Properties of three-dimensional bubbles of constant mean curvature

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Abstract

Foam bubbles of constant mean curvature with a number of faces between 3 and 32 have been computer-generated. The surface area per unit volume is only weakly dependent upon the number of faces, which allows an approximate equation for the energy of a three-dimensional foam to be derived. A further relation between the rate of change of the volume of a bubble due to gas diffusion and its number of faces is compared to existing theory.

It was recently argued by Graner *et al.* (2001) that the surface energy E of an unstrained two-dimensional (2D) dry foam consisting of many bubbles with given areas is almost independent of topology. Here the energy is equal to the product of the perimeter of the films and the film tension: $E = P\gamma$. Graner *et al.* (2001) then proposed a simple equation to estimate P from the areas A_i of the individual bubbles in a large unstrained cluster, namely

$$P \approx \frac{3.72}{2} \sum_i \sqrt{A_i}. \tag{1}$$

This is based upon calculations of the ratio $e(n) = P/\sqrt{A}$ for individual 'regular' bubbles consisting of n identical circular films meeting internally at 120°. For these $n-$sided bubbles, e is weakly dependent on n, changing monotonically from $e \approx 3.78$ for $n = 2$ to $e \approx 3.71$ as $n \to \infty$. The value in (1) is $e(n = 6)$. Note that the factor of $\frac{1}{2}$ appears because each film is counted twice. If we denote their (constant) curvature by κ, then the geometry of these regular 2D bubbles is such that

$$\sqrt{A}\kappa = \frac{\pi(n-6)}{3e(n)} \approx 0.282(n-6). \tag{2}$$

Vaz *et al.* (2002) have shown that (1) provides a good estimate of the energy of large 2D free bubble clusters. Its accuracy increases as the widths of the distributions of n and A_i decrease. It should also apply to unbounded periodic clusters in their unstrained minimum-energy configuration.

Our aim is to investigate the variation of the surface area S of 3D bubbles with unit volume and constant mean curvature, i.e. the 3D counterpart of the 2D regular bubbles. If the area is weakly dependent on the number of faces, as in 2D, then we may write the following simple equation to estimate the energy of a large unstrained 3D foam:

$$E = \gamma\, S \approx \gamma\, \frac{\lambda}{2} \sum_i V_i^{2/3} \tag{3}$$

for some constant λ which is independent of the number of faces F. Note that the film tension γ can be taken as twice the liquid surface tension. We shall determine λ from calculations using the Surface Evolver (Brakke 1992). Previous authors (Monnereau *et al.* 2000, Kraynik *et al.* 2002) have used the Evolver to simulate foams in order to find the curvatures of the bubbles. We take a different approach, and simulate a single bubble with constant curvature, which we believe to be representative of bubbles with the same number of faces in an unstrained cluster. Some guidance in the values of λ to be expected is given by the calculations of Kraynik *et al.* (2002) on monodisperse random foams: they find values of λ between 5.288 and 5.380. The Kelvin tetrakaidecahedron with fourteen faces of zero mean curvature has $\lambda = 5.306$, and the Weaire-Phelan foam has $\lambda = 5.288$ (Kraynik *et al.* 2002).

[Insert table 1 about here]

In contrast to 2D, for given F there are, in general, various (usually non-regular) topologies for a (trivalent) bubble (i.e. with three edges meeting at each vertex). We have chosen to analyse bubbles which, in their unrelaxed planar version, have edges of the same length and, where possible, have identical faces (i.e. with the same number of edges per face). The topology of these bubbles is indicated for each F in table 1 with the usual notation (e.g. $4_6 6_8$ indicates a bubble with six square faces and eight hexagonal faces). For $F = 3, 4, 6$ and 12 these are 3D regular bubbles, with identical faces meeting at $120°$ and equal edge lengths. We have also included the analytic calculations for a two-sided bubble, composed of a pair of spherical caps, and the 'ideal' regular plane-faced bubble (Isenberg 1992) which has 13.39 faces. For $F \geq 14$ each equi-curvature bubble has faces with two or three different numbers of sides, and those with fewer sides tend to be smaller.

[Insert figure 1 about here]

For various values of F between 3 and 32, as indicated in table 1, we start with the plane-faced polyhedron and use the Surface Evolver to minimize the area of the films for fixed volume. Each bubble is created by connecting it, with films of the same tension, to a frame with the same topology, as illustrated for a cube in figure 1 (taken from the Evolver example given by Brakke – see `http://www.susqu.edu/brakke/evolver/examples/examples.htm`). The pressure outside the bubble is defined to be zero. The resulting bubble has faces with the same constant mean curvature that meet at $120°$; we shall refer to them as equi-curvature bubbles. We show these (relaxed) bubbles in figure 2, with the exception of the ideal bubble.

[Insert figure 2 about here]

There is a slight difficulty: we seek the value of $\lambda = S/V^{2/3}$, which we expect to be independent of the volume V. By introducing a frame, however, we introduce into the calculation another length scale, that of the side-length of the frame. This is not significant for the regular bubbles ($F = 3, 4, 6, 12$) due to their symmetry. But for the non-regular bubbles we find that as the volume of the bubble increases, so as to eventually fill the frame, the value of λ decreases slightly. This is because for non-regular bubbles the films connecting the bubble to the frame are not necessarily perpendicular to the bubble; the most straightforward way to make the films close to being perpendicular is to reduce the gap between bubble and frame. We therefore take the minimum value of λ, corresponding to a bubble that almost fills the frame.

The values of $\lambda = S/V^{2/3}$, $L/V^{1/3}$ and $S\kappa/V^{1/3}$ are given in table 1, for surface area S, total edge-length L and mean curvature $\kappa = p/(2\gamma)$, where p is the internal pressure of the bubble. For $F = 2$ the analytical values are $S/V^{2/3} = (1152\pi/25)^{1/3}$, $L/V^{1/3} = \sqrt{3}(12\pi^2/5)^{1/3}$ and $S\kappa/V^{1/3} = (96\pi^2/5)^{1/3}$. For the ideal bubble, $F = 13.39$, the values are found by treating the bubble as a regular polyhedron with the required tetrahedral angle at the vertices (see the reference to the work of Ogeé in Phelan (1994)). If a is the edge length and $\eta_F = 6 - 12/F$ the number of edges per face, then the surface area of the ideal bubble is (Zwillinger 1996)

$$S = \frac{F\eta_F\, a^2}{4} \cot\left(\pi/\eta_F\right) \tag{4}$$

and the volume is

$$V = \frac{F\eta_F\, a^3}{24} \frac{\cot^2\left(\pi/\eta_F\right)}{\sqrt{4\sin^2\left(\pi/\eta_F\right) - 1}}. \tag{5}$$

The results show that $S/V^{2/3}$ is fairly constant: for the regular bubbles it is always close to 5.27, while the average over all equi-curvature bubbles is $\lambda = 5.29$. Note that $S/V^{2/3}$ does not change monotonically with F as $P/\sqrt{A}$ does with n in 2D. Indeed, the 32-faced bubble approaches very closely the value for the pentagonal dodecahedron. We have excluded the two bubbles whose values are calculated analytically, $F = 2$ and $F = 13.39$, as unrealistic, although they show the lowest values of $S/V^{2/3}$. The latter is of course not a 'real' bubble, since it has a non-integer number of faces. The two-faced bubble presents an interesting question: in two-dimensions, it has been shown that a two-sided bubble is metastable (Weaire and Kermode 1983), and lowers its energy by merging into a vertex and gaining a side. In 3D, the stability of the bubble will depend upon the Gaussian curvature of the surface in which it resides, and it may be that the two-faced bubble will be (globally) stable in a region of the surface with an extremum in Gaussian curvature.

We can therefore write (3) as

$$S \approx \frac{5.29}{2} \sum_i V_i^{2/3}, \tag{6}$$

which is the 3D counterpart of (1). We again expect that the accuracy of (6) for a 3D bubble cluster increases as the width of the distributions of F and V_i decrease.

We can also find the dependence of the total edge length of the equi-curvature bubbles on the number of faces F. We expect that the average face area $\bar{A}$ will scale with S/F and the perimeter of each face will scale with $\sqrt{\bar{A}} = \sqrt{S/F}$. Thus the total edge length L should be proportional to $\sqrt{S\ F}$. In figure 3 we therefore plot

$$\frac{(L/V^{1/3})}{(\sqrt{F\ S/V^{2/3}})} = \frac{L}{\sqrt{FS}}$$

against F. It is almost constant, and close to the prefix of equation (1), $3.72/2 = 1.86$, suggesting that each face of an equi-curvature bubble is well approximated by a regular planar 2D bubble. The only exception is the two-faced bubble, since each face has only one edge.

[Insert figure 3 about here]

The results of table 1 can be used to evaluate the rate of change of volume of an equi-curvature bubble with F faces due to gas diffusion to and from the surrounding bubbles. This is the problem of coarsening of three-dimensional foams. Coarsening laws in 3D are not exact, though many attempts have been made to find a suitable approximation (Glazier 1993, Weaire and Glazier 1993, Sire 1994). Hilgenfeldt *et al.* (2001) have shown how to use a result of Minkowski for convex polyhedra, which relates the curvature of a face to the caliper radius. They find the following equation for the rate of change of volume of a 3D bubble:

$$\frac{\mathrm{d}}{\mathrm{d}t} V^{2/3} = \frac{3}{2^{1/3}} \left[(F-2)\tan\left(\frac{\pi}{\eta_F}\right) \right]^{2/3} \tan^{1/3}\left(\frac{\chi_F}{2}\right) \left(\frac{\pi}{3} - \chi_F\right) \tag{7}$$

where $\chi_F = 2\tan^{-1}\sqrt{4\sin^2(\pi/\eta_F) - 1}$ and η_F is as above. Rather than the linear dependence on F usually suggested for the coarsening problem, this expression approaches square-root behaviour for large F.

For a 2D bubble with n sides, the product of perimeter and curvature is

$$P\,\kappa = \frac{\pi}{3}(n-6) \tag{8}$$

which allowed von Neumann (1952) to express the rate of change of the area of a bubble as

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \nu_2(n-6) \quad \text{where } \nu_2 = \frac{\pi}{3}\xi\gamma. \tag{9}$$

This equation is exact, and applies to any bubble (not necessarily regular). ξ is the permeability of the films which make up the interface (i.e. the volume of gas which diffuses across a unit area of film in unit time due to a unit pressure difference).

We shall now perform the same procedure to try and find an analogous law for 3D bubbles, using the values of $S\kappa/V^{1/3}$ from table 1. The straight-line fit, shown in figure 4, is

$$\frac{S\kappa}{V^{1/3}} \approx -0.311(F-F_0) \quad \text{with } F_0 = 15.86. \tag{10}$$

Since its curvature is the same everywhere, the rate of change of volume of an equi-curvature bubble in 3D is

$$-\frac{\mathrm{d}V}{\mathrm{d}t} = S\kappa\,\xi\gamma. \tag{11}$$

Thus

$$-\frac{\mathrm{d}}{\mathrm{d}t}V^{2/3} = -\frac{2}{3}V^{-1/3}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{2}{3}\frac{S\kappa}{V^{1/3}}\,\xi\gamma. \tag{12}$$

This leads to the following coarsening law for 3D bubbles:

$$\frac{\mathrm{d}}{\mathrm{d}t}V^{2/3} \approx \nu_3(F-15.86) \quad \text{where } \nu_3 = 0.207\,\xi\gamma. \tag{13}$$

The value of F_0 is slightly higher than the experimental value of 13.5 found by Monnereau and Vignes-Adler (1998), who also find a linear dependence on F, but in close agreement with the Monte Carlo (Potts model) simulations of Glazier (1993) with $F_0 = 15.8$. The straight-line fit provides a useful quick estimate for mean-field simulations of coarsening.

[Insert figure 4 about here]

Our data show good agreement with the formula (7) due to Hilgenfeldt *et al.* (2001), also shown in figure 4. As they point out, their approximations break down for small F, where there is significant deviation from our data. Finally, note that an accurate coarsening law must depend upon the second moment of the distribution of faces, $\mu_2 = \langle F^2\rangle - \langle F\rangle^2$, for which we can suggest no correction based upon the present work.

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Table and Figure captions

Table 1: Surface area S, total edge length L and mean curvature κ data for the equi-curvature bubbles of figure 2. Each bubble has F faces and topology denoted by a_b, meaning b faces with a edges. The first six are regular bubbles, i.e. the faces are identical. The values for a two-sided bubble and the ideal bubble are based on analytic calculations. The last four bubbles are relaxed starting from polyhedra with equal edge lengths.

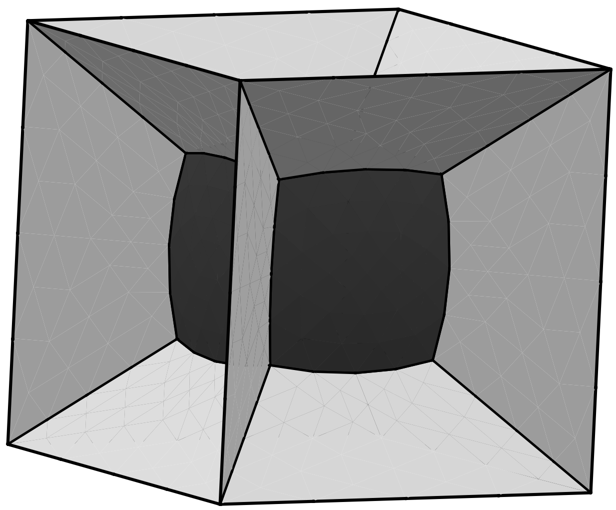
Figure 1: An example, for $F = 6$, of the frame used to generate the equi-curvature bubbles with the Surface Evolver. All films have the same tension.

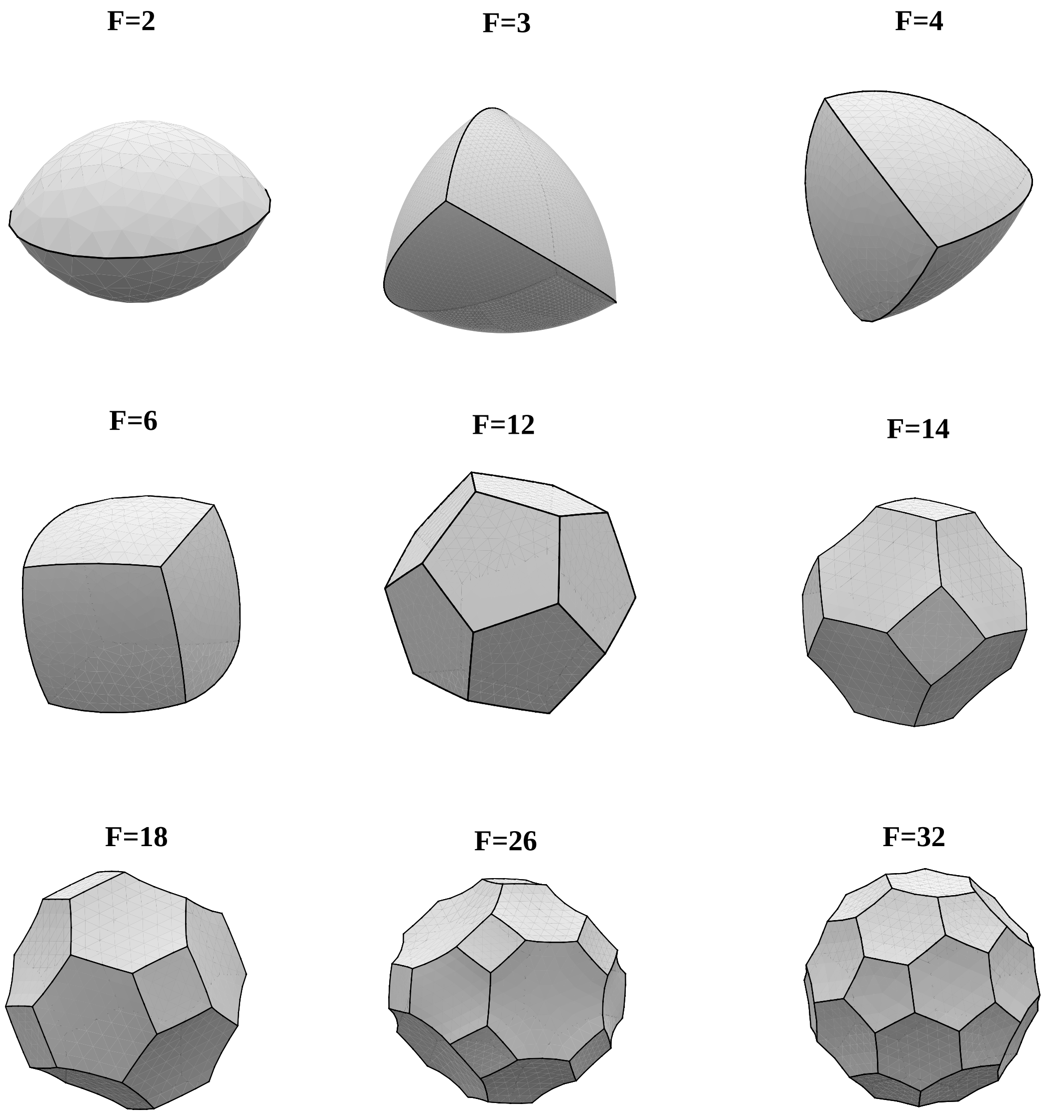
Figure 2: Equi-curvature bubbles with given number of faces F.

Figure 3: The scaled edge length of an equi-curvature bubble with more than two faces is remarkably close to the value 1.86 for a plane 2D regular bubble from (1).

Figure 4: The rate of change of volume of the equi-curvature bubbles. Also shown is the straight-line fit (13) and the theoretical formula (7).

	F	Topology	$S/V^{2/3}$	$L/V^{1/3}$	$S\kappa/V^{1/3}$
Paired spherical caps	2	1_2	5.251	4.974	5.744
Trihedron	3	2_3	5.316	7.303	4.656
Tetrahedron	4	3_4	5.287	8.526	3.970
Cube	6	4_6	5.272	10.610	2.848
Pentagonal Dodecahedron	12	5_{12}	5.256	15.088	0.452
Ideal bubble	13.39	$5.10_{13.39}$	5.254	15.940	0.000
Tetrakaidecahedron	14	$4_6 6_8$	5.334	16.002	0.092
Truncated Rhombic Dodecahedron	18	$4_6 6_{12}$	5.301	18.237	−1.138
Rhombitruncated Cubeoctahedron	26	$4_{12} 6_8 8_6$	5.410	21.284	−2.613
Truncated Icosahedron	32	$5_{12} 6_{20}$	5.261	24.372	−4.458

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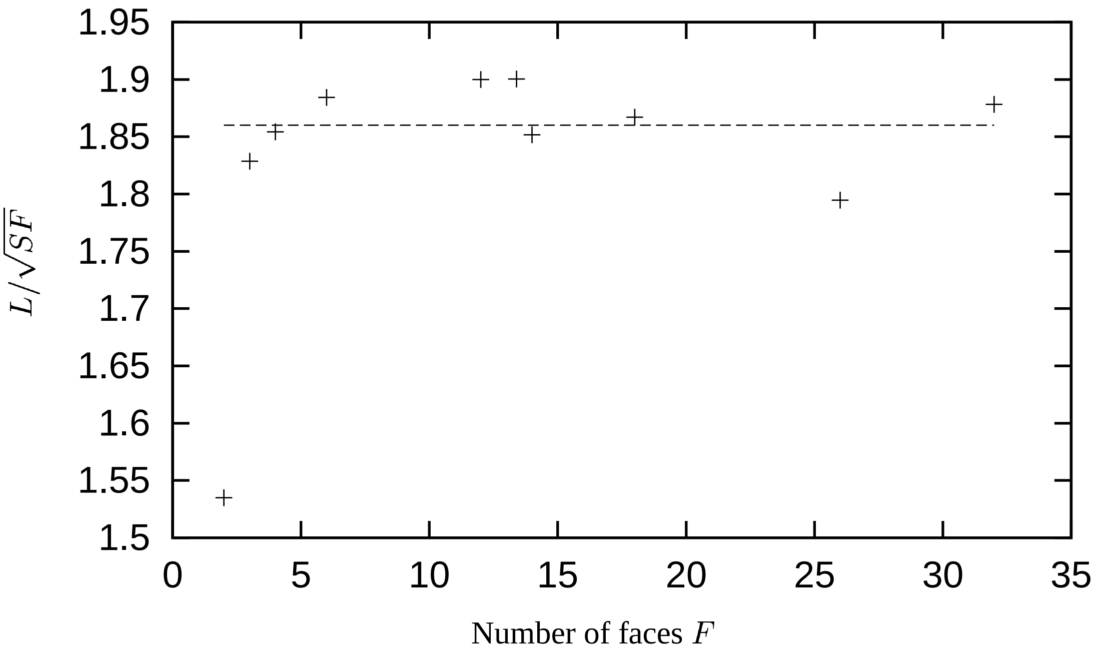
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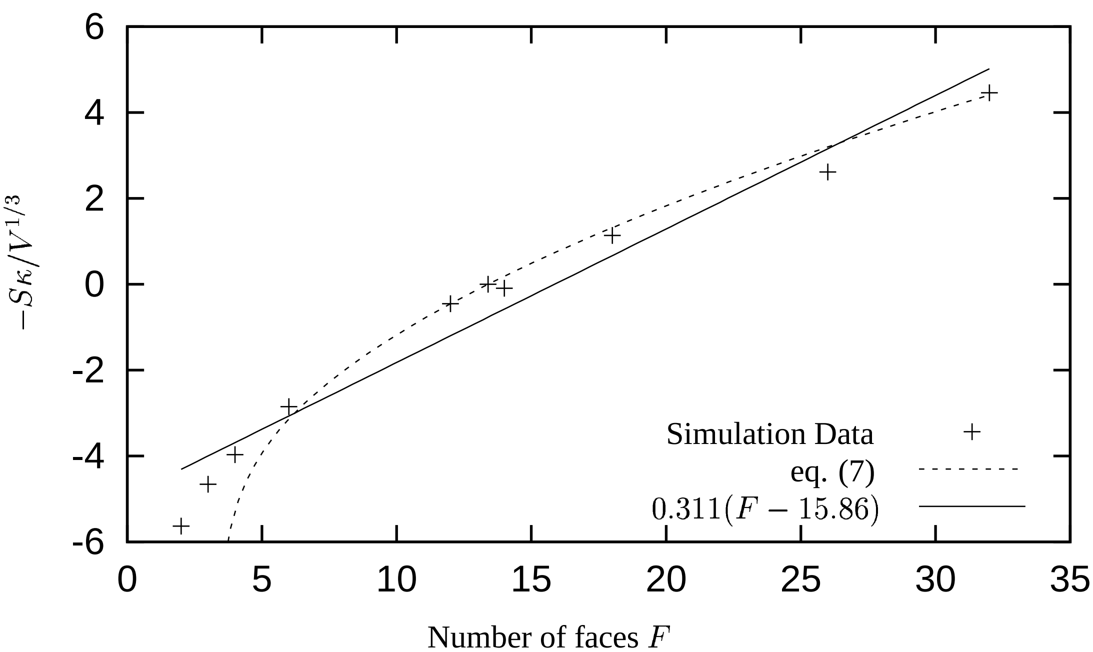

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$-S\kappa/V^{1/3}$
Number of faces F
Simulation Data
eq. (7)
$0.311(F - 15.86)$