MA10110 Assignment 3: Vectors – Lines and Products
SOLUTIONS

1. (a) $\overrightarrow{AB} = \underline{b} - \underline{a}$; (b) $\overrightarrow{AC} = \frac{3}{4}(\underline{b} - \underline{a})$; (c) $\overrightarrow{CB} = \frac{1}{4}(\underline{b} - \underline{a})$; (d) $\overrightarrow{OC} = \underline{a} + \frac{3}{4}(\underline{b} - \underline{a}) = \frac{1}{4}\underline{a} + \frac{3}{4}\underline{b}$. [2,2,2,2]

2. (a) $\overrightarrow{OB} = \underline{c} + 3\underline{a}$; (b) $\overrightarrow{AB} = -\underline{a} + \underline{c} + 3\underline{a} = 2\underline{a} + \underline{c}$;
(c) $\overrightarrow{OD} = \underline{a} + \frac{1}{2}\overrightarrow{AB} = 2\underline{a} + \frac{1}{2}\underline{c}$; (d) $\overrightarrow{CD} = 3\underline{a} - \frac{1}{2}\overrightarrow{AB} = 2\underline{a} - \frac{1}{2}\underline{c}$. [2,2,2,2]

3. (a) $\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA} = -\underline{c} + \underline{a}$;
(b) $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = \underline{c} - \underline{a} + 2\underline{a} = \underline{c} + \underline{a}$;
(c) $\overrightarrow{ED} = \overrightarrow{EB} + \overrightarrow{BD} = \frac{1}{2}\overrightarrow{CB} + \frac{1}{2}\overrightarrow{BA} = \frac{1}{2}(2\underline{a}) + \frac{1}{2}(-\underline{c} - \underline{a}) = \frac{1}{2}(-\underline{c} + \underline{a})$.
Therefore $\overrightarrow{CA} = 2\overrightarrow{ED}$, as required.

4. (a) $\overrightarrow{OC} = \frac{1}{2}\underline{b}$, $\overrightarrow{OD} = \underline{a} + \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}$, and $\overrightarrow{AC} = -\underline{a} + \frac{1}{2}\underline{b}$.
Then we can construct $\overrightarrow{AF}$ in two ways: $\overrightarrow{AF} = h(-\underline{a} + \frac{1}{2}\underline{b})$ and $\overrightarrow{AF} = -\underline{a} + k(\frac{1}{2}\underline{a} + \frac{1}{2}\underline{b})$.
Comparing coefficients of A and B gives $-h = -1 + \frac{k}{2}$ and $h = k$, giving $h = k = \frac{2}{3}$.
(b) $\overrightarrow{BF} = -\underline{b} + \frac{2}{3}\overrightarrow{OD} = \frac{1}{3}\underline{a} - \frac{2}{3}\underline{b} = \frac{2}{3}(\frac{1}{2}\underline{a} - \underline{b}) = \frac{2}{3}\overrightarrow{BE}$, so $\overrightarrow{BF}$ and $\overrightarrow{BE}$ are parallel and since they share a common point (B), the points B, F, and E must be collinear. [6,4]

5. (a) $\overrightarrow{AB} = -\underline{i} - \underline{j} + 2\underline{k}$; (b) $|\overrightarrow{AB}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$; (c) $\hat{\underline{v}} = -\frac{1}{\sqrt{6}}\underline{i} - \frac{1}{\sqrt{6}}\underline{j} + \frac{2}{\sqrt{6}}\underline{k}$;
(d) $\underline{r} = 2\underline{i} + 3\underline{j} - 5\underline{k} + \lambda(-\underline{i} - \underline{j} + 2\underline{k}) = (2 - \lambda)\underline{i} + (3 - \lambda)\underline{j} + (2\lambda - 5)\underline{k}$. [2,2,2,2]

6. (a) $\overrightarrow{AB} = 4\underline{i} + 3\underline{j}$, so $\underline{r} = 2\underline{i} + 4\underline{j} + \lambda(4\underline{i} + 3\underline{j}) = (2 + 4\lambda)\underline{i} + (4 + 3\lambda)\underline{j}$. When $\lambda = 3$, we have $\underline{r} = 14\underline{i} + 13\underline{j}$ as required.
(b) $\overrightarrow{OC} = \lambda(3\underline{i} - 4\underline{j})$; it is orthogonal to $\overrightarrow{AB}$ if $(3\underline{i} - 4\underline{j}).(4\underline{i} + 3\underline{j}) = 0$, which is the case.
(c) The lines intersect when $(2 + 4\lambda_1)\underline{i} + (4 + 3\lambda_1)\underline{j} = \lambda_2(3\underline{i} - 4\underline{j})$; then $2 + 4\lambda_1 = 3\lambda_2$ and $4 + 3\lambda_1 = -4\lambda_2$, so $\lambda_1 = -\frac{4}{5}$ and $\lambda_2 = -\frac{2}{5}$ and N has position vector $-\frac{6}{5}\underline{i} + \frac{8}{5}\underline{j}$.
The length of $\overrightarrow{ON}$ is $\sqrt{\frac{6^2 + 8^2}{25}} = 2$, which is the perpendicular distance of AB from O.

7. $\overrightarrow{BA} = -\underline{b} + \underline{a}$, so this line is $\underline{r} = \underline{a} + \lambda_1(\underline{a} - \underline{b})$. $\overrightarrow{CD} = 3\underline{a} - 2\underline{b}$, so this line is $\underline{r} = -3\underline{a} + \lambda_2(3\underline{a} - 2\underline{b})$.
They intersect when $\underline{a} + \lambda_1(\underline{a} - \underline{b}) = -3\underline{a} + \lambda_2(3\underline{a} - 2\underline{b})$, which gives $\lambda_1 = 8, \lambda_2 = 4$.
This gives the position vector of the point L as $\underline{a} + 8(\underline{a} - \underline{b}) = 9\underline{a} - 8\underline{b}$. [6,4,2]

8. $\overrightarrow{AC} = -5\underline{a} + 5\underline{b}$, so $\underline{r} = 3\underline{a} + 2\underline{b} + \lambda(\underline{b} - \underline{a}) = (3 - \lambda)\underline{a} + (2 + \lambda)\underline{b}$.
We need to show that B lies on $\overrightarrow{AC}$: put $\lambda = 4$ to give $\underline{r} = -\underline{a} + 6\underline{b}$, which is B.
$\overrightarrow{AB} = -4\underline{a} + 4\underline{b}$ and $\overrightarrow{BC} = -\underline{a} + \underline{b}$, so $\overrightarrow{AB} = 4\overrightarrow{BC}$ and hence $AB : BC = 4 : 1$. [3,2,2]

9. Write $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$, $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ and $\underline{c} = c_1\underline{i} + c_2\underline{j} + c_3\underline{k}$.

(a) Then, because addition of scalars is commutative, we can write

$$\underline{c} + \underline{b} = (c_1 + b_1)\underline{i} + (c_2 + b_2)\underline{j} + (c_3 + b_3)\underline{k} = (b_1 + c_1)\underline{i} + (b_2 + c_2)\underline{j} + (b_3 + c_3)\underline{k} = \underline{b} + \underline{c}.$$

(b) In the same way, because scalars are associative, we can write

$$\begin{aligned} (\underline{a} + \underline{b}) + \underline{c} &= ((a_1 + b_1) + c_1)\underline{i} + ((a_2 + b_2) + c_2)\underline{j} + ((a_3 + b_3) + c_3)\underline{k} \\ &= (a_1 + (b_1 + c_1))\underline{i} + (a_2 + (b_2 + c_2))\underline{j} + (a_3 + (b_3 + c_3))\underline{k} \\ &= \underline{a} + (\underline{b} + \underline{c}). \end{aligned}$$

(c) Finally

$$\lambda(\underline{a}+\underline{b}) = \lambda(a_1+b_1)\underline{i}+\lambda(a_2+b_2)\underline{j}+\lambda(a_3+b_3)\underline{k} = (\lambda a_1+\lambda b_1)\underline{i}+(\lambda a_2+\lambda b_2)\underline{j}+(\lambda a_3+\lambda b_3)\underline{k} = \lambda\underline{a}+\lambda\underline{b}.$$

[4,4,4]

10. If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$, $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ and $\underline{c} = c_1\underline{i} + c_2\underline{j} + c_3\underline{k}$, then

    (a) $\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3$, since scalars commute; this is equal to $\underline{b}.\underline{a}$ as required.

    (b) $\underline{a}.(\underline{b} + \underline{c}) = a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) = a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 = a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3 = \underline{a}.\underline{b} + \underline{a}.\underline{c}$.

    (c) Consider the $\underline{i}$ component of $\underline{a} \wedge (\underline{b} + \underline{c})$, since all components are similar. We have $[\underline{a} \wedge (\underline{b} + \underline{c})]_{\underline{i}} = a_2(b_3 + c_3) - a_3(b_2 + c_2) = a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2 = (a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2) = [\underline{a} \wedge \underline{b}]_{\underline{i}} + [\underline{a} \wedge \underline{c}]_{\underline{i}}$.

11. $|\underline{a}| = \sqrt{5^2 + 1^2 + 2^2} = \sqrt{30}$, $|\underline{b}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$, $\underline{a}.\underline{b} = 10 + 3 + 2 = 15$; so $\cos\theta = \dfrac{15}{\sqrt{30}\sqrt{14}} = \sqrt{\dfrac{15}{28}}$.

12. (a) $|\underline{a}| = \sqrt{x^2 + y^2} = 5 \Rightarrow x^2 + y^2 = 25$.

    (b) $x = 3\lambda$ and $y = 2\lambda$ so $\frac{y}{x} = \frac{2}{3} \Rightarrow y = \frac{2}{3}x$.

    (c) $\underline{a}.\underline{b} = 3x + 2y = 0$, so $y = -\frac{3}{2}x$.

    (d) $(\underline{a} - \underline{b}).(4\underline{i} + \underline{j}) = ((x - 3)\underline{i} + (y - 2)\underline{j}).(4\underline{i} + \underline{j}) = 4(x - 3) + y - 2 = 4x + y - 14 = 0 \Rightarrow y = 14 - 4x$. [2,2,2,4]

13. (a) $\underline{a}.\underline{b} = 2 - 2 + 3 = 3$; (b) $\underline{b} \wedge \underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix} = \underline{i} + 3\underline{j} + 5\underline{k}$;

    (c) $\underline{a}.(\underline{b} \wedge \underline{c}) = 1 - 6 - 15 = -20$; (d) $\underline{a} \wedge (\underline{b} \wedge \underline{c}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & -3 \\ 1 & 3 & 5 \end{vmatrix} = -\underline{i} - 8\underline{j} + 5\underline{k}$.

14. Write $\underline{a} = \underline{i} + 2\underline{j} + 2\underline{k}$ and $\underline{b} = 2\underline{i} - 3\underline{j} + 6\underline{k}$. Then $\hat{\underline{a}} = \frac{1}{3}\underline{i} + \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}$, and $\text{proj}_{\underline{a}}\underline{b} = \dfrac{\underline{a}.\underline{b}}{|\underline{a}|}\hat{\underline{a}} = \dfrac{2 - 6 + 12}{3}\hat{\underline{a}} = \dfrac{8}{3}\hat{\underline{a}} = \dfrac{8}{9}\underline{i} + \dfrac{16}{9}\underline{j} + \dfrac{16}{9}\underline{k}$.

15. Volume = $|\underline{a}.(\underline{b} \wedge \underline{c})|$.
    $\underline{a}.(\underline{b} \wedge \underline{c}) = \begin{vmatrix} 3 & -2 & -1 \\ 1 & 3 & 4 \\ 2 & 1 & -2 \end{vmatrix} = 3 \times (-10) - (-2) \times (-10) + (-1) \times (-5) = -45$ and the volume is 45.