2.4 Products of vectors

We will describe four products of vectors (some more useful than others).

2.4.1 Scalar product

Definition 2.4. *The* scalar product[2] $\underline{a}.\underline{b}$ *of two vectors* $\underline{a}$ *and* $\underline{b}$ *is the scalar quantity* $|\underline{a}||\underline{b}|\cos\theta$*, where* θ *is the angle between* $\underline{a}$ *and* $\underline{b}$*:*

$$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta.$$

In component form it is $\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Example 2.12. *Find the cosine of the angle between the vectors* $\underline{u} = \underline{i} - 2\underline{j} - 2\underline{k}$ *and* $\underline{v} = 6\underline{i} + 3\underline{j} + 2\underline{k}$.

Remarks:
- Two non-zero vectors are perpendicular if their scalar product is zero (since $\cos\pi/2 = 0$);
- For any vector $\underline{a}$, $\underline{a}.\underline{a} = |\underline{a}||\underline{a}|\cos(0) = |\underline{a}|^2$;
- Using components, it is possible to prove commutativity of the scalar product: $\underline{a}.\underline{b} = \underline{b}.\underline{a}$;
- Using components, it is possible to prove distributivity of the scalar product: $\underline{a}.(\underline{b}+\underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$.

These last two proofs are left as an *exercise*.

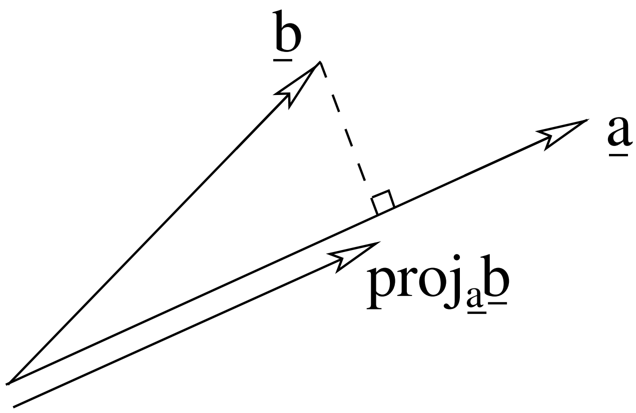
Example 2.13. *Give the vector equation of the line through the point* $\underline{a} = 2\underline{i} + 3\underline{j}$ *perpendicular to the vector* $\underline{b} = -\underline{i} + 2\underline{j}$.

Example 2.14. *Points* A*,* B*,* C*, and* D *have position vectors* $-2\underline{i}+3\underline{j}$*,* $3\underline{i}+8\underline{j}$*,* $7\underline{i}+6\underline{j}$*, and* $7\underline{i}-4\underline{j}$ *respectively. Show that* $\overrightarrow{AC}$ *is perpendicular to* $\overrightarrow{BD}$.

2.4.2 Projection of vectors

Definition 2.5. *The projection of a vector* $\underline{b}$ *onto a vector* $\underline{a}$ *is defined to be the component of* $\underline{b}$ *in the direction of* $\underline{a}$*, given by*

$$\mathrm{proj}_{\underline{a}}\underline{b} = |\underline{b}|\cos\theta\hat{\underline{a}} = \frac{\underline{a}.\underline{b}}{|\underline{a}|}\hat{\underline{a}} = \frac{\underline{a}.\underline{b}}{|\underline{a}|^2}\underline{a}.$$



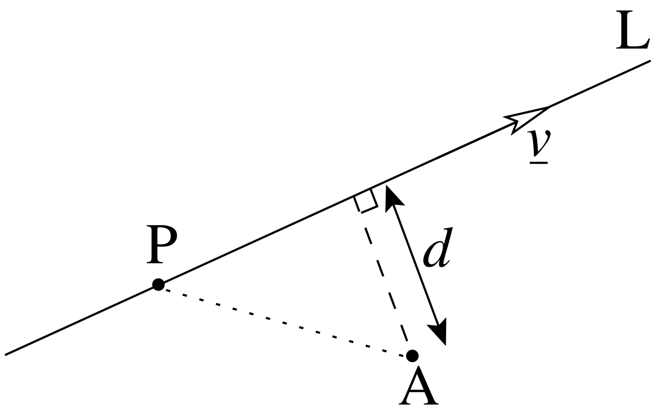
Its length – the scalar projection of $\underline{b}$ *on* $\underline{a}$ *– is* $|\mathrm{proj}_{\underline{a}}\underline{b}| = |\underline{b}|\cos\theta = \dfrac{|\underline{a}.\underline{b}|}{|\underline{a}|}$.

Example 2.15. *Find the projection of* $\underline{a} = 2\underline{i} + 5\underline{j}$ *on the vector* $\underline{b} = \underline{i} + \underline{j}$.

Example 2.16. *Given* $\underline{a} = \underline{i} - 2\underline{j} + 3\underline{k}$ *and* $\underline{b} = 3\underline{i} - 4\underline{k}$*, find the scalar projection of* $\underline{a}$ *on* $\underline{b}$.

[2] Also called the dot product

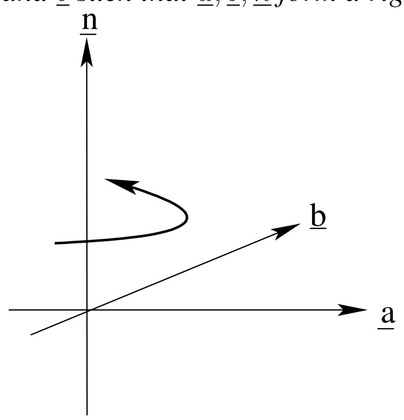
Example 2.17. *Find the perpendicular distance of the point A with position vector $4\underline{i} + 5\underline{j}$ from the line L with vector equation $\underline{r} = -3\underline{i} + \underline{j} + \lambda(\underline{i} + 2\underline{j})$.*



Example 2.18. *Find the perpendicular distance of the point C with position vector $5\underline{i} - 9\underline{j} + 2\underline{k}$ from the line $\underline{r} = 5\underline{i} + 3\underline{j} - \underline{k} + \lambda(2\underline{i} + 5\underline{j} + 7\underline{k})$.*

2.4.3 Vector product

Definition 2.6. *The vector product[3] $\underline{a} \wedge \underline{b}$ of two vectors $\underline{a}$ and $\underline{b}$ is the vector $|\underline{a}||\underline{b}| \sin\theta \hat{\underline{n}}$, where $\hat{\underline{n}}$ is a unit vector perpendicular to the plane of $\underline{a}$ and $\underline{b}$ such that $\underline{a}, \underline{b}, \hat{\underline{n}}$ form a right-handed set.*



In component form, the vector product can be written as a determinant:

$$\underline{a} \wedge \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example 2.19. *For vectors $\underline{a} = 5\underline{i} - \underline{j} + 2\underline{k}$ and $\underline{b} = \underline{i} - 2\underline{j} + 3\underline{k}$, the scalar product of $\underline{a}$ and $\underline{b}$ is*

$$\underline{a}.\underline{b} = 5 \times 1 + (-1) \times (-2) + 2 \times 3 = 13$$

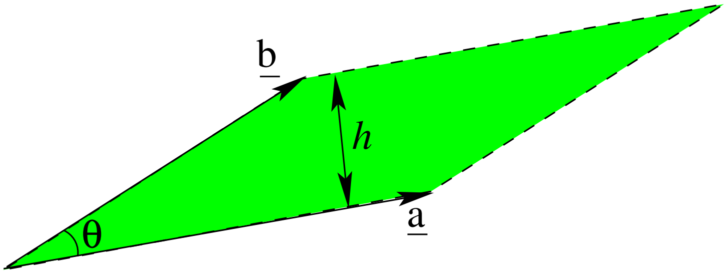
and the vector product of $\underline{a}$ and $\underline{b}$ is

$$\underline{a} \wedge \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & -1 & 2 \\ 1 & -2 & 3 \end{vmatrix} = \underline{i} - 13\underline{j} - 9\underline{k}.$$

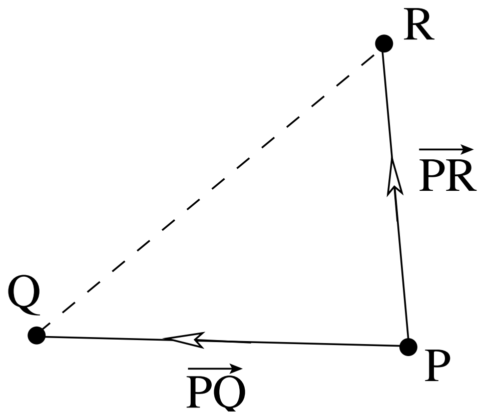
[3] Also called the cross product

Remarks:

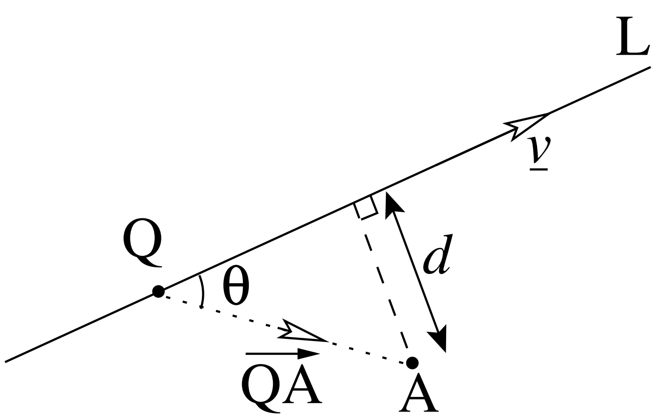
• The length of the resulting vector is $|\underline{a} \wedge \underline{b}| = |\underline{a}||\underline{b}||\sin\theta|$;
• Two non-zero vectors are parallel if their vector product is zero (since $\sin 0 = 0$);
• The vector product does *not* commute. Instead, $\underline{b} \wedge \underline{a} = -(\underline{a} \wedge \underline{b})$;
• The vector product is distributive: $\underline{a} \wedge (\underline{b} + \underline{c}) = \underline{a} \wedge \underline{b} + \underline{a} \wedge \underline{c}$;
(*Exercise: test this using components*.)
• For the unit vectors in the directions of the coordinate axes we have: $\underline{i} \wedge \underline{j} = \underline{k}$, $\underline{j} \wedge \underline{k} = \underline{i}$, and $\underline{k} \wedge \underline{i} = \underline{j}$, but $\underline{i} \wedge \underline{k} = -\underline{j}$ etc.;
• $|\underline{a} \wedge \underline{b}|$ is the area of the parallelogram formed by $\underline{a}$ and $\underline{b}$. The height of the parallelogram is $h = |\underline{b}| \sin\theta$.



Example 2.20. *Find a vector perpendicular to the plane containing the points* $P : (1, -1, 0)$*,* $Q : (2, 1, -1)$*,* $R : (-1, 1, 2)$*, and the area of the triangle with these vertices.*



Example 2.21. *Repeat example 2.17 (find the perpendicular distance of the point with position vector* $4\underline{i} + 5\underline{j}$ *from the line* $\underline{r} = -3\underline{i} + \underline{j} + \lambda(\underline{i} + 2\underline{j})$*) with the vector product.*



2.4.4 Triple products

Definition 2.7. *The* scalar triple product *of vectors* $\underline{a}$, $\underline{b}$, $\underline{c}$ *is*

$$[\underline{a},\underline{b},\underline{c}] = \underline{a}.(\underline{b}\wedge\underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

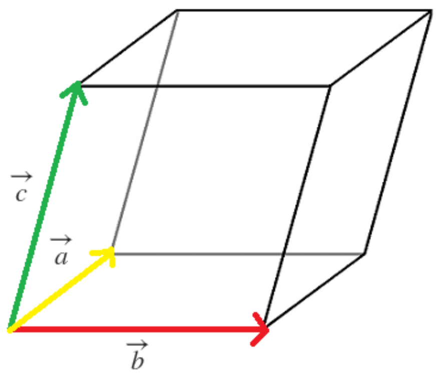
Remarks:

• To see that this is indeed equal to the given determinant, note that

$$\begin{aligned}\underline{a}.(\underline{b}\wedge\underline{c}) &= \left(a_1\underline{i}+a_2\underline{j}+a_3\underline{k}\right).\left[(b_2c_3-b_3c_2)\underline{i}-(b_1c_3-b_3c_1)\underline{j}+(b_1c_2-b_2c_1)\underline{k}\right] \\ &= a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1),\end{aligned}$$

which is the result of evaluating the given determinant.

• The result is a scalar, which represents the volume of the parallelepiped formed by the three vectors.



• A special case is when the points at the ends of the three vectors all lie in the plane $z = 1$. The tetrahedron formed by these three points and the origin has one-sixth the volume of the parallelepiped and, since its perpendicular height is one, its volume is also equal to one-third the area of the triangle in the plane $z = 1$. This proves the result we used in §1.1.6, that the area of the triangle is one half of the triple scalar product.

• Repeated vectors: $[\underline{a},\underline{a},\underline{b}] = 0$ since $\underline{a}\wedge\underline{b}$ is perpendicular to $\underline{a}$.

• If $\underline{a}$, $\underline{b}$, $\underline{c}$ are non-zero and *co-planar*, then $[\underline{a},\underline{b},\underline{c}] = 0$.

Example 2.22. *Calculate the scalar triple product of vectors* $\underline{a} = \underline{i}-2\underline{j}+\underline{k}$, $\underline{b} = 3\underline{i}+\underline{j}+2\underline{k}$ *and* $\underline{c} = -\underline{i}-2\underline{j}+5\underline{k}$.

Solution:

$$[\underline{a},\underline{b},\underline{c}] = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & 2 \\ -1 & -2 & 5 \end{vmatrix} = 1(5+4)+2(15+2)+1(-6+1) = 9+34-5 = 38$$

□

Definition 2.8. *The* vector triple product *of vectors* $\underline{a}$, $\underline{b}$, $\underline{c}$ *is*

$$\underline{a}\wedge(\underline{b}\wedge\underline{c}) = (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}.$$

We can demonstrate this result using components. Without loss of generality, we show that it is true for the $\underline{i}$ component:

$$\begin{aligned}
[\underline{a}\wedge(\underline{b}\wedge\underline{c})]_{\underline{i}} &= \left[\underline{a}\wedge\left((b_2c_3-b_3c_2)\underline{i}+(b_3c_1-b_1c_3)\underline{j}+(b_1c_2-b_2c_1)\underline{k}\right)\right]_{\underline{i}} \\
&= a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3) \\
\text{(expand)} &= a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3 \\
&\quad \text{Now we add and subtract } a_nb_nc_n \text{ to the } n^{\text{th}} \text{ component:} \\
&= a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3+a_1b_1c_1-a_1b_1c_1 \\
\text{(collect)} &= b_1(a_1c_1+a_2c_2+a_3c_3)-c_1(a_1b_1+a_2b_2+a_3b_3) \\
&= [(\underline{a}.\underline{c})\underline{b}-(\underline{a}.\underline{b})\underline{c}]_{\underline{i}}\,.
\end{aligned}$$

Example 2.23. *Calculate the vector triple product of* $\underline{a}=\underline{i}-2\underline{j}+\underline{k}$, $\underline{b}=3\underline{i}+\underline{j}+2\underline{k}$ *and* $\underline{c}=-\underline{i}-2\underline{j}+5\underline{k}$.

Solution: First

$$\underline{b}\wedge\underline{c}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 3 & 1 & 2\\ -1 & -2 & 5\end{vmatrix}=9\underline{i}-17\underline{j}-5\underline{k}.$$

Then

$$\underline{a}\wedge(\underline{b}\wedge\underline{c})=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & -2 & 1\\ 9 & -17 & -5\end{vmatrix}=27\underline{i}+14\underline{j}+\underline{k}.$$

As a check, note that $\underline{a}.\underline{c}=8$ and $\underline{a}.\underline{b}=3$, so $8\underline{b}-3\underline{c}=24\underline{i}+8\underline{j}+16\underline{k}-(-3\underline{i}-6\underline{j}+15\underline{k})=27\underline{i}+14\underline{j}+\underline{k}$. □