our interest in rheology.

Is the study of *two-dimensional* (2D) foams, or froths, useful? As in many fields, the restriction to 2D allows easier visualization of both experiment and simulation, and often simplifies the theory, allowing us to isolate and study new phenomena (though we must check for spurious 2D effects).

To develop tools for 2D simulations, we first consider the foam's liquid fraction  $\Phi_l$ , *i.e.* its fractional liquid content by volume. The dry limit of low liquid fraction,  $\Phi_l \rightarrow 0$ , is a natural idealization, in which soap films are infinitesimally thin arcs of circles. The Laplace equation then determines the curvature  $\kappa$  of a film from the pressure difference  $\Delta p$  between neighbouring bubbles:

$$\Delta p = \gamma \kappa, \tag{1}$$

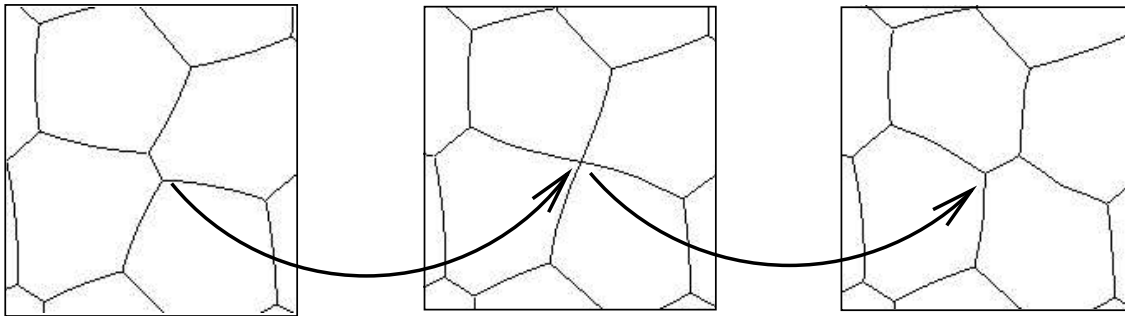
where  $\gamma$  is the surface tension of the films. At equilibrium, these films meet in three-fold vertices at  $120^\circ$ .

Minimization of film or line-length completely determines a 2D foam's equilibrium structure, since its energy is proportional to the product of surface tension and film length. A progressively strained foam's energy increases until two three-fold vertices approach one another and undergo a T1 neighbour exchange as in figure 1. The T1 reduces the pattern energy: the films connecting two three-fold vertices are shorter than those for a four-fold vertex, which rapidly decays. The location and statistics of T1s thus determine the inelastic properties of the foam and how it releases energy during flow.

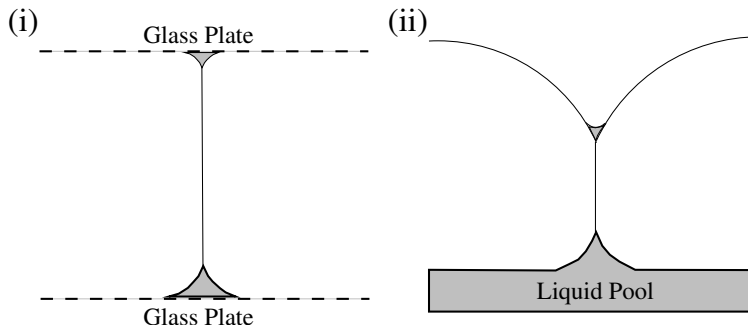
Simulations must handle a range of liquid fractions up to the wet limit of  $\Phi_l \approx 18\%$  (Weaire and Hutzler 1999). As the liquid fraction increases, the vertices swell into triangular Plateau borders, named after the Belgian scientist who pioneered the study of soap film structures (Plateau 1873). At the wet limit, a *rigidity loss transition* occurs (Bolton and Weaire 1992), leaving separated circular bubbles.

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**Fig. 1.** A T1 topological transformation, or neighbour exchange, is the mechanism by which the foam dissipates energy in the quasi-static regime. The length of a short edge between two vertices shrinks to zero, forming a four-fold vertex which immediately dissociates into two three-fold vertices. Two of the adjacent bubbles lose an edge and the other two gain an edge.



**Fig. 2.** Possible experimental configurations for a 2D foam consist of (i) trapping the foam between two close, parallel horizontal plates and (ii) floating a bubble-raft on the surface of a liquid.

The most common experimental realizations of a 2D foam, illustrated in figure 2, consist of either trapping bubbles between two glass plates, as pioneered by Smith (1952), or allowing bubbles to form a floating raft on a liquid pool. Bragg and Nye (1947) used the latter as an analogy for crystal structure. A further possibility is to trap bubbles floating on a liquid beneath a glass plate (Smith 1952, Vaz and Fortes 1997). Such foams are not truly 2D, since the meniscus or solid surface can change their behaviour (Cox *et al* 2003). However, they often approximate the ideal 2D froth of simulations.

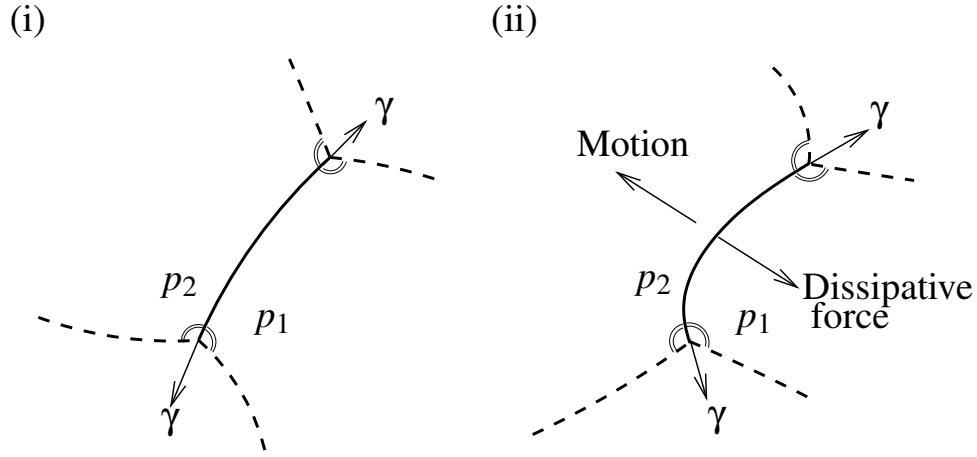
The development of computer codes to model foam flow has followed the work of Princen and co-workers in the 1980s, reviewed in Princen (2000), on the deformation of the ideal hexagonal (honeycomb) network. Early work concentrated on the coarsening dynamics of foams (Weaire and Kermode 1983, Kermode and Weaire 1990, Herdtle and Aref 1992), in which diffusion transfers gas between bubbles. The quasi-static PLAT code of Bolton and Weaire (1992), which relaxes the foam between small increments in strain, allowed Hutzler *et al* (1995) to model foams of arbitrary liquid fraction under extensional flow.

Okuzono *et al* (1993) used a *vertex* model, which included viscous dissipation at the vertices. This dry model connects vertices by straight films, causing some inaccuracy in resolving T1s, although Cantat and Delannay (2003) have used it successfully. Jiang *et al* (1999) used the Potts model to include viscous dissipation in the dry limit. Despite certain caveats about the definition of viscosity, the model gives good, computationally inexpensive, results for T1 statistics. Durian's wet-limit bubble model (Durian 1995) approximates bubbles as circular discs connected by springs; the results compare well with experimental data for wet foams.

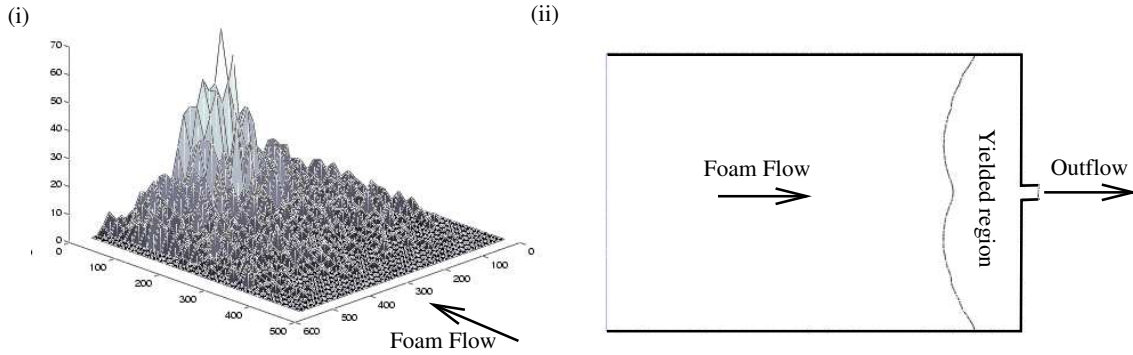
We next describe some experiments and simulations designed to clarify the physics of foam rheology. Approaches to 2D foam flows have progressed from continuum approximations (§2) which subsume all structural detail into a macroscopic yield stress, to quasi-static simulations (§3) which assume that the foam has sufficient time to relax between each small increment in strain, to our viscous froth model (§4), which combines an improved version of the viscous dissipation of the continuum approximation with the precise foam structure of the quasi-static approximation (figure 3) and accounts realistically for the pressures to give greater insight into the dynamics of moving foams.

## 2. Continuum approximations

Before we analyze how to exploit the structure of a foam to predict its behavior, we consider the continuum limit in which a foam behaves like a yield-stress or shear-thinning fluid. Kraynik (1988) showed how to infer values for the



**Fig. 3.** (i) Quasi-static and (ii) viscous froth dynamics applied to a single film in a 2D froth. In the quasi-static case, after each increment in strain the pattern relaxes to a new equilibrium in which the pressure difference  $\Delta p = p_1 - p_2$  balances the surface tension  $\gamma$  according to the Laplace Law (1). In the viscous froth model a viscous drag on the film proportional to the velocity alters the force balance to the form in (3). In both cases the films meet at  $120^\circ$ , but only in the quasi-static case do they form circular arcs.



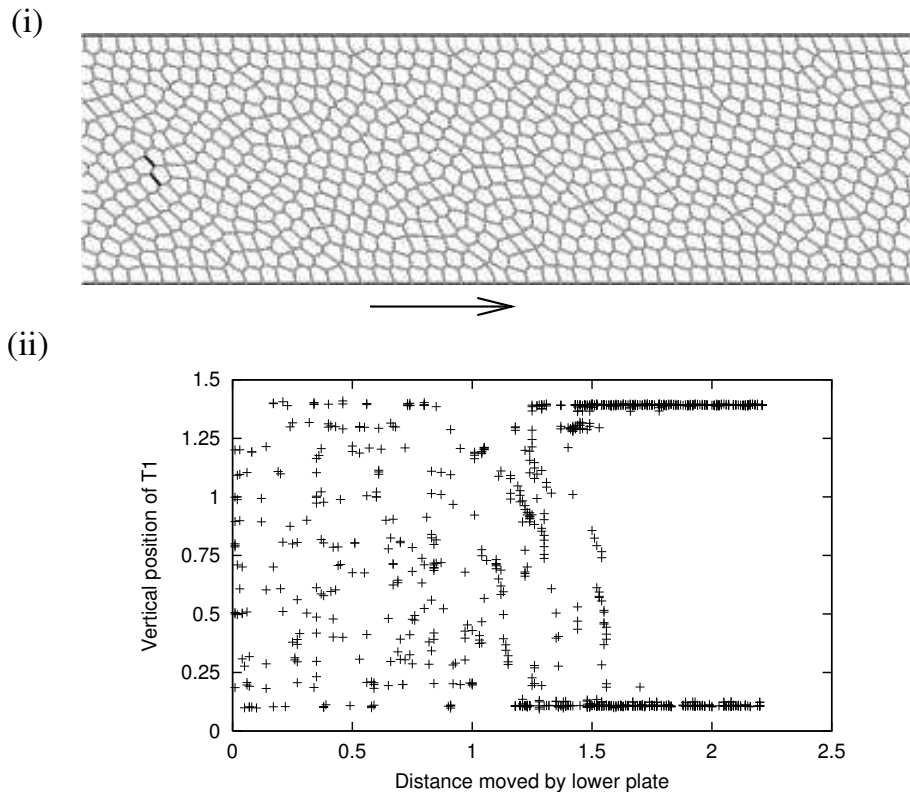
**Fig. 4.** The Bingham approximation for a yield-stress fluid indicates the correlation between macroscopic dissipation and local topological events. In this experiment of Glazier and co-workers (Jiang *et al* 2000, Asipauskas *et al* 2003) a foam flows steadily through a twenty-to-one contraction. The resulting distribution of T1s in the experiment (i) (showing the density of T1s per unit time per unit area, from Asipauskas *et al* (2003)) is comparable to the regions in which a Bingham fluid yields (ii) (obtained using the CFD package *Fluent* with slip boundary conditions).

yield stress  $\tau_y$  and viscosity  $\mu$  of a foam, and to model it as a *Bingham* fluid in which the stress  $\tau$  varies as:

$$\tau = \begin{cases} \tau_y & \tau \leq \tau_y \\ \tau_y + \mu \dot{\gamma} & \tau \geq \tau_y. \end{cases} \quad (2)$$

In this relationship the strain-rate is  $\dot{\gamma}$ . The variation of stress with strain can be generalised to a *Herschel-Bulkley* relationship with an arbitrary exponent  $n$  on the strain-rate. The yield-stress depends upon the liquid fraction, and the value of the exponent  $n$  is still debated.

We can solve the Navier-Stokes equations for a Bingham fluid and compare the results to experiments on flowing foams. In the *contraction* flow of Jiang *et al* (2000) (see also Asipauskas *et al* (2003)) a slow, steady flow of bubbles between two horizontal glass plates pushes through a 20:1 contraction (see figure 4). Image analysis gives both the streamlines of the flow and the positions of the T1s which dissipate the foam's excess energy. Comparing the simulated and experimental streamlines is not particularly useful, though the positions of the T1s and the areas in which the Bingham fluid yields correspond as in figure 4 (ii). A more meaningful comparison examines the structure of the foam itself (even in a quasi-static model which neglects viscosity and relaxes the energy between small step increments of strain). To motivate this study, we describe two recent experiments which reach conclusions concerning the nature of Couette shear on a 2D foam.



**Fig. 5.** (i) Moving the lower boundary shears a foam of 750 bubbles in a rectangular box periodic in the  $\hat{x}$  direction (quasi-static Surface Evolver calculation). (ii) After an initial transient, corresponding approximately to a unit shear, the T1s occur close to both the moving and the stationary plates.

### 3. Couette shear

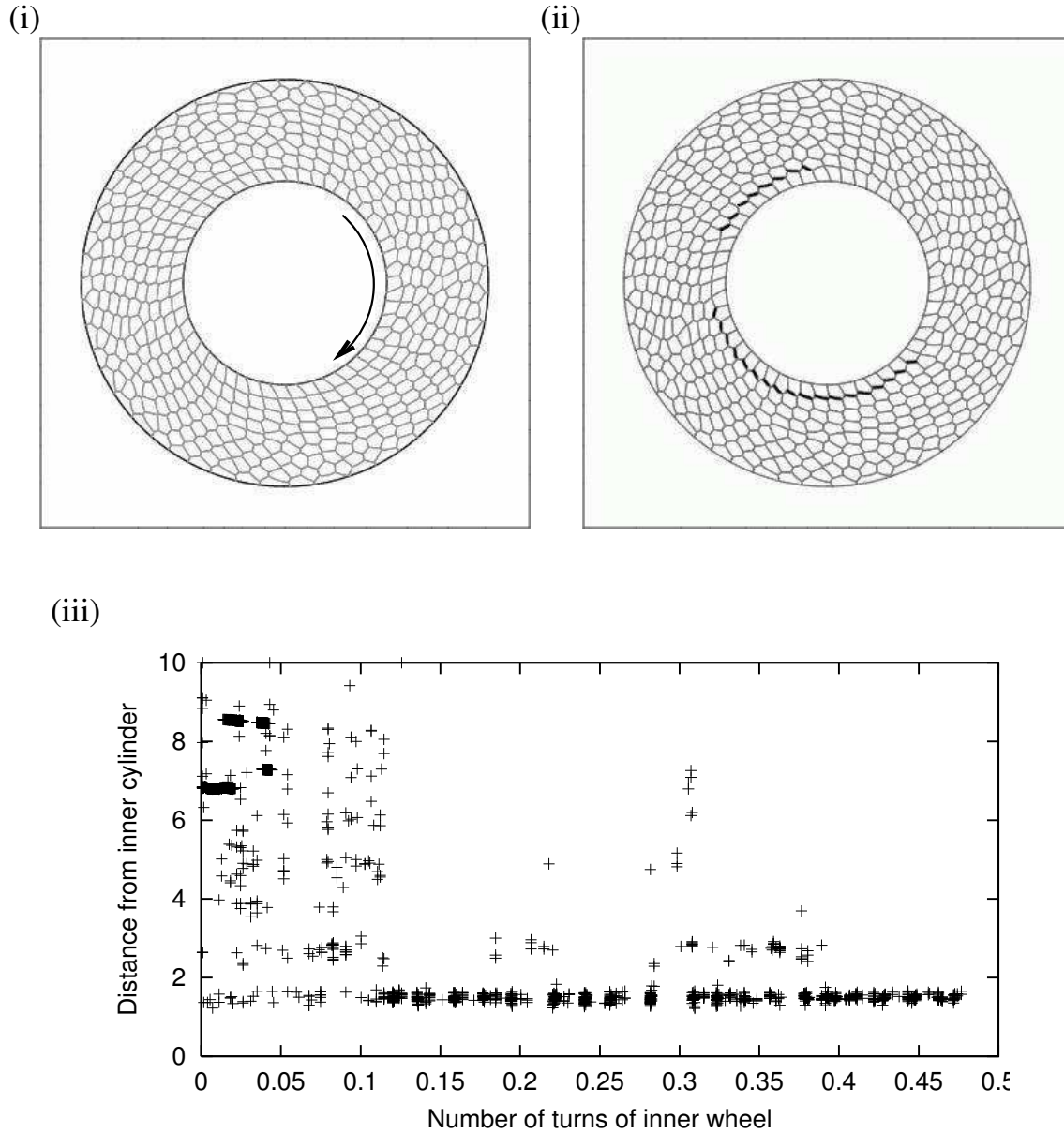
The Debregeas *et al* (2001) experiment traps a reasonably monodisperse (equal-area) foam between horizontal glass plates in the annular region between two cylinders *i.e.* in a Couette viscometer. The inner cylinder rotates in small increments, allowing ample time for full relaxation between each step, satisfying the requirements of the quasi-static approximation. The roughness of the surfaces pins the bubbles close to each of the cylinder walls. T1s almost all occur within about five bubble diameters of the inner (moving) cylinder. Such shear-banding or localization, although usual in granular materials, had not been previously been seen in foams. Lauridsen *et al* (2002)'s similar experiment using Bragg and Nye (1947)'s bubble-raft configuration moved the outer cylinder in a quasi-static fashion, and did not produce shear-banding. In both cases the foam's liquid fraction was about 8%, well away from either dry or wet limits. The unexplained discrepancy between these two experiments motivated our Couette simulations.

Perhaps the best tool for simulating equilibrium non-viscous dry foams is Brakke (1992)'s Surface Evolver. Surface Evolver minimizes the line-length of the foam structure subject to constraints, such as fixed bubble areas and incorporates a circular-arc mode, which models soap films precisely, rather than as a collection of short straight segments. A quasi-static simulation in Surface Evolver repeatedly applies a small strain increment followed by relaxation to equilibrium.

Kabla and Debregeas (2003) performed Surface Evolver-like simulations of Debregeas *et al* (2001)'s experiment using 768 bubbles with straight edges and slightly disperse areas with a markedly jagged lower boundary to a 2D rectangular domain periodic in one direction. They also found a shear-band in the inner layer of five bubbles.

To simulate the effect of the non-zero liquid fraction in the experiment, they introduced a cut-off film length below which they triggered a T1, mimicking the effect of the Plateau borders, which cause vertices to touch at a separation greater than a dry model would predict.

Our own quasi-static simulations using the Surface Evolver with vertices fixed to a flat boundary in the same rectangular domain as Kabla and Debregeas (2003) do not show localization. In agreement with naive analysis, the slip occurs only within one bubble width of *both* the outer and inner surfaces, as figure 5 shows. Why should symmetry break and the T1s occur close to only *one* wall? For monodisperse foams and smooth boundaries, the T1s do not propagate into the foam bulk, so we must vary the degree of surface roughness, the cut-off length used to trigger T1s, and the area-dispersity to seek conditions which produce shear-banding.



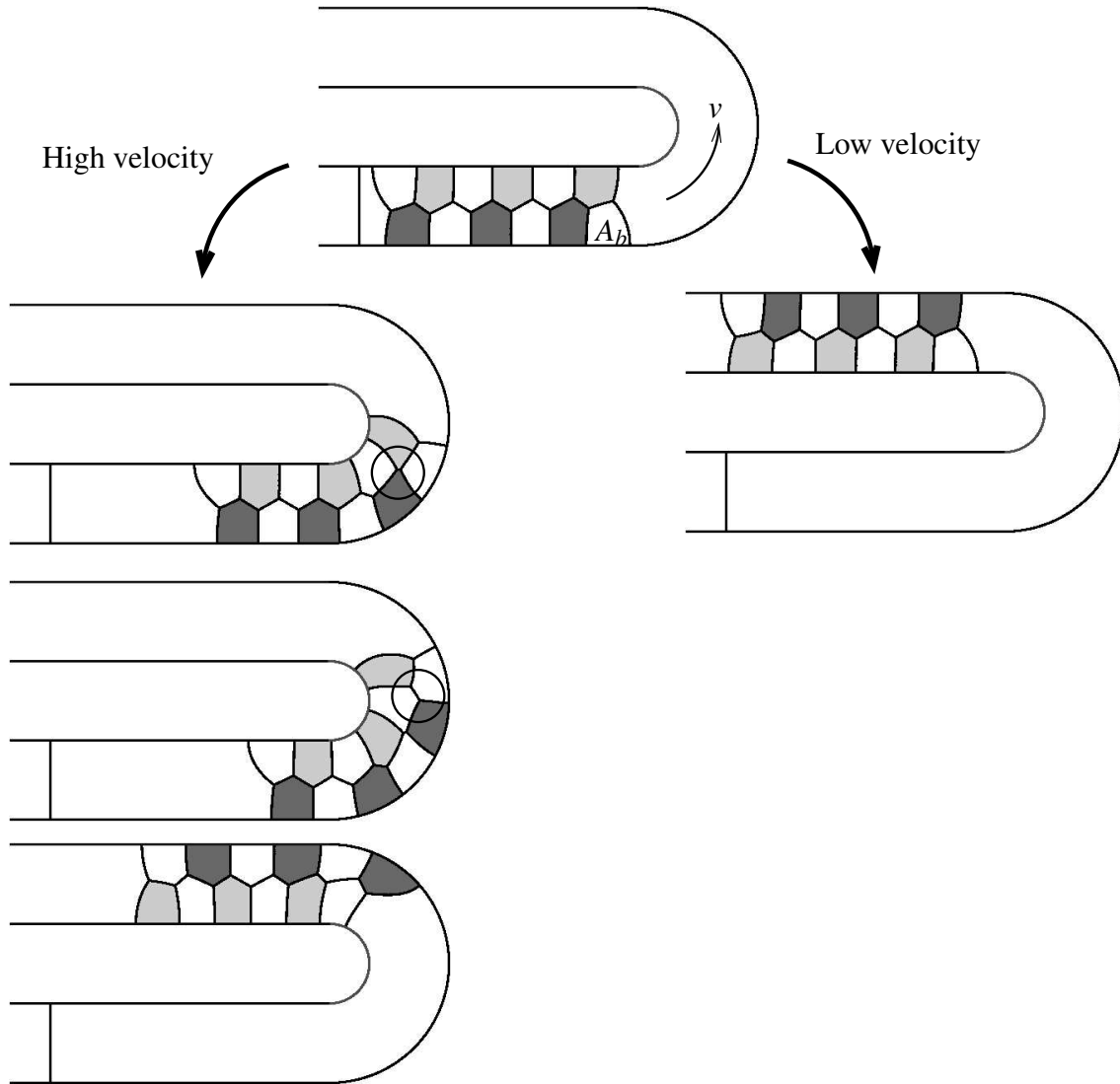
**Fig. 6.** The inner cylinder of a Couette viscometer rotates quasi-statically to strain a foam of 500 bubbles (Surface Evolver calculation). Energy dissipates close to the moving wall, in small avalanches of T1s. (i) and (ii) show consecutive images from the evolution: the black internal edges are those created by T1s. (iii) After an initial transient, the majority of the T1s lie close to the inner boundary of the cell.

To break symmetry we employ an annular geometry closely resembling the experimental configuration (figure 6). After an initial transient, when the foam relaxes towards a more ordered (predominantly hexagonal) state, T1s only occur within one bubble width of the smooth inner, moving wall.

Exploring so many free parameters (area dispersity, cut-off length, boundary profile *etc.*) requires many simulations. We have also neglected the effects of viscous dissipation. In slow Couette-shear experiments, viscosity should not be too important. But, as we show in the next section, explaining some experimental results requires us to include the viscous drag that soap films experience when they move across a surface.

#### 4. Viscous Froth

Unlike other models of viscous 2D foam flow, we aim to develop a predictive simulation tool that accurately represents both the curvature of the soap films and the viscous forces. Rather than solving the Laplace equation (1) for



**Fig. 7.** Viscous froth simulations of a 2D ordered foam flowing around a sample  $180^\circ$  bend. The initial configuration is at the top. Low velocities (right-hand-side) produce no T1s and the final structure is identical to that for a quasi-static calculation. At higher velocities (left-hand-side) the bubbles successively swap neighbours due to a T1 (circled) as they pass close to the apex of the bend. As the bubble size increases, the topological transformation occurs at lower velocities.

each film, we now use it to determine the velocity  $v$  of the film according to (Kern *et al* 2003),

$$\Delta p - \gamma\kappa = \lambda v. \quad (3)$$

The parameter  $\lambda$  includes the viscous dissipation due to dragging the film along a surface (figure 3). Bretherton (1961) showed that the velocity  $v$  in (3), which is normal to the film, should be raised to the  $2/3$  power. First we consider the linear case, which is easier to solve, yet agrees qualitatively with experiment (Kern *et al* 2003).

We have applied the viscous froth model to T1 dynamics. Our experiments on the flow of ordered foam structures in 2D channels (Weaire *et al* 2003) include the situation in figure 7 in which a 211 staircase structure moves around a  $180^\circ$  bend. At low velocities the structure flows elastically around the bend, while at higher velocities it experiences a single T1 at the apex of the bend.

Quasi-static calculations failed to predict this T1, so we implemented the viscous froth equation (3) on a discretized network of films, using Surface Evolver for all necessary book-keeping (such as deleting short edges and keeping track of the discretization after T1s) and to include adherence to the channel walls. Kern *et al* (2003) describe an alternative implementation for a periodic 2D foam. Glazier *et al.* have also used the Potts model to replicate these results.

Our simulations to ascertain the critical velocity as a function of bubble area  $A_b$  at which T1s begin suggest that  $v$  scales as  $A_b^{-1}$ .

To establish a direct correspondence between the 2D viscous froth model and a particular physical realization we will have to include nonlinear viscous drag on the top and bottom plates and side-wall drag. Additional viscous effects, associated with the elongation of films, result in deviations from the  $120^\circ$  angles at vertices (Reinelt and Kraynik 1989) and may be the dominant dissipative mechanism in three dimensions. In 2D the large viscous drag on the confining surfaces probably dominates. Since any additional effects due to film elongation must depend upon the height of the foam sample, careful measurement of vertex angles might help to validate our model.

## 5. Summary

2D experiments and simulations provide an excellent environment in which to explore foam dynamics. Simulation using the new generation of viscous froth models shows close qualitative agreement with experiment and accurately predicts dissipation events. We intend to use this simulation technique to predictively design these and other experiments. Industrial applications include micro-fluidics, in particular the transport of small separated volumes of liquid using bubbles.

What are the difficulties in three dimensions? While visualization is difficult, due to the opacity of liquid foams, the sophisticated experiments of Rouyer *et al* (2003) allow comparison with Surface Evolver simulations, such as those of Reinelt and Kraynik (1993, 1996).

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## References

- Asipauskas, M., Aubuoy, M., Glazier, J.A., Graner, F. and Jiang, Y. 2003. A texture tensor to quantify deformations: the example of two-dimensional flowing foams. *Granular Matter* **5**: 71–74.
- Bolton, F. and Weaire, D. 1992. The effects of Plateau borders in the two-dimensional soap froth .II. General simulation and analysis of rigidity loss transition. *Phil. Mag. B* **65**: 473–487.
- Bragg, L. and Nye, J.F. 1947. A dynamical model of a crystal structure. *Proc. R. Soc. Lond.* **A190**: 474–481.
- Brakke, K. 1992. The Surface Evolver. *Exp. Math.* **1**: 141–165.
- Bretherton, F.P. 1961. The motion of long bubbles in tubes. *J. Fluid Mech.* **10**: 166–188.
- Cantat, I. and Delannay, R. 2003. Dynamical transition induced by large bubbles in two-dimensional foam flows. *Phys. Rev. E* **67**: 031501.
- Cox, S.J., Vaz, M.F. and Weaire, D. 2003. Topological changes in a two-dimensional foam cluster. *Eur. Phys. J. E* **11**: 29–35.
- Debregeas, G., Tabuteau, H. and di Meglio, J.M. 2001. Deformation and flow of a two-dimensional foam under continuous shear. *Phys. Rev. Lett.* **87**: 178305.
- Durian, D.J. 1995. Foam Mechanics at the Bubble Scale. *Phys. Rev. Lett.* **75**: 4780–4783.
- Herttle, T. and Aref, H. 1992. Numerical experiments on 2-dimensional foams. *J. Fluid Mech.* **241**: 233–260.
- Hutzler, S., Weaire, D. and Bolton, F. 1995. The effects of Plateau borders in the two-dimensional soap froth .III. Further results. *Phil. Mag. B* **71**: 277–289.
- Jiang, Y., Asipauskas, M., Glazier, J.A., Aubuoy, M., Graner, F. and Jiang, Y. 2000. *Ab initio* Derivation of Stress and Strain in Fluid Foams. In P. Zitha, J. Banhart and G. Verbist, (eds), *Foams, emulsions and their applications*. MIT-Verlag, Bremen pp. 297–304.
- Jiang, Y., Swart, P.J., Saxena, A., Asipauskas, M. and Glazier, J.A. 1999. Hysteresis and Avalanches in two-dimensional foam Rheology simulations. *Phys. Rev. E* **59**: 5819–5832.
- Kabla, A. and Debregeas, G. 2003. Local stress relaxation and shear banding in a dry foam under shear. *Phys. Rev. Lett.* **90**: 258303.
- Kermode, J.P. and Weaire, D. 1990. 2D-FROTH: a program for the investigation of 2-dimensional froths. *Comp. Phys. Commun.* **60**: 75.
- Kern, N., Weaire, D., Martin, A., Hutzler, S. and Cox, S.J. 2003. The two-dimensional viscous froth model for foam dynamics. *Submitted to Phys. Rev. E*.
- Kraynik, A.M. 1988. Foam flows. *Ann. Rev. Fluid Mech.* **20**: 325–357.

- Lauridsen, J., Twardos, M. and Dennin, M. 2002. Shear-induced stress relaxation in a two-dimensional wet foam. *Phys. Rev. Lett.* **89**: 098303.
- Okuzono, T., Kowasaki, K. and Nagai, T. 1993. Rheology of Random Foams. *J. Rheol.* **37**: 571–586.
- Plateau, J.A.F. 1873. *Statique Expérimentale et Théorique des Liquides Soumis aux Seules Forces Moléculaires*. Gauthier-Villars, Paris.
- Princen, H.M. 2000. The Structure, Mechanics and Rheology of Concentrated Emulsions and Fluid Foams. In Sjöblom, J. (eds), *Encyclopedic Handbook of Emulsion Technology*. Marcel Dekker, New York. pp. 243–278.
- Reinelt, D.A. and Kraynik, A.M. 1989. Viscous Effects in the Rheology of Foams and Concentrated Emulsions. *J. Colloid Interf. Sci* **132**: 491–503.
- Reinelt, D.A. and Kraynik, A.M. 1993. Large Elastic Deformations of Three-Dimensional Foams and Highly Concentrated Emulsions. *J. Colloid Interf. Sci* **159**: 460–470.
- Reinelt, D.A. and Kraynik, A.M. 1996. Simple shearing flow of a dry Kelvin soap foam. *J. Fluid Mech.* **311**: 327–342.
- Rouyer, F., Cohen-Addad, S., Vignès-Adler, M. and Höhler, R. 2003. Dynamics of yielding observed in a three-dimensional aqueous dry foam. *Phys. Rev. E* **167**: 021405.
- Smith, C.S. 1952. Grain shapes and other metallurgical applications of topology. In *Metal Interfaces*. American Society for Metals, Cleveland, OH. pp. 65–108.
- Vaz, M.F. and Fortes, M.A. 1997. Experiments on defect spreading in hexagonal foams. *J. Phys.: Condens. Matter* **9**: 8921–8935.
- Weaire, D. and Hutzler, S. 1999. *The Physics of Foams*. Clarendon Press, Oxford.
- Weaire, D., Hutzler, S., Cox, S., Kern, N., Alonso, M.D. and Drenckhan, W. 2003. The Fluid Dynamics of Foams. *J. Phys.: Condens. Matter* **15**: S65–S73.
- Weaire, D. and Kermode, J.P. 1983. Computer simulation of a two-dimensional soap froth I. Method and motivation. *Phil. Mag. B* **48**: 245–249.