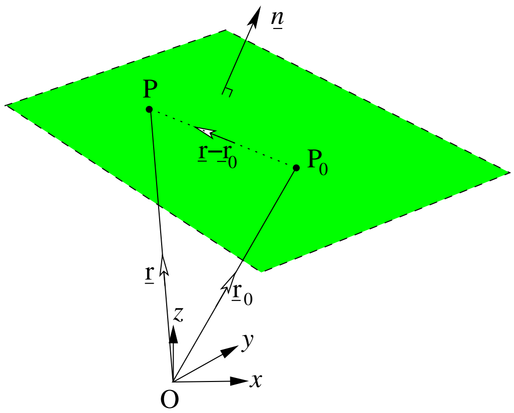
2.5 Planes

We wrote an equation for a line by finding a point on the line and its direction. To find an equation for a plane we find a point on the plane and the direction *normal* to the plane.

Given a point P_0 in the plane with position vector $\underline{r}_0$ and a vector $\underline{n}$ normal (or perpendicular) to the plane, then if P is an arbitrary point in the plane, the vector from P to P_0 must be perpendicular to $\underline{n}$. If we write the position vector of P as $\underline{r}$, then $\overrightarrow{P_0P} = \underline{r} - \underline{r}_0$ and the vector equation of the plane is

$$(\underline{r} - \underline{r}_0).\underline{n} = 0,$$

or $\underline{r}.\underline{n} = \underline{r}_0.\underline{n}$.



In components, if we write $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, $\underline{r}_0 = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$ and $\underline{n} = n_x\underline{i} + n_y\underline{j} + n_z\underline{k}$ then the equation of the plane becomes $n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0$ or

$$n_x x + n_y y + n_z z = d \tag{1}$$

for constant $d = \underline{n}.\underline{r}_0 = n_x x_0 + n_y y_0 + n_z z_0$.

Example 2.24. *Find the equation of the plane passing through* $(0, 2, 1)$*, normal to* $3\underline{i} - 2\underline{j} - \underline{k}$*.*

Example 2.25. *Find the equation of the plane passing through the points A:(0,0,1), B:(2,0,0), and C:(0,3,0).*

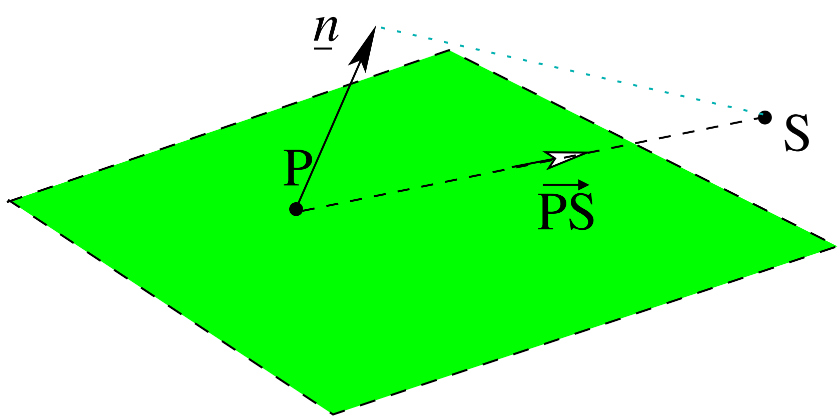
Example 2.26. *Find the (vector) equation of the line formed by the intersection of the planes* $x + y - z = 1$ *and* $2x + 3y + z = -3$*.*

Remark: The angle between two intersecting planes is the acute angle made by their normal vectors, which can be calculated using the scalar product.
For example, the cosine of the angle between the planes $x = 0$ and $y = 0$ is found from the scalar product of their unit normals, $\cos\theta = \underline{i}.\underline{j} = 0$, so $\theta = \frac{\pi}{2}$.

Example 2.27. *Find the cosine of the angle between the planes* $x + 6y - 2z = 0$ *and* $2x - y - 6z = 0$*.*

Remark: The shortest distance of a point S:(x', y', z') from a plane $ax + by + cz = d$ is found by taking a point P:(x, y, z) in the plane and calculating the scalar projection of $\overrightarrow{PS}$ on to the unit normal, i.e.

$$\Delta = \frac{\left|\overrightarrow{PS}.\underline{n}\right|}{|\underline{n}|},$$



Remark: Equivalently, we can mirror the expression for the distance of a point from a line (§1.1.3) to planes, by writing $\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP}$. We have

$$\Delta = \frac{|ax' + by' + cz' - d|}{\sqrt{a^2 + b^2 + c^2}},$$

because $d = \overrightarrow{OP}.\underline{n}$.

Example 2.28. *As a check, note that the distance of the point S:(0,0,1) from the plane* $z = 0$ *should be 1. If we take* P *to be (0,0,0), then* $\overrightarrow{PS} = \underline{k}$*. The unit normal to the plane is* $\underline{n} = \underline{k}$*. So* $\overrightarrow{PS}.\underline{\hat{n}} = \underline{k}.\underline{k} = 1$ *as expected. Alternatively, in the equation of the plane we have* $c = 1$ *and* $a = b = d = 0$*. For S we have* $x' = y' = 0$ *and* $z' = 1$*. So* $\Delta = \frac{1 \times 1}{\sqrt{1^2}} = 1$*.*

Example 2.29. *Find the distance of the point (2,-3,4) from the plane* $x + 2y + 2z = 13$*.*