MA10110 Assignment 2: Coordinate Geometry – Circles, Polars, Loci and Conics
SOLUTIONS

1. (a) If $A:(-3,8)$ and $B:(5,2)$ then $|AB| = 10$ so the radius is 5. The midpoint of AB is (1,5). Then $(x-1)^2+(y-5)^2 = 25$, or $x^2+y^2-2x-10y+1=0$.

   (b) Substitute $x=3, y=-1$ in the equation to give: $3^2+1-\lambda-2=0$, so $\lambda=8$. Then $x^2+y^2+8y-2=0$.

2. (a) Let $S(x,y) = x^2+y^2-x-y-\frac{1}{2}$. In standard form we have $a=-\frac{1}{2}$, $b=-\frac{1}{2}$ and $c=-\frac{1}{2}$. Therefore the centre of the circle is at $(\frac{1}{2},\frac{1}{2})$ and its radius is $r^2=\frac{1}{4}+\frac{1}{4}-\left(-\frac{1}{2}\right)=1$, so $r=1$.

   (b) The equation of the tangent to a circle at (x_1,y_1) is $(x_1+a)x+(y_1+b)y+ax_1+by_1+c=0$. In this case we have $(x_1-\frac{1}{2})x+(y_1-\frac{1}{2})y-\frac{1}{2}x_1-\frac{1}{2}y_1-\frac{1}{2}=0$.

   (i) $S(\frac{3}{2},\frac{1}{2})=0$, so this point is on the circle. Equation of tangent is $2x-3=0$.

   (ii) $S(\frac{1}{2}(1+\sqrt{3}),0)=0$, so this point is on the circle. Equation of tangent is $2\sqrt{3}x-2y-(3+\sqrt{3})=0$.

3. (a) Write $S(x,y)=x^2+y^2+2x+4y-2$. Comparing to standard form, we have $a=1$, $b=2$, and $c=-2$ so the centre is at (-1,-2) and the radius is $r=\sqrt{1^2+2^2-(-2)}=\sqrt{7}$.

   (b) Now $S(4,3)=16+9+8+12-2=43$, so the tangential distance is $\sqrt{43}$.

4. We have $x=5$ and $y=12$. So $r=\sqrt{5^2+12^2}=13$ and $\theta=\tan^{-1}(12/5)$ (which is $\approx 67°$).

5. $xy=x+y$ becomes $r^2\sin\theta\cos\theta=r(\cos\theta+\sin\theta)$, so $r=\dfrac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}=\dfrac{1}{\sin\theta}+\dfrac{1}{\cos\theta}=\mathrm{cosec}\theta+\sec\theta$, as required.

6. Expand the $\cos 2\theta$ to give $r^2(\cos^2\theta-\sin^2\theta)=1+r\cos\theta$. Now substitute Cartesians to give $x^2-y^2=1+x$.

7. Substitute x and y in polars: $(r\cos\theta-3)^2+(r\sin\theta+4)^2=2$. This expands to $r^2-6r(\cos\theta-\frac{4}{3}\sin\theta)+23=0$. This can be written $r^2-6r\cos(\theta+\theta_0)/\cos\theta_0+23=0$, where $\tan\theta_0=\frac{4}{3}$ and so $\cos\theta_0=\frac{3}{5}$. Hence the polar equation for the circle is $r^2-10r\cos(\theta+\theta_0)+23=0$ with $\theta_0=\tan^{-1}(\frac{4}{3})$.

8. Assume that the circle is at the origin: $x^2+y^2=16$. Write $S(x,y)=x^2+y^2-16$, then the tangential distance is $3=\sqrt{S(x,y)}=\sqrt{x^2+y^2-16}$. Expanding gives $x^2+y^2=16+9=25$ which is a circle with the same centre but radius 5.

9. W.l.o.g. move the circle to the origin and let it have unit radius; we can shift and magnify afterwards if required.

   If P has coordinates $(\hat{x},\hat{y})$ then $\hat{x}^2+\hat{y}^2=1$. Write the coordinates of K as (a,b). Call M the midpoint of PK, with coordinates (x,y), since it is the motion of M that we seek.

   From the definition of the midpoint we have $x=\frac{1}{2}(a+\hat{x})$ and $y=\frac{1}{2}(b+\hat{y})$. Now substitute for $P:(\hat{x},\hat{y})$ to give $(2x-a)^2+(2y-b)^2=1$. Rearrange to find $x^2+y^2-ax-by+\frac{1}{4}(a^2+b^2-1)=0$, which is a circle with centre $(\frac{1}{2}a,\frac{1}{2}b)$, i.e. half way between K and the centre of the circle, and radius $r=\frac{1}{2}$, i.e. half the radius of the original circle.

10. (a) Perpendicular bisector of AB;

    (b) Two semi-infinite straight lines from A at angle θ to AB;

    (c) Angle $\widehat{APB}$ is therefore a right angle, so P is on a circle with diameter AB;

    (d) Circle of radius $\frac{1}{2}|AB|+1$. Also, if $|AB|>2$, a circle of radius $\frac{1}{2}|AB|-1$.

    (e) Two arcs of circles with AB as chord (long arcs if $\theta<\pi/2$, short arcs if $\theta>\pi/2$);

11. Write $P:(x_1,y_1)$, $Q:(x_2,y_2)$ and the midpoint of PQ as $M:(x,y)$. We have $Ax_1+By_1+C=0$ and $Ax_2+By_2+D=0$. The midpoint has coordinates $\left(\frac{1}{2}(x_1+x_2),\frac{1}{2}(y_1+y_2)\right)$, so we can substitute $x_1=2x-x_2$ and $y_1=2y-y_2$ into $Ax_1+By_1+C=0$ to give $Ax+By+\frac{1}{2}(C-Ax_2-By_2)=0$. But we know that $-Ax_2-By_2=D$, so the equation of the line is $Ax+By+\frac{1}{2}(C+D)=0$; its slope is $-\frac{A}{B}$, which is the same as the slope of the two given lines, so they are indeed parallel.

12. Put $K : (a, b)$, so that $P : (x = \frac{1}{2}(a+2), y = \frac{1}{2}b)$.

   (a) Here $a = 0$, so this is the line $x = 1$, parallel to the y-axis.

   (b) We have $b = a - 1$, so $2x = a + 2$ and $2y = a - 1$; eliminating a gives the line $2x - 2y - 3 = 0$ parallel to the given line.

   (c) Now $a^2 + b^2 = 1$, so $(2x-2)^2 + (2y)^2 = 1$, or $4x^2 + 4y^2 - 8x + 4 = 1$, or $x^2 + y^2 - 2x + \frac{3}{4} = 0$. This is a circle, centre $(1, 0)$ and radius $\frac{1}{2}$.

13. Eccentricity is distance to focus divided by distance to directrix.

   (a) $e = \frac{1}{2}$, ellipse;

   (b) $e = \frac{3}{3} = 1$, parabola;

   (c) $e = \frac{\sqrt{5}}{4}$, ellipse;

   (d) $e = \frac{5}{6}$, ellipse.

14. W.l.o.g. take the focus to be at the origin, then in polars the conic is $r = \dfrac{ke}{1 + e\cos\theta}$. The value of $\Delta = \frac{1}{AF} + \frac{1}{BF}$ is the sum of the reciprocals of the distances from the origin of each of the points A and B. Then if θ denotes the inclination of ℓ we have:

$$\Delta = \frac{1}{r(\theta)} + \frac{1}{r(\theta+\pi)} = \frac{1 + e\cos\theta}{ke} + \frac{1 + e\cos(\theta+\pi)}{ke} = \frac{(1 + e\cos\theta) + (1 - e\cos\theta)}{ke} = \frac{2}{ke}$$

   which is constant for any given conic, since every conic has fixed k and e.

15. The slope of the tangent is given by $\frac{\mathrm{d}y}{\mathrm{d}x}$; implicit differentiation yields $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2y}$, which is equal to $\frac{1}{2}$ at the given point. Then the tangent has equation $y - 1 = \frac{1}{2}(x - 1)$, or $x - 2y + 1 = 0$.

16. (a) Slope of the tangent is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4a}{2y} = \frac{2a}{y} = \frac{1}{t}$. The slope of the normal is therefore $-t$.
   Tangent: $y - 2at = \frac{1}{t}(x - at^2)$, or $x - ty + at^2 = 0$;
   Normal: $y - 2at = -t(x - at^2)$, or $tx + y - 2at - at^3 = 0$.

   (b) The directrix lies along $x = -a$, so any point on the directrix has coordinates $(-a, b)$ for some $b \in \mathbb{R}$. The equation of the tangent to the parabola is $x - ty + at^2 = 0$. Substitution gives $-a - tb + at^2 = 0$ which is a quadratic for t with solution $t^{\pm} = \frac{1}{2a}(b \pm \sqrt{b^2 + 4a^2})$. This always has two real solutions, so there are indeed two possible tangents.
   Each has slope 1/t, and if they are orthogonal, then the product of their slopes should be -1. This product is

$$\frac{1}{t^+t^-} = \frac{2a}{(b + \sqrt{b^2+4a^2})}\frac{2a}{(b - \sqrt{b^2+4a^2})} = \frac{4a^2}{b^2 - (b^2 + 4a^2)} = -1,$$

   as required.

   (c) For each value of $t^{\pm}$, the tangent hits the curve at $(a{t^{\pm}}^2, 2at^{\pm})$. The slope of the line between these points is

$$m = \frac{2a(t^+ - t^-)}{a({t^+}^2 - {t^-}^2)} = \frac{2}{t^+ + t^-} = \frac{2a}{b},$$

   and therefore its equation is $y - 2at^+ = \frac{2a}{b}(x - a{t^+}^2)$ or

$$y = \frac{2a}{b}x + 2at^+ - \frac{2a^2}{b}{t^+}^2 = \frac{2a}{b}x + \frac{2a}{b}(bt^+ - a{t^+}^2) = \frac{2a}{b}x - \frac{2a^2}{b}$$

   from the quadratic for t. This equation does indeed have $y = 0$ when $x = a$, as required.

17. The tangent has slope $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{x^2} = -\dfrac{1}{t^2}$ and therefore its equation is $y - \frac{1}{t} = -\frac{1}{t^2}(x - t)$, or $x + t^2y - 2t = 0$. It meets the x axis at $A : (2t, 0)$ and the y axis at $B : (0, \frac{2}{t})$. The area of the triangle AOB is $\frac{1}{2} \times 2t \times \frac{2}{t} = 2$, which is independent of t.

18. The slope of the tangent $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ can be calculated from implicit differentiation of the equation of the ellipse: $\dfrac{2x\mathrm{d}x}{a^2} + \dfrac{2y\mathrm{d}y}{b^2} = 0$. Since it is perpendicular to the tangent, the slope of the normal is $-\dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{a^2}{b^2}\dfrac{y}{x} = \dfrac{a^2}{b^2}\dfrac{b\sin\theta}{a\cos\theta} = \dfrac{a}{b}\tan\theta$, as required.