MA10110 Assignment 3: Vectors – Lines and Products
Attempt all questions

Solutions to questions marked with an asterisk ($*$) should be written out neatly, scanned, and uploaded to Blackboard.
Ensure that you include your name and AU email address,

1. $*$ OAB is a triangle with $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$. C is a point on AB such that $AC : CB = 3 : 1$. Express the following vectors in terms of $\underline{a}$ and $\underline{b}$:

$$(a)\overrightarrow{AB}, \quad (b)\overrightarrow{AC}, \quad (c)\overrightarrow{CB}, \quad (d)\overrightarrow{OC}.$$

[2,2,2,2]

2. $OABC$ is a trapezium with $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OC} = \underline{c}$ and $\overrightarrow{CB} = 3\underline{a}$. D is the mid-point of AB. Express the following vectors in terms of $\underline{a}$ and $\underline{c}$:

$$(a)\overrightarrow{OB}, \quad (b)\overrightarrow{AB}, \quad (c)\overrightarrow{OD}, \quad (d)\overrightarrow{CD}.$$

[2,2,2,2]

3. $OABC$ is a trapezium with $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OC} = \underline{c}$ and $\overrightarrow{CB}$ parallel to, and twice as long as, $\overrightarrow{OA}$. The points D and E are the midpoints of $\overrightarrow{AB}$ and $\overrightarrow{CB}$ respectively. Find the following vectors in terms of $\underline{a}$ and $\underline{c}$:

$$(a)\overrightarrow{CA}, \quad (b)\overrightarrow{AB}, \quad (c)\overrightarrow{ED}.$$

Hence show that $\overrightarrow{CA}$ is parallel to, and twice as long as, $\overrightarrow{ED}$.

4. $*$ OAB is a triangle with $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$; C is the mid-point of OB, D is the mid-point of AB and E is the mid-point of OA; OD and AC intersect at F.
If $\overrightarrow{AF} = h\overrightarrow{AC}$ and $\overrightarrow{OF} = k\,\overrightarrow{OD}$ show that
   (a) $h = k = 2/3$,
   (b) B, F and E are collinear with $\overrightarrow{BF} = 2/3\overrightarrow{BE}$. [6,4]

5. $*$ The points A and B have position vectors $2\underline{i} + 3\underline{j} - 5\underline{k}$ and $\underline{i} + 2\underline{j} - 3\underline{k}$, respectively, relative to an origin O. Determine:
   (a) $\overrightarrow{AB}$
   (b) The length AB
   (c) The unit vector in the direction of $\overrightarrow{AB}$.
   (d) The vector equation of the line through A parallel to AB. [2,2,2,2]

6. The position vectors of two points A and B relative to an origin O are $2\underline{i} + 4\underline{j}$ and $6\underline{i} + 7\underline{j}$, respectively.
   (a) Find the vector equation of the line through A and B and show that the point with position vector $14\underline{i} + 13\underline{j}$ lies on this line.
   (b) If C is the point with position vector $3\underline{i} - 4\underline{j}$, find the vector equation of the line OC and show that OC is orthogonal to AB.
   (c) Find the position vector of the point N at which AB meets OC. Hence find the perpendicular distance from O to AB.

7. The position vectors of the four points A, B, C, D relative to an origin O are $\underline{a}$, $\underline{b}$, $-3\underline{a}$, and $-2\underline{b}$ respectively. The lines BA produced and CD produced meet at L. Show that the position vector of L is $9\underline{a} - 8\underline{b}$. [12]

8. * Given the three points A, B, C with position vectors $3\underline{a} + 2\underline{b}$, $-\underline{a} + 6\underline{b}$, and $-2\underline{a} + 7\underline{b}$ respectively, write down the vector equation of the line through A and C. Hence show that A, B, C are collinear and find the ratio $AB : BC$. [3,2,2]

9. Show, in terms of the components of vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, that

   (a) $\underline{c} + \underline{b} = \underline{b} + \underline{c}$ (commutative)

   (b) $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$ (associative)

   (c) $\lambda(\underline{a} + \underline{b}) = \lambda\underline{a} + \lambda\underline{b}$ (distributive)

   for any $\lambda \in \mathbb{R}$. [4,4,4]

10. Use components to show that:

    (a) the scalar product is commutative, i.e. $\underline{a}.\underline{b} = \underline{b}.\underline{a}$;

    (b) the scalar product is distributive, i.e. $\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$;

    (c) the vector product is distributive, i.e. $\underline{a} \wedge (\underline{b} + \underline{c}) = \underline{a} \wedge \underline{b} + \underline{a} \wedge \underline{c}$.

11. If $\underline{a} = 5\underline{i} - \underline{j} + 2\underline{k}$ and $\underline{b} = 2\underline{i} - 3\underline{j} + \underline{k}$, find the lengths of the vectors $\underline{a}$ and $\underline{b}$. Evaluate the scalar product $\underline{a}.\underline{b}$ and hence the cosine of the angle between the vectors $\underline{a}$ and $\underline{b}$.

12. * If $\underline{a} = x\underline{i} + y\underline{j}$ and $\underline{b} = 3\underline{i} + 2\underline{j}$, write down a relationship between x and y in each of the following cases:

    (a) $\underline{a}$ has length 5.

    (b) $\underline{a}$ and $\underline{b}$ are parallel.

    (c) $\underline{a}$ is orthogonal to $\underline{b}$.

    (d) $\underline{a} - \underline{b}$ is orthogonal to the vector $4\underline{i} + \underline{j}$. [2,2,2,4]

13. If $\underline{a} = \underline{i} - 2\underline{j} - 3\underline{k}$, $\underline{b} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{c} = \underline{i} + 3\underline{j} - 2\underline{k}$, find

    (a) $\underline{a}.\underline{b}$;

    (b) $\underline{b} \wedge \underline{c}$;

    (c) $\underline{a}.(\underline{b} \wedge \underline{c})$;

    (d) $\underline{a} \wedge (\underline{b} \wedge \underline{c})$.

14. Find the projection of the vector $2\underline{i} - 3\underline{j} + 6\underline{k}$ on the vector $\underline{i} + 2\underline{j} + 2\underline{k}$.

15. Find the volume of the parallelepiped spanned by the vectors $\underline{a} = 3\underline{i} - 2\underline{j} - \underline{k}$, $\underline{b} = \underline{i} + 3\underline{j} + 4\underline{k}$ and $\underline{c} = 2\underline{i} + \underline{j} - 2\underline{k}$.