MA10110 Assignment 1: Coordinate Geometry – Lines and Circles
SOLUTIONS

1. (a) $y = \frac{3}{2}x + \frac{7}{2}$, (b) $y = -\frac{5}{3}x + \frac{1}{3}$, (c) $x = -\frac{3}{7}$,
   The slope of the line $\alpha x + 2y - 1 = 0$ is $m = -\frac{\alpha}{2}$. This is equal to $\frac{3}{2}$ when $\alpha = -3$. [1,1,1,2]

2. (a) The length is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(7-3)^2 + (-3-9)^2} = \sqrt{16 + 144} = \sqrt{160} = 4\sqrt{10}$.
   (b) $M : \left(\frac{7+3}{2}, \frac{-3+9}{2}\right) = (5, 3)$.
   (c) The slope of AB is $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{12}{-4} = -3$. The slope of the perpendicular is therefore $\frac{1}{3}$ because the product of the slopes must be -1. Then we have $y - 3 = \frac{1}{3}(x - 5)$, or $x - 3y + 4 = 0$.
   (d) C is on a circle, centre M, with AB a diameter.
   Therefore the length MC = AM = MB = $\sqrt{(3-5)^2 + (9-3)^2} = \sqrt{40} = 2\sqrt{10}$.

3. The line through the intersection point has equation $3x - y + 3 + \lambda(x + y - 1) = 0$ for some real number λ.
   We write this in the form $(3 + \lambda)x + (\lambda - 1)y + 3 - \lambda = 0$. which has slope $m = \frac{3+\lambda}{1-\lambda}$.
   Setting $m = 1$ gives $\lambda = -1$, so the required line is $x - y + 2 = 0$. [1,2,2]

4. $y - 3 = m(x + 2)$, where m is the slope. In general form this is $mx - y + 2m + 3 = 0$.

5. The line through the intersection point has equation $3x - 5y + 41 + \lambda(5x + 6y - 8) = 0$, or $(3+5\lambda)x - (5-6\lambda)y + 41 - 8\lambda = 0$.
   To pass through (8,3), we need $3 \times 8 - 5 \times 3 + 41 + \lambda(5 \times 8 + 6 \times 3 - 8) = 0$, giving $\lambda = -\frac{50}{50} = -1$.
   Then the line is $2x + 11y - 49 = 0$. [2,2,1]

6. The slope of $ax + by + c = 0$ is $m = -a/b$. The line is therefore $y - q = -\frac{a}{b}(x - p)$. Rearrange to find the given answer. [5]

7. The line through the intersection point has equation $2x-3y+1+\lambda(5x+8y-3) = 0$, or $(2+5\lambda)x-(3-8\lambda)y+1-3\lambda = 0$.
   To pass through (-2,1) requires that $-4 - 3 + 1 + \lambda(-10 + 8 - 3) = -6 - 5\lambda = 0$.
   Thus $\lambda = -\frac{6}{5}$ and substituting for λ shows that the line is $-4x - \left(3 + \frac{48}{5}\right) y + 1 + \frac{18}{5} = 0$, or $20x + 63y - 23 = 0$.
   [2,1,2]

8. The line through the intersection point has equation $3x - 2y + 10 + \lambda(4x + 7y - 3) = 0$, or $(3+4\lambda)x - (2-7\lambda)y + 10 - 3\lambda = 0$.
   To pass through (0,1) requires that $-2 + 10 + \lambda(7 - 3) = 8 + 4\lambda = 0$.
   Thus $\lambda = -2$ and substituting for λ shows that the line is $5x + 16y - 16 = 0$. [2,3]

9. The slope of $3x - 6y + 1 = 0$ is $m = \frac{1}{2}$. The orthogonal line has slope -2, since the product of the slopes must be -1.
   Thus the required line has equation $y + 1 = -2(x - 3)$, or $2x + y - 5 = 0$. [1,2,2]

   or

   The line orthogonal to $3x - 6y + 1 = 0$ is of the form $6x + 3y + d = 0$ for some d. Substitute for $x = 3, y = -1$ to give $d = -15$. Then $6x + 3y - 15 = 0$, which reduces to the answer above. [1,2,2]

10. The midpoint of (5,-3) and (3,7) is M:(4,2).
    The slope of the line joining the points is $m = \frac{7+3}{3-5} = -5$, so the slope of the perpendicular is $\frac{1}{5}$.
    Then the line is $y - 2 = \frac{1}{5}(x - 4)$, or $x - 5y + 6 = 0$. [1,2,2]

11. The line $3x - 6y - 5 = 0$ has slope $m = \frac{1}{2}$.

    (a) The orthogonal line is of the form $6x + 3y + d = 0$ for some d. Substitute for $x = 3, y = -5$ to give $d = -3$.
    Then $6x + 3y - 3 = 0$, or $2x + y - 1 = 0$.

(b) With slope $\frac{1}{2}$, passing through (3,-5), the parallel line has equation is $y+5=\frac{1}{2}(x-3)$ or $x-2y-13=0$. [1,2,2]

12. We use the formula $\Delta=\frac{ax'+by'+c}{\sqrt{a^2+b^2}}$ for the distance of the point (x', y') from the line $l : ax+by+c=0$. Here $x'=2, y'=3, a=4, b=-3, c=5$, giving $\Delta=\frac{5+8-9}{\sqrt{4^2+3^2}}=\frac{4}{5}$. [2,3]

13. Write $F(x,y)=3x-9y+112$.
Calculate $F(-30,8)=-90-72+112=-50<0$, and $F(4,11)=12-99+112=25>0$. Since the signs are different, the points lie on opposite sides of the line.
$F(0,0)=112>0$, so (4,11) is on the same side of the line as the origin.
We need $0<F(5,c)=15-9c+112=127-9c$, giving $c<\frac{127}{9}$. [2,1,2]

14. Write $F(x,y)=4x-y+c$, with $F(-3,-6)=-12+6+c=c-6$ and $F(7,20)=8+c$.
We require these two expressions to have opposite signs, i.e. either $-8<c<6$ or $c<-8, c>6$.
A sketch indicates that the latter is not possible, so $-8<c<6$ is the required solution. [2,1,2]

15. For each line we evaluate $\dfrac{ax+by+c}{\sqrt{a^2+b^2}}$: $\dfrac{4x-3y+5}{5}$ and $\dfrac{x+4}{1}$. Now equate with different signs:
(i) positive: $4x-3y+5=5x+20 \Rightarrow x+3y+15=0$.
(ii) negative: $4x-3y+5=-5x-20 \Rightarrow 9x-3y+25=0$.

16. The line $y=3x-5$ has slope $\tan\theta_1=3$. The line $x+4y=1$ has slope $\tan\theta_2=-\frac{1}{4}$.
The angles $\phi^+=\theta_2-\theta_1$ and $\phi^-=\theta_1-\theta_2$ are given by

$$\tan\phi^+=\frac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_2\tan\theta_1}=\frac{-\frac{13}{4}}{\frac{1}{4}}=-13 \text{ and } \tan\phi^-=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_2\tan\theta_1}=\frac{\frac{13}{4}}{\frac{1}{4}}=13.$$

(So $\phi^+\approx 94.4^\circ$ and $\phi^-\approx 85.6^\circ$.)

17. Any line through the intersection of $U_2=0$ and $U_3=0$ has equation $U_2+\lambda U_3=0$. Its slope is $-\dfrac{a_2+\lambda a_3}{b_2+\lambda b_3}$, which we equate to $-\frac{a_1}{b_1}$ to give $\lambda=\dfrac{a_2b_1-a_1b_2}{a_1b_3-a_3b_1}$. Substitute this value for λ in $U_2+\lambda U_3=0$ and multiply by the denominator of λ (i.e. $a_1b_3-a_3b_1$) to give the required result.

18. The line $ax+by+c=0$ has slope $m=-\frac{a}{b}$. The orthogonal line has slope $\frac{b}{a}$, and therefore equation $y=\frac{b}{a}x+\bar{d}$, for arbitrary $\bar{d}$. Rearranging gives $bx-ay+d=0$, with d replacing $a\bar{d}$.
If this line passes through (p,q) then we must have $bp-aq+d=0$, which gives $d=aq-bp$, as required.

19. The line $y=2x-1$ has slope 2; if its inclination is θ, then $\tan\theta=2$.
We seek lines with inclination $\phi_1=\theta+\hat{\theta}$ and $\phi_2=\theta-\hat{\theta}$, with $\hat{\theta}=30^\circ$ and therefore $\tan\hat{\theta}=\frac{1}{\sqrt{3}}$.
The slopes of these lines are therefore

$$\tan\phi_i=\tan(\theta\pm\hat{\theta})=\frac{\tan\theta\pm\tan\hat{\theta}}{1\mp\tan\theta\tan\hat{\theta}}=\frac{2\pm\frac{1}{\sqrt{3}}}{1\mp\frac{2}{\sqrt{3}}}=\frac{2\sqrt{3}\pm 1}{\sqrt{3}\mp 2}=\frac{(2\sqrt{3}\pm 1)(\sqrt{3}\pm 2)}{(\sqrt{3}\mp 2)(\sqrt{3}\pm 2)}=\frac{8\pm 5\sqrt{3}}{-1}=-8\mp 5\sqrt{3},$$

as required.

20. The area of the triangle is given by the determinant

$$\mathcal{A}=\frac{1}{2}\begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 5 & 1 & 1 \end{vmatrix}=\frac{1}{2}\left|2(-2)+3(-4)+6\right|=\left|\frac{-10}{2}\right|=5.$$