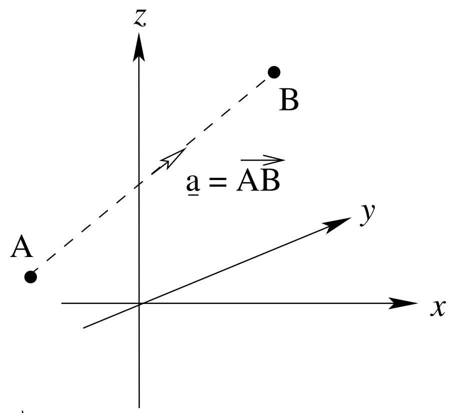
2 Vectors

Quantities that can be defined by a single number, such as temperature and speed, are known as *scalars*. This is not true of quantities like force and velocity, which, in addition to magnitude (size), have a *direction*. These are vectors. Since they describe such fundamental things, it is worth understanding how to interpret and manipulate them. We also extend our analysis to three-dimensional space.

2.1 Basic concepts

Definition 2.1.
A vector is a directed line segment. The directed line segment $\overrightarrow{AB}$ has initial point A and end point B.
A vector is often written as an underlined *(or* **bold***) lower-case letter, e.g. $\overrightarrow{AB} = \underline{a}$ or $\overrightarrow{AB} = \boldsymbol{a}$.*



The length of a vector is denoted $|\overrightarrow{AB}| = |\underline{a}|$. If $\underline{a}$ has components (a_1, a_2, a_3) in the x, y, and z directions respectively, then its length (or modulus, or magnitude) is $|\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. A vector with modulus equal to one is called a unit vector, sometimes written with a hat: $\hat{\underline{a}}$. By definition, for any vector $\hat{\underline{a}} = \dfrac{\underline{a}}{|\underline{a}|}$.

Example 2.1. *Find a unit vector in the direction $\underline{a} = \underline{i} + 2\underline{j}$.*

On Cartesian axes $Oxyz$, we can resolve any vector into its components in the three coordinate directions. We write $\underline{i}$, $\underline{j}$, and $\underline{k}$ for unit vectors lying along Ox, Oy, and Oz respectively; e.g. $\underline{i}$ goes from (0,0,0) to (1,0,0). Then if A is any point with coordinates (a_1, a_2, a_3), its position vector *is $\underline{a} = \overrightarrow{OA} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$.*

Definition 2.2. *Two vectors are equal iff they have the same length and direction. In component form, this means that the vectors $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ are equal iff $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$. If $\underline{a} = \underline{b}$ then $\underline{b} - \underline{a} = \underline{0}$, the zero vector.*

Example 2.2. *If A is the point with position $(1, 1)$, with $\overrightarrow{OA} = \underline{i} + \underline{j}$ and B is at $(2, 3)$ with position vector $\overrightarrow{OB} = 2\underline{i} + 3\underline{j}$, then $\overrightarrow{AB} = \underline{i} + 2\underline{j}$. If we write $\underline{c} = \overrightarrow{AB}$ then in components $c_1 = 1$ and $c_2 = 2$.*

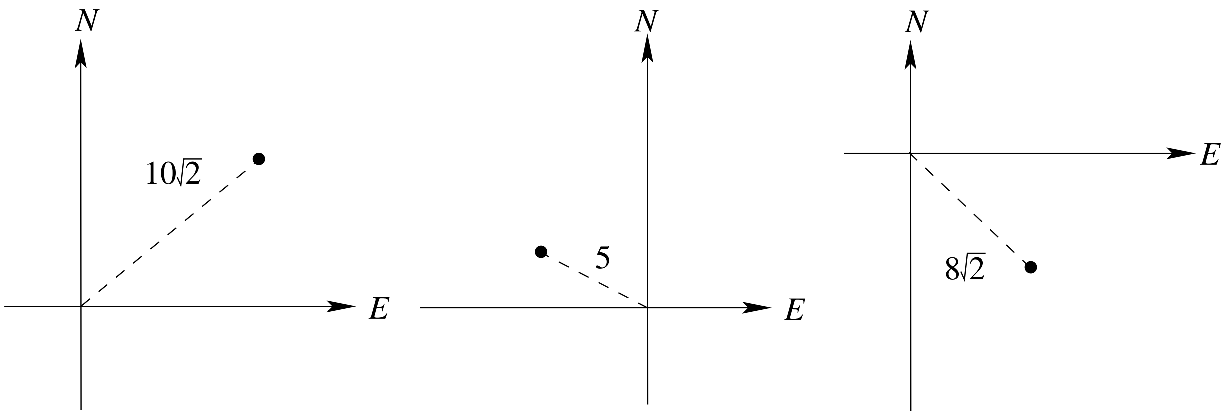
Remark: How is $\overrightarrow{BA}$ related to $\overrightarrow{AB}$? These two vectors should have the same length but opposite direction. If $\overrightarrow{AB} = \underline{a}$, then $\overrightarrow{BA} = -\underline{a}$, or $\overrightarrow{BA} = -\overrightarrow{AB}$.

Definition 2.3. *Two vectors are* parallel *if one is a scalar multiple of the other; i.e. $\underline{a}$ is parallel to $\underline{b}$ if $\underline{b} = \lambda\underline{a}$ for some $0 \neq \lambda \in \mathbb{R}$.*
If $\lambda > 0$, the vectors are parallel and in the same direction.
If $\lambda < 0$, the vectors are parallel and in opposite directions (antiparallel).

Example 2.3. *Given vectors* $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$, $\underline{b} = 2\underline{i} + \underline{k}$, *and* $\underline{c} = -\underline{i} + 3\underline{j} + 5\underline{k}$, *then*
(a) $\underline{a} + \underline{b} = 3\underline{i} + 2\underline{j} + 4\underline{k}$;
(b) $2\underline{a} - \underline{b} = 4\underline{j} + 5\underline{k}$;
(c) $\underline{a} - \underline{b} + \underline{c} = -2\underline{i} + 5\underline{j} + 7\underline{k}$;
(d) the unit vector $\hat{\underline{b}} = \frac{2}{\sqrt{5}}\underline{i} + \frac{1}{\sqrt{5}}\underline{k}$.

Example 2.4. *Given vectors* $\underline{a} = \underline{i} + \underline{j} - 2\underline{k}$, $\underline{b} = \underline{i} + \underline{k}$, *and* $\underline{c} = 2\underline{i} - \underline{j} + 3\underline{k}$, *then*
(a) $\underline{a} + \underline{b} - 3\underline{c} = -4\underline{i} + 4\underline{j} - 10\underline{k}$;
(b) $|\underline{a} + \underline{b} + \underline{c}| = |4\underline{i} + 2\underline{k}| = \sqrt{16 + 4} = \sqrt{20}$;
(c) $\underline{a} - 2\underline{b} + \underline{c} = \underline{i} - \underline{k}$;
(d) $|2\underline{a} + \underline{b} + 2\underline{c}| = |7\underline{i} + 3\underline{k}| = \sqrt{49 + 9} = \sqrt{58}$;
(e) unit vectors in the directions of $\underline{a}$, $\underline{b}$ *and* $\underline{c}$ *are* $\frac{1}{\sqrt{6}}\underline{i} + \frac{1}{\sqrt{6}}\underline{j} - \frac{2}{\sqrt{6}}\underline{k}$, $\frac{1}{\sqrt{2}}\underline{i} + \frac{1}{\sqrt{2}}\underline{k}$, *and* $\frac{2}{\sqrt{14}}\underline{i} - \frac{1}{\sqrt{14}}\underline{j} + \frac{3}{\sqrt{14}}\underline{k}$ *respectively.*

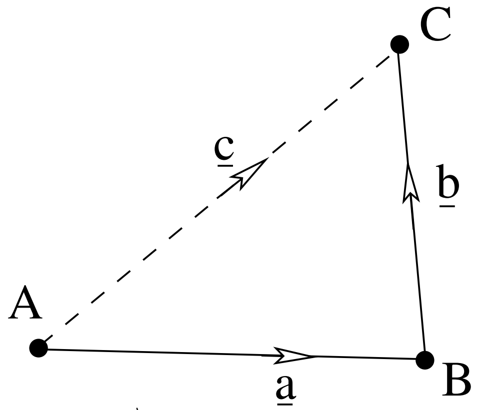
Example 2.5. *Express the following vectors in terms of* $\underline{i}$ *and* $\underline{j}$, *unit vectors due east and north respectively.*
(a) $10\sqrt{2}$ *units in a direction north east;*
(b) 5 units in a direction N 60° *W;*
(c) $8\sqrt{2}$ *units in a direction N* 135° *E.*



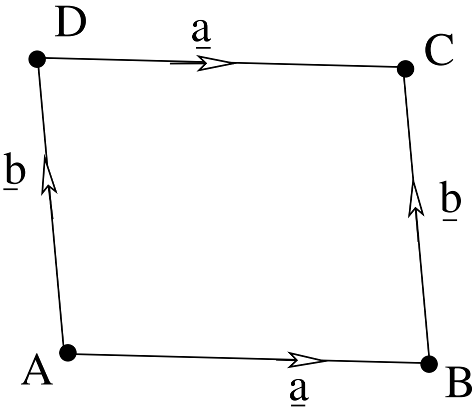
Solutions: (a) $10\underline{i} + 10\underline{j}$; (b) $-5\cos(30^\circ)\underline{i} + 5\sin(30^\circ)\underline{j} = -\frac{5\sqrt{3}}{2}\underline{i} + \frac{5}{2}\underline{j}$; (c) $8\underline{i} - 8\underline{j}$. □

2.2 Addition of vectors

If $\underline{a}$ is the vector from A to B ($\overrightarrow{AB}$) and $\underline{b}$ is the vector from B to C ($\overrightarrow{BC}$), what is $\overrightarrow{AC}$?



For example, if $\overrightarrow{AB} = \underline{i}$ and $\overrightarrow{BC} = \underline{j}$, then $\overrightarrow{AC} = \underline{i} + \underline{j}$.
The result is that we can add vectors according to the "triangle law" of addition:
$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \underline{a} + \underline{b}$.
Does the order in which we add the vectors make any difference (is addition of vectors commutative)?
Demonstration 1: Consider a parallelogram $ABCD$, with $\overrightarrow{AB} = \underline{a}$ and $\overrightarrow{BC} = \underline{b}$. We have that $\overrightarrow{DC}$ is parallel to $\overrightarrow{AB}$, and has the same length, so $\overrightarrow{DC} = \underline{a}$. Similarly, $\overrightarrow{AD} = \underline{b}$. Starting from A there are two ways to get to C: $\overrightarrow{AC} = \overrightarrow{AB}+\overrightarrow{BC} = \underline{a}+\underline{b}$ and $\overrightarrow{AC} = \overrightarrow{AD}+\overrightarrow{DC} = \underline{b}+\underline{a}$ ("parallelogram law" of addition). Therefore $\underline{a}+\underline{b} =$



$\underline{b}+\underline{a}$ and the order doesn't matter.
Demonstration 2: We demonstrate using components, since we know that the order in which we add scalars does not matter. Write $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$. Then

$$\underline{a} + \underline{b} = (a_1 + b_1)\underline{i} + (a_2 + b_2)\underline{j} + (a_3 + b_3)\underline{k} = (b_1 + a_1)\underline{i} + (b_2 + a_2)\underline{j} + (b_3 + a_3)\underline{k} = \underline{b} + \underline{a}.$$

2.2.1 Proving results about vectors

Using the component form allows us to prove many other results about vectors. For example, that they are *associative*:

$$(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c}).$$

So brackets do not matter, and we can add vectors in any order.

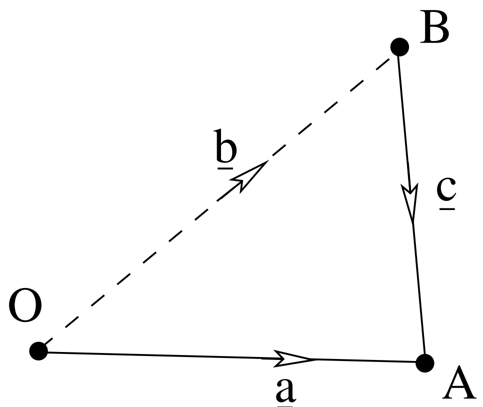
Further, they obey the *distributive* law:

$$\lambda(\underline{a} + \underline{b}) = \lambda\underline{a} + \lambda\underline{b}$$

for multiplication by a scalar λ.

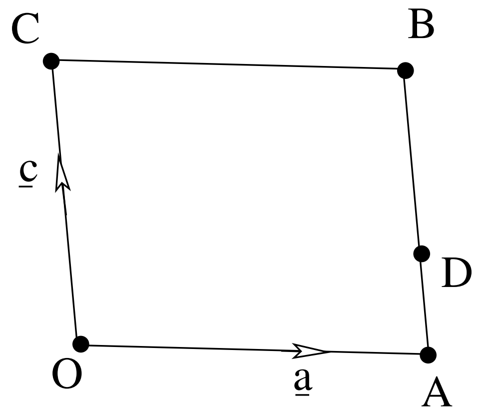
Remark: Subtraction works in the same way as addition, since $\underline{a} - \underline{b} = \underline{a} + (-\underline{b})$. For example, in the triangle OAB with $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$ we have

$$\underline{c} = \overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -\overrightarrow{OB} + \overrightarrow{OA} = \overrightarrow{OA} - \overrightarrow{OB} = \underline{a} - \underline{b}.$$



Remark: Now consider the vector (line segment) $\overrightarrow{AB}$. Denote by C its midpoint. Then $\overrightarrow{AC} = \frac{1}{2}\overrightarrow{AB}$.

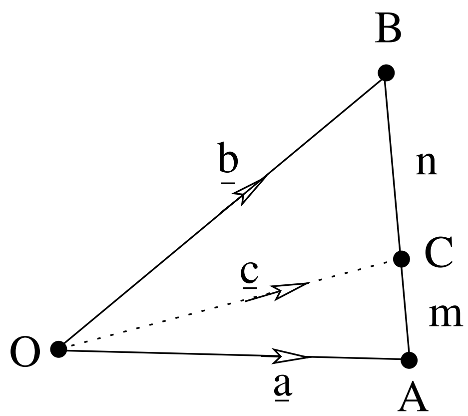
Example 2.6. *In the parallelogram $OABC$, $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OC} = \underline{c}$. The point D lies on $\overrightarrow{AB}$ and divides it such that $AD : DB = 1 : 2$.*



Express the following in terms of $\underline{a}$ and $\underline{c}$:
(a) $\overrightarrow{CB}$; (b) $\overrightarrow{BC}$; (c) $\overrightarrow{AB}$; (d) $\overrightarrow{AD}$; (e) $\overrightarrow{OD}$; (f) $\overrightarrow{DC}$.

Proposition 1. *Let $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$. If C is a point which divides AB into two segments which are in the ratio $m : n$, then the vector $\overrightarrow{OC} = \underline{c}$ is given by*

$$\underline{c} = \frac{n\underline{a} + m\underline{b}}{m+n}.$$



Proof. Since $\overrightarrow{AB} = \underline{b} - \underline{a}$, we have

$$\overrightarrow{OC} = \overrightarrow{OA} + \frac{m}{m+n}\overrightarrow{AB} = \underline{a} + \frac{m}{m+n}(\underline{b} - \underline{a}) = \frac{(m+n-m)\underline{a} + m\underline{b}}{m+n} = \frac{n\underline{a} + m\underline{b}}{m+n}.$$

□

Example 2.7. *In the parallelogram $OABC$, $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OC} = \underline{c}$. D is a point on AB such that $AD : DB = 2 : 1$. $\overrightarrow{OD}$ produced meets $\overrightarrow{CB}$ produced at E. Further $\overrightarrow{DE} = h\overrightarrow{OD}$, $h \in \mathbb{R}$, and $\overrightarrow{BE} = k\overrightarrow{CB}$, $k \in \mathbb{R}$. Find (a) $\overrightarrow{BE}$ in terms of $\underline{a}$ and k; (b) $\overrightarrow{DE}$ in terms of h, $\underline{a}$ and $\underline{c}$; (c) the values of h and k.*