MA10110 Assignment 4: Vectors – Planes and Kinematics
SOLUTIONS

1. $z = 0$.

2. $2(x-5) - 2(y-1) + 4(z-2) = 0$ or $x - y + 2z = 8$. Changing the sign of $\underline{n}$ makes no difference.

3. (a) First find the normal: $\underline{n} = (\underline{b} - \underline{a}) \wedge (\underline{c} - \underline{a}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -3 & 0 & 0 \\ -1 & 3 & 2 \end{vmatrix} = 6\underline{j} - 9\underline{k}$.
   Then the equation of the plane is $0(x-5) + 6(y-1) - 9(z-2) = 0$ or $2y - 3z + 4 = 0$.
   (b) Evaluating $2y - 3z$ at (10,3,1) gives 3, so the parallel plane has equation $2y - 3z - 3 = 0$.
   (c) The distance between them is given by $\Delta = \overrightarrow{PS}.\hat{\underline{n}}$, with $\overrightarrow{PS} = (10\underline{i}+3\underline{j}+\underline{k})-(5\underline{i}+\underline{j}+2\underline{k})$ and $\hat{\underline{n}} = \frac{2}{\sqrt{13}}\underline{j} - \frac{3}{\sqrt{13}}\underline{k}$.
   Thus $\Delta = \dfrac{(10-5)\times 0 + (3-1)\times 2 + (1-2)\times(-3)}{\sqrt{13}} = \dfrac{7}{\sqrt{13}}$.

4. First find the normal: $\underline{n} = (\underline{b} - \underline{a}) \wedge (\underline{c} - \underline{a}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -2 & 5 & 5 \\ -1 & 3 & -1 \end{vmatrix} = -20\underline{i} - 7\underline{j} - \underline{k}$.
   Then the equation of the plane is $-20(x-3) - 7(y+2) - (z+1) = 0$ or $20x + 7y + z = 45$.
   The distance from the origin is $\Delta = |\underline{a}.\hat{\underline{n}}| = \dfrac{|-60+14+1|}{\sqrt{450}} = \sqrt{\dfrac{45}{10}} = \dfrac{3}{\sqrt{2}}$.

5. (a) First find a point on the intersection line between A and B; for example put $z = 0$ and solve the simultaneous equations $3x - 4y = 2$ and $-2x + y = 1$ to give $(-\frac{6}{5}, -\frac{7}{5}, 0)$.
   The direction of the line is parallel to $\underline{n}_A \wedge \underline{n}_B = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -4 & 1 \\ -2 & 1 & -3 \end{vmatrix} = 11\underline{i} + 7\underline{j} - 5\underline{k}$,
   so the line is $\underline{r} = \left(-\frac{6}{5} + 11\lambda\right)\underline{i} + \left(-\frac{7}{5} + 7\lambda\right)\underline{j} - 5\lambda\underline{k}$.
   (b) Now repeat for A and C: a point on the line is $(-\frac{1}{5}, 0, \frac{13}{5})$.
   $\underline{n}_A \wedge \underline{n}_C = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -4 & 1 \\ 1 & -5 & 2 \end{vmatrix} = -3\underline{i} - 5\underline{j} - 11\underline{k}$, so the line is $\underline{r} = \left(-\frac{1}{5} + 3\lambda\right)\underline{i} + 5\lambda\underline{j} + \left(\frac{13}{5} + 11\lambda\right)\underline{k}$, absorbing a minus sign into λ.
   (c) The angle between B and C is the angle between their normals. So $\cos\theta = \hat{\underline{n}}_B.\hat{\underline{n}}_C = \dfrac{-2-5-6}{\sqrt{14}\sqrt{30}} = -\dfrac{13}{\sqrt{420}}$.
   For the acute angle, $\cos\theta = \frac{13}{\sqrt{420}}$.

6. (a) Take the vector product of the vectors to give a perpendicular one: $\underline{n}^* = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -3 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 2\underline{i} + 3\underline{j} + 7\underline{k}$.
   A unit vector is $\hat{\underline{n}}^* = \frac{2}{\sqrt{62}}\underline{i} + \frac{3}{\sqrt{62}}\underline{j} + \frac{7}{\sqrt{62}}\underline{k}$.
   (b) A plane perpendicular to the intersection of these two planes has normal $\underline{n}^*$. If it passes through (1,1,1) then its equation is $2(x-1) + 3(y-1) + 7(z-1) = 0$, or $2x + 3y + 7z = 12$.

7. The distance is 1.
   Formally, write $S : (3,2,1)$ and choose a point in the plane, e.g. $P : (0,0,0)$. The normal to $z = 0$ is $\underline{n} = 1\underline{k}$. So $\overrightarrow{PS} = 3\underline{i} + 2\underline{j} + \underline{k}$ and $|\overrightarrow{PS}.\hat{\underline{n}}| = 3\times 0 + 2\times 0 + 1\times 1 = 1$.

8. We can use the formula $\Delta = |ax + by + cz - d|/\sqrt{a^2+b^2+c^2}$ with $a = 3, b = -\sqrt{15}, c = 5$ and $x = 2, y = \sqrt{15}, z = -1$ to give $\Delta = |6 - 15 - 5 - 12|/\sqrt{9+15+25} = 26/7$.

9. The normal to the plane is $\underline{n} = 20\underline{i}+7\underline{j}+\underline{k}$, with magnitude $\sqrt{450}$. A point in the plane is P:(0,0,55), so $\overrightarrow{PS} = -55\underline{k}$.
   Then $|\overrightarrow{PS}.\hat{\underline{n}}| = |-\frac{55}{\sqrt{450}}| = \frac{11}{3\sqrt{2}}$.

10. Time is 200 seconds (note the units!)

11. $\underline{v} = 10\underline{i} + 10\underline{j} + 2t\underline{k}$. At $t = 10$ this is $\underline{v} = 10\underline{i} + 10\underline{j} + 20\underline{k}$.
$\underline{a} = 2\underline{k}$, with magnitude 2 at all times.

12. $\underline{r}_1 = \underline{i} + 5\underline{j} + t\underline{i}$ and $\underline{r}_2 = 3\underline{i} - 1\underline{j} + 2t\underline{j}$. The square of the distance between them is

$$|\underline{r}_2 - \underline{r}_1|^2 = |(3-(1+t))\underline{i} + (-1+2t-5)\underline{j}|^2 = |(2-t)\underline{i} + (-6+2t)\underline{j}|^2 = (2-t)^2 + (2t-6)^2 = 5t^2 - 28t + 40.$$

This is minimized when $10t - 28 = 0$, or $t = 2.8$, which is the time at which they are closest (2 hours and 48 minutes). Their distance at that time is $\sqrt{\frac{8}{10}} = \frac{2}{\sqrt{5}}$.

13. The gift's position vector is $\underline{r}_G = 7\underline{i} + 3\underline{j}$, and for the children we have $\underline{r}_D = \underline{i} - 3\underline{j}$ and $\underline{r}_B = 8\underline{j}$.
To get to the present, Doris moves in the direction $\underline{r}_G - \underline{r}_D = 6\underline{i} + 6\underline{j}$, i.e. at $45°$ to the $\underline{i}$ direction ("north-east"), a distance $\sqrt{72} = 6\sqrt{2}$.
Boris moves in the direction $\underline{r}_G - \underline{r}_B = 7\underline{i} - 5\underline{j}$, i.e. at an angle $\tan^{-1}(5/7)$ "south" of the $\underline{i}$ direction, a distance $\sqrt{74}$.
Since $\sqrt{72} < \sqrt{74}$, Doris arrives first.

14. $\underline{v}_{Santa} = \frac{30}{\sqrt{2}}\underline{i} + \frac{30}{\sqrt{2}}\underline{j}$ and $\underline{v}_{ship} = -\frac{10}{\sqrt{2}}\underline{i} + \frac{10}{\sqrt{2}}\underline{j}$.
Therefore the relative velocity is $\underline{v}_{sS} = \underline{v}_{ship} - \underline{v}_{Santa} = -\frac{40}{\sqrt{2}}\underline{i} - \frac{20}{\sqrt{2}}\underline{j}$.
So to Santa it appears that the ship is moving at a speed $\sqrt{\frac{40^2+20^2}{2}} = 10\sqrt{10}$ in a direction given by $\tan\theta = \frac{20}{40} = \frac{1}{2}$, i.e. about $26°$ south of west.

15. We differentiate $\underline{r} = \frac{1}{2}(t^2 - 3t)\underline{i} + 2t\underline{j}$ to get $\underline{v} = (t - \frac{3}{2})\underline{i} + 2\underline{j}$ and $\underline{a} = \underline{i}$.
The square of the speed is $|\underline{v}|^2 = (t - \frac{3}{2})^2 + 2^2 = t^2 - 3t + \frac{25}{4}$. This is minimized when $2t - 3 = 0$, i.e. $t = \frac{3}{2}$. Then $|\underline{v}_{\min}| = \sqrt{\frac{9}{4} - \frac{9}{2} + \frac{25}{4}} = \sqrt{\frac{16}{4}} = 2$, as required.

16. The velocity of sleigh A relative to the wind is $\underline{v}_{Aw} = (\frac{3}{5}\underline{i} + \frac{4}{5}\underline{j})v$. Its position is $\underline{r}_A = 60\underline{i} + 90\underline{j} + t((\frac{3}{5}\underline{i} + \frac{4}{5}\underline{j})v + \underline{w})$, since $\underline{v}_{Aw} = \underline{v}_A - \underline{w}$, with t in hours.
When $t = 1$ hour, we have $132\underline{i} + 175\underline{j} = (60 + \frac{3}{5}v)\underline{i} + (90 + \frac{4}{5}v)\underline{j} + \underline{w}$ (†).
The velocity of sleigh B relative to the wind is $\underline{v}_{Bw} = (\frac{7}{25}\underline{i} - \frac{24}{25}\underline{j})v$. Its position is $\underline{r}_B = 60\underline{i} + 90\underline{j} + t((\frac{7}{25}\underline{i} - \frac{24}{25}\underline{j})v + \underline{w})$ since $\underline{v}_{Bw} = \underline{v}_B - \underline{w}$.
When $t = 1$ hour, we have $100\underline{i} - \underline{j} = (60 + \frac{7}{25}v)\underline{i} + (90 - \frac{24}{25}v)\underline{j} + \underline{w}$.
Subtracting this expression for sleigh B from the expression for sleigh A (†) gives $32\underline{i} + 176\underline{j} = \frac{8}{25}v\underline{i} + \frac{44}{25}v\underline{j}$. Thus $v = 100$ km/h.
Now substitute back in to (†) to find $\underline{w} = (132 - (60 + 60))\underline{i} + (175 - (90 + 80))\underline{j} = 12\underline{i} + 5\underline{j}$ km/h.