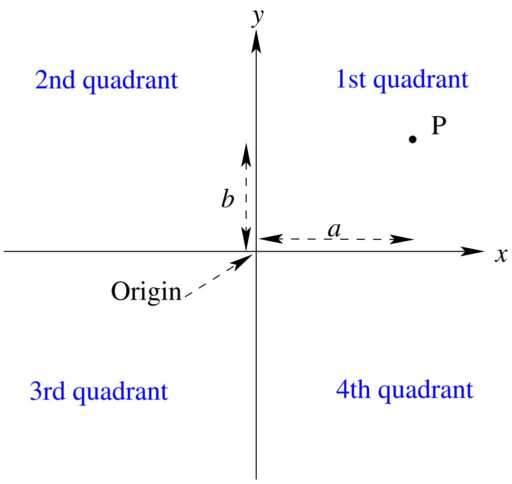
1 Cartesian Geometry (or analytic plane geometry)

1.1 Points and Lines

1.1.1 Points

Consider a point P:(a, b), where a and b can be any real numbers ($a, b \in \mathbb{R}$), referred to perpendicular axes x (abscissa) and y (ordinate); a is the x-coordinate of P and b the y-coordinate. With $a > 0, b > 0$ P is drawn in the *first quadrant*.



1.1.2 Lines

Definition 1.1. General equation of a line*: A line between two points P & Q is represented by the linear equation* $ax + by + c = 0$*, where* $a, b, c \in \mathbb{R}$ *are constant and* a *and* b *cannot both be zero.*

Definition 1.2. Slope-intercept equation*: If* $b = 0$ *then we have the vertical line* $x = \hat{c}$*; otherwise we can write the equation in the unique form* $y = mx + c$*, where* m *is called the* slope *and* c *the* intercept.

Example 1.1. *The line* $2x + 3y - 5 = 0$ *can be written* $y = \dfrac{-2x+5}{3} = -\dfrac{2}{3}x + \dfrac{5}{3}$*. It has slope* $-\frac{2}{3}$ *and intercept* $\frac{5}{3}$*.*

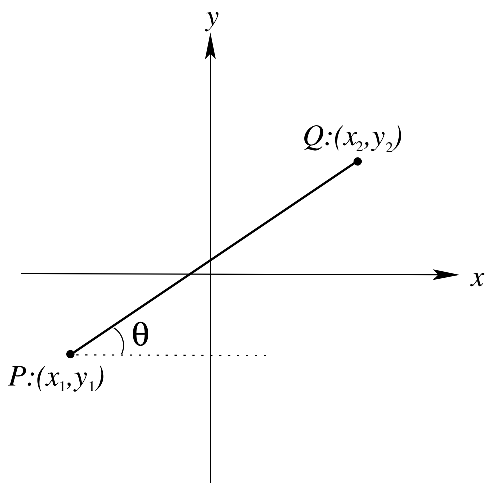
Example 1.2. *The line* $6x + 9y - 15 = 0$ *is the same one, since in slope-intercept form it is also* $y = -\dfrac{2}{3}x + \dfrac{5}{3}$*. It is useful to always cancel common factors in equations to help us identify when two geometric entities (e.g. lines) are equivalent.*

Example 1.3. *The line* $3x + 4 = 0$ *can be written* $x = -\frac{4}{3}$*; it is a vertical line with infinite slope.*

Remark: In general, $ax + by + c = 0$ has slope $\begin{cases} -a/b & \text{if } b \neq 0 \\ \infty & \text{if } b = 0 \end{cases}$.

Definition 1.3. *Two lines are parallel iff they have the same slope. Thus lines parallel to* $ax + by + c = 0$ *all have equation* $ax + by + d = 0$ *for arbitrary* d*.*

Equivalently, $ax + by + c = 0$ *is parallel to* $a'x + b'y + c' = 0$ *iff* $-\dfrac{a}{b} = -\dfrac{a'}{b'}$ *iff* $a'b = ab'$*.*



Definition 1.4. *Given two points P:*(x_1, y_1) *and Q:*(x_2, y_2) *we have:*
- *the distance PQ is* $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$*;*
- *the midpoint of PQ is* $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$*;*
- *the slope of PQ is* $\tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = m$*.*

Remark: The line of slope m through the point (x_1, y_1) has equation

$$y - y_1 = m(x - x_1)$$

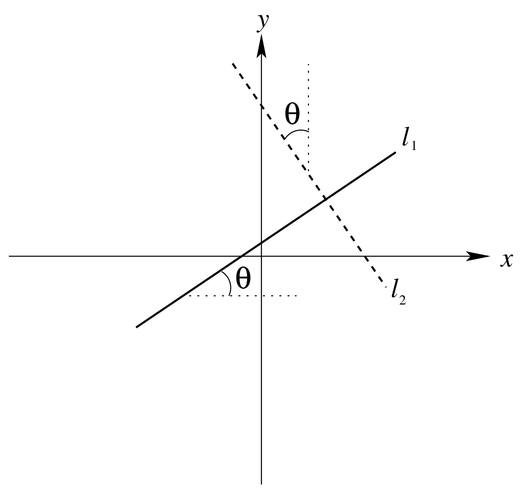
or $y - mx = y_1 - mx_1$ or, in general form, $mx - y + (y_1 - mx_1) = 0$.

Remark: The line joining the points (x_1, y_1) and (x_2, y_2), with $x_1 \neq x_2$, has equation

$$y - y_2 = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)(x - x_2),$$

with slope $m = \frac{y_1 - y_2}{x_1 - x_2}$.

Definition 1.5. *If two lines* $l_1 : y = m_1 x + c_1$ *and* $l_2 : y = m_2 x + c_2$ *are* **perpendicular** *then their slopes satisfy* $m_1 m_2 = -1$.



To see this, note that if we write $m_1 = \tan(\theta)$, then

$$m_2 = \tan(\pi/2 + \theta) = \frac{\sin(\pi/2 + \theta)}{\cos(\pi/2 + \theta)} = \frac{\cos(\theta)}{-\sin(\theta)} = -\frac{1}{\tan(\theta)}.$$

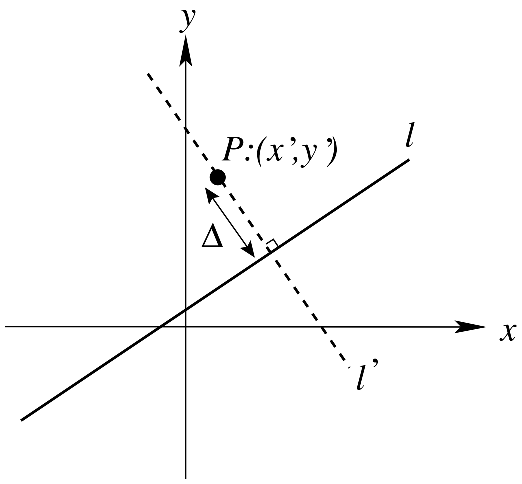
Remark: Therefore a line perpendicular or *orthogonal* to the line $l_1 : ax+by+c = 0$ has equation $bx-ay+d = 0$.

Example 1.4. *Find the equations of the lines which are parallel and orthogonal to* $2x + 3y + 5 = 0$ *and pass through P:(2,-5).*

Remark: We can write $F(x, y) = ax + by + c$. (F is a function of two variables, x and y. For example, $F(1, 0) = a + c$.) The equation of a line is then $F(x, y) = 0$.

1.1.3 Shortest distance from a point to a line

How do we find the shortest distance from a point P:(x', y') to a line $l : ax + by + c = 0$? The shortest distance should be along a line that is perpendicular to l, going through P. So we find the equation of the perpendicular line through P, and the point (x_1, y_1) where it intersects l. The distance from the point to the line is then just the distance between these two points.



The perpendicular takes the form $l' : bx - ay + d = 0$. We find d by substituting the coordinates of P to give $l' : bx - ay - bx' + ay' = 0$. We find the intersection of l and l' (simultaneous equations) to be

$$(x_1, y_1) = \left(\frac{-ac + b^2x' - aby'}{a^2 + b^2}, \frac{-bc - abx' + a^2y'}{a^2 + b^2} \right).$$

Then (by definition 1.4) the distance between P and (x_1, y_1) is

$$\begin{aligned}
\Delta &= \sqrt{(x' - x_1)^2 + (y' - y_1)^2} \\
&= \sqrt{\left(\frac{(a^2 + b^2)x' + ac - b^2x' + aby'}{a^2 + b^2}\right)^2 + \left(\frac{(a^2 + b^2)y' + bc + abx' - a^2y'}{a^2 + b^2}\right)^2} \\
&= \frac{1}{a^2 + b^2}\sqrt{(a^2x' + ac + aby')^2 + (b^2y' + bc + abx')^2} \\
&= \frac{1}{a^2 + b^2}\sqrt{a^2 (ax' + c + by')^2 + b^2 (by' + c + ax')^2} \\
&= \frac{c + ax' + by'}{\sqrt{a^2 + b^2}}.
\end{aligned}$$

If we write $F(x, y) = ax + by + c$, then $\Delta = \dfrac{F(x', y')}{\sqrt{a^2 + b^2}}$.

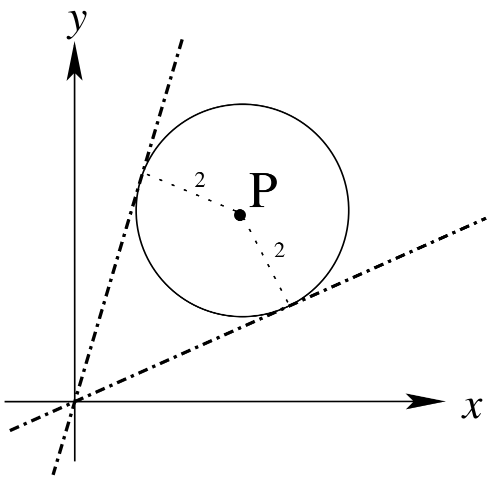
Since the distance must be a positive number, we should take the absolute value, distance = $|\Delta|$.

Remark: Note that we can evaluate F for any other point Q; if the sign of $F(x_Q, y_Q)$ is the same as the sign of $F(x_P, y_P)$, then P & Q lie on the same side of the line.

Example 1.5. *If l is the line $5x - 7y + 3 = 0$, then we put $F(x, y) = 5x - 7y + 3$.*
Now $F(0, 0) = 3$, $F(-3, -2) = 2$, $F(-3, -1) = -5$, and $F(5, 4) = 0$.
So (5,4) is on the line l, (-3,-2) is on the same side of the line as the origin, and (-3,-1) is on the other side.

Example 1.6. *Evaluate the perpendicular distance of the point P:(2,5) from the line $4x + 7y + 14 = 0$. For what values of c is the point $(c, 5)$ on the same side of the line as P?*

Example 1.7. *Find the equations of the lines through the origin whose perpendicular distance from P:(3,4) is 2.*



1.1.4 Intersecting lines

Consider two lines k and l with equations $F(x, y) = 0$ and $G(x, y) = 0$ respectively. Since F and G are functions of two variables, so is $F(x, y) + \lambda G(x, y)$, for some constant $\lambda \in \mathbb{R}$. Moreover, as F and G are linear, so is $F + \lambda G$; therefore $F(x, y) + \lambda G(x, y) = 0$ is the equation of a line, call it h.

If k and l are not parallel, then they intersect at some point P:(x_1, y_1), and at that point $F(x_1, y_1) = 0$, $G(x_1, y_1) = 0$, and so $F(x_1, y_1) + \lambda G(x_1, y_1) = 0$. Thus (x_1, y_1) is on the line h.

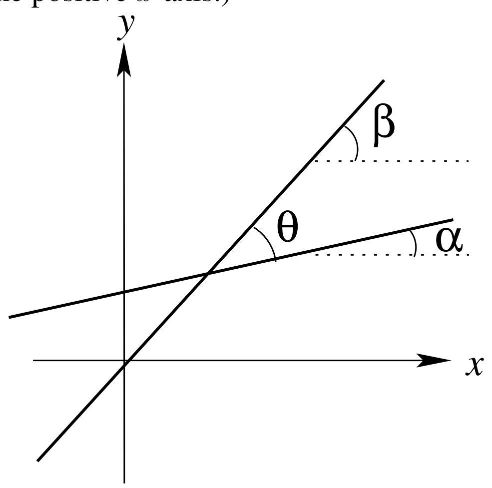
Definition 1.6. *We have shown that the* **general equation of a line through the intersection of two lines** *with equations $F(x, y) = 0$ and $G(x, y) = 0$ is $F(x, y) + \lambda G(x, y) = 0$.*

Example 1.8. *Find the equation of the line through the point Q:(1,-1) and the intersection of the lines $2x - 7y + 3 = 0$ and $5x + y - 8 = 0$.*

Example 1.9. *Find the equation of the line of slope 5 through the intersection of the lines $2x - 7y + 3 = 0$ and $5x + y - 8 = 0$.*

1.1.5 Angles between lines

Consider two lines that pass through the same point (intersect). What is the angle between them? That is, given line 1 with slope $m_1 = \tan\alpha$ and line 2 with slope $m_2 = \tan\beta$, what is $\tan(\beta - \alpha)$? (Note that angles are measured anticlockwise from the positive x-axis.)



We write $\theta = \beta - \alpha$, then

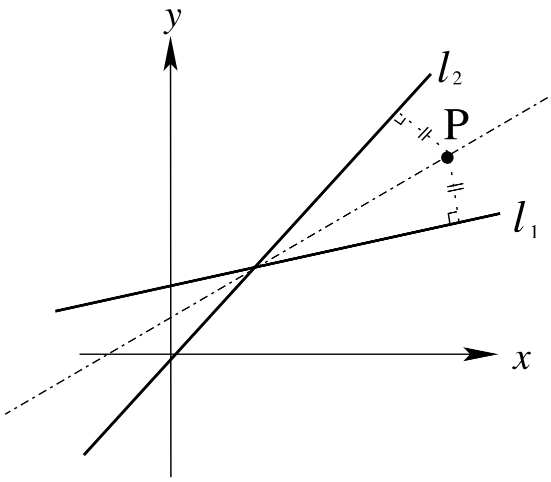
$$\tan\theta = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta} = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

Thus, two lines with slopes m_1 and m_2 meet at an angle $\theta = \tan^{-1}\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$.

Note that the other valid answer is $\theta' = \pi - \theta$, given by $\tan\theta' = \frac{m_1 - m_2}{1 + m_1 m_2}$.

Example 1.10. *Find the angle between the lines $3x - y + 1 = 0$ and $x + 2y - 1 = 0$.*

Definition 1.7. *The* **angular bisector** *of two lines is the line consisting of points which have the same perpendicular distance from each line.*



Since the distance of a point P:(x, y) from the line $ax + by + c = 0$ is $\Delta = \pm\frac{ax + by + c}{\sqrt{a^2 + b^2}}$ (if we take the other sign, then we will bisect the obtuse angle instead), then the equation of the angular bisector of $l_1 : ax + by + c = 0$ and $l_2 : a'x + b'y + c' = 0$ is

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm\frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}.$$

Example 1.11. *Find the angular bisectors (acute and obtuse) of the lines* $3x-4y+2 = 0$ *and* $12x+5y-2 = 0$.

1.1.6 Collinearity

Given a line through two points A:(x_1, y_1) and B:(x_2, y_2), how do we determine if the point P:$(\bar{x}, \bar{y})$ is on it? One possibility is to evaluate the area of the triangle APB: if it is zero, then the points are collinear. Another is to substitute for P in the equation of the line to see if it is satisfied. We will show that these are equivalent.

The equation of the line through A and B is

$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1),$$

or $(y_2 - y_1)x - (x_2 - x_1)y + y_1x_2 - x_1y_2 = 0$ in general form. So the condition for A, B and P to be collinear is

$$(y_2 - y_1)\bar{x} - (x_2 - x_1)\bar{y} + y_1x_2 - x_1y_2 = 0.$$

Definition 1.8. *The area of the triangle with vertices A, B and P can be evaluated from the following determinant*[1]*:*

$$\mathcal{A} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \bar{x} & \bar{y} & 1 \end{vmatrix} = \frac{1}{2}\left[x_1(y_2 - \bar{y}) - y_1(x_2 - \bar{x}) + x_2\bar{y} - y_2\bar{x}\right].$$

If $\mathcal{A} = 0$, then we have $x_1(y_2 - \bar{y}) - y_1(x_2 - \bar{x}) + x_2\bar{y} - y_2\bar{x} = 0$ which, after rearrangement, is the same as above.

The determinant therefore provides an easy way to determine the equation of a line: put $x \quad y \quad 1$ in the first row and 1s in the third column and complete the determinant with the coordinates of two points through which the line passes.

[1] we will explain this later in the module using vectors

Example 1.12. *Given points A:(3,-1) and B:(-2,3), the equation of the line through A and B is*

$$0 = \begin{vmatrix} x & y & 1 \\ 3 & -1 & 1 \\ -2 & 3 & 1 \end{vmatrix} = (-1-3)x - (3+2)y + (9-2) = -4x - 5y + 7.$$

Given A, B, and C:(4,7), the area of the triangle ABC is

$$A = \frac{1}{2} \begin{vmatrix} 3 & -1 & 1 \\ -2 & 3 & 1 \\ 4 & 7 & 1 \end{vmatrix} = \frac{1}{2} \left| 3(3-7) + (-2-4) + (-14-12) \right| = \frac{1}{2} \left| -12 - 6 - 26 \right| = 22.$$

So A, B and C are not *collinear.*