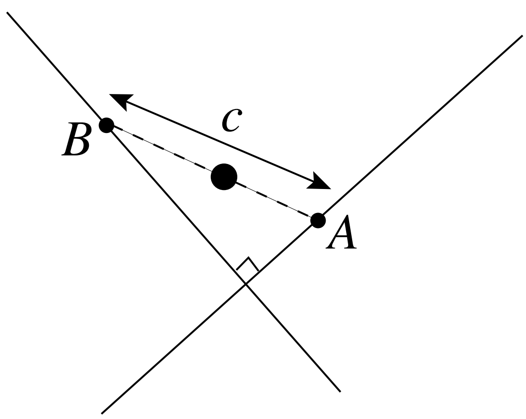
1.3 Locus of a point

The *path* of a point P that moves subject to certain conditions is called the locus of P. For example,
(i) the locus of a point that is always at a distance R from the fixed point (a, b) is a circle of radius R with centre (a, b);
(ii) if l is a fixed line, the locus of a point P which is always a fixed distance d from l consists of two lines parallel to l;
(iii) if k and l are distinct lines and P is at the same distance from both k and l, then there are two cases to consider: either k and l are parallel, in which case the locus of P is a straight line parallel to l and k and halfway between them; or if k and l are not parallel, then the locus of P is the two angular bisectors of k and l.

Example 1.20. *A point A moves on a line k and a point B moves on a line l perpendicular to k such that the length AB is a constant, c. What is the locus of the midpoint of AB?*



Example 1.21. *A point P moves such that its distance from the point (1,1) is always half the distance from the point (-1,1). Find its locus.*

Example 1.22. *A point P moves such that it is the same distance from the line $y = -1$ as from the point (0,2). Find its locus.*

Example 1.23. *A point P moves such that its distance from the point (2,0) is always two less than its distance from the point (-2,0). Find its locus.*

Solution: Write P:(x, y). Then we have $2 + \sqrt{(x-2)^2 + y^2} = \sqrt{(x+2)^2 + y^2}$. First, square both sides. Then rearrange to get the remaining square-root on one side of the equation; then square again. After some manipulation this becomes $x^2 - \frac{1}{3}y^2 = 1$, or $y = \pm\sqrt{3(x^2-1)}$, which is a hyperbola.
(Note that the other branch, at negative x, would be appropriate if the distance from (2,0) was two *more* than the distance from (-2,0).) □

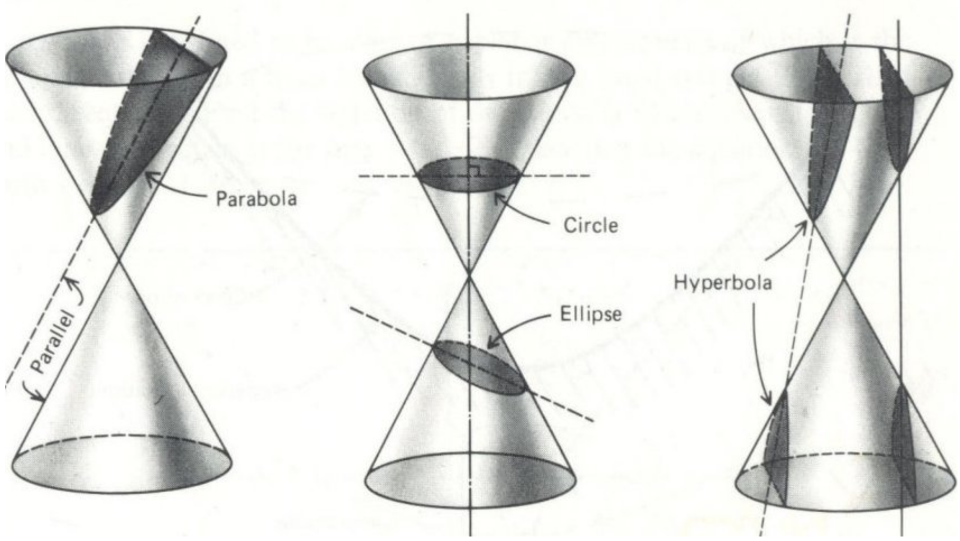
Example 1.24. *A point P moves such that the sum of its distances from the points (0,2) and (0,-2) is a constant, $2c$. Find its locus if $c > 2$.*

Solution: Write P:(x, y). Then we have $\sqrt{x^2 + (y-2)^2} + \sqrt{x^2 + (y+2)^2} = 2c$. Move one square-root to the other side of the equation and square both sides. Then rearrange to get the remaining square-root on one side of the equation; then square again. After some manipulation this becomes $\frac{x^2}{c^2-4} + \frac{y^2}{c^2} = 1$ which is an ellipse.
Note that we must have $c > 2$, else the locus would be empty. □

1.4 Conic Sections

1.4.1 Focus and directrix

Parabolas, ellipses (including the special case of the circle) and hyperbolas are called *conic sections*, or conics, because they can be visualized as cuts (or sections) through a double cone, as shown below.



An alternative definition, as we have seen above, can be given in terms of the locus of a point (see figure 1):

Definition 1.13.
The **parabola** *is the set of points in the plane that are equidistant from a given fixed point (the focus) and a given fixed line (the directrix). In* **standard form** *the focus is at (a,0), the directrix is at* $x = -a$, *and the equation of a parabola is* $y^2 = 4ax$.

The **ellipse** *is the set of points in the plane whose distance from two given fixed points (the foci) have a constant sum. In* **standard form** *the foci are at (±c,0), the sum of distances from the foci is* $2a$, *and the equation of an ellipse is* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *with* $b^2 = a^2 - c^2$.

The **hyperbola** *is the set of points in the plane whose distance from two given fixed points (the foci) have a constant difference. In* **standard form** *the foci are at (±c,0), the difference in distances from the foci is* $2a$, *and the equation of a hyperbola is* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *with* $b^2 = c^2 - a^2$.

Example 1.25. *Show that the equation* $y^2 = k(x + k)$ *represents a parabola. Find its focus and directrix.*

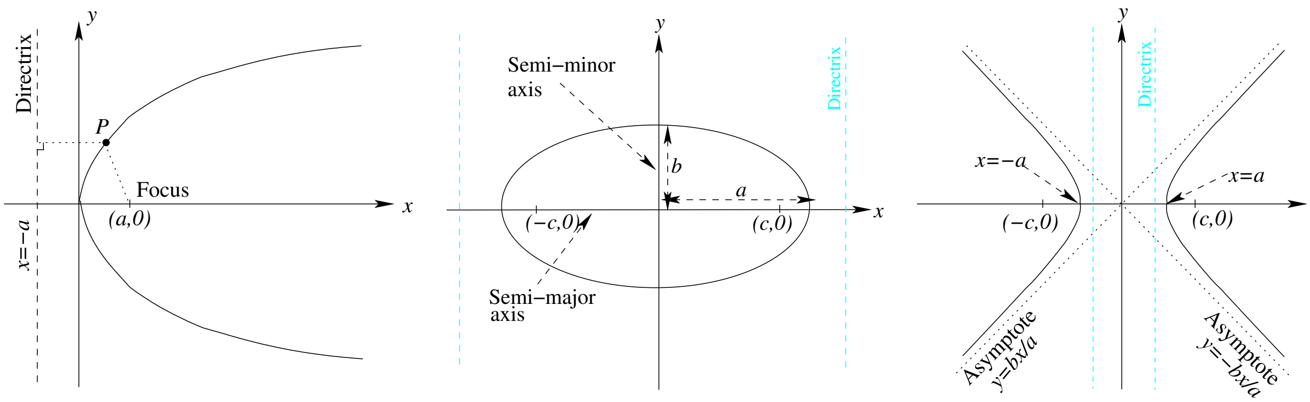
Example 1.26. *Show that the equation* $x^2 - 3y^2 = 2\alpha y - \alpha^2$ *represents a hyperbola. Find its foci and asymptotes.*

1.4.2 Eccentricity

The eccentricity of an ellipse measures how far from a circle it is. We define the dimensionless number

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2} < 1.$$

So when $b = a$ we have a circle, with $e = 0$, and in the limit $b \to 0$ we find $e \to 1$.



(a) (b) (c)

Figure 1: (a) Parabola. (b) Ellipse, with foci at $x = \pm c = \pm\sqrt{a^2 - b^2}$. (c) Hyperbola, with foci at $x = \pm c = \pm\sqrt{a^2 + b^2}$.

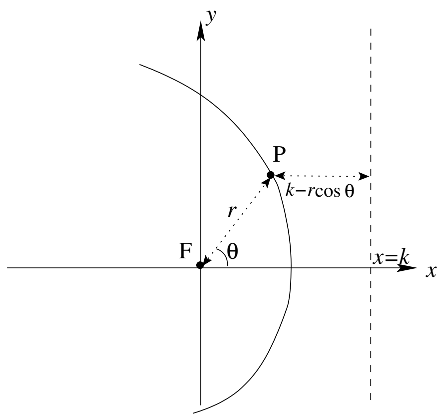
This definition of eccentricity can be extended to a parabola ($e = 1$) and a hyperbola $e = \sqrt{1 + \left(\frac{b}{a}\right)^2} > 1$ if we define an ellipse and a hyperbola to have (notional) directrices at $x = \pm\frac{a}{e}$. Then the eccentricity is the ratio of the distance of a point on the conic from the focus to the distance from the directrix:

$$e = \frac{\text{distance to focus}}{\text{distance to directrix}}.$$

Thus each point on an ellipse is closer to the (nearest) focus than to the (nearest) directrix ($e < 1$);
each point on a hyperbola is farther from the (nearest) focus than from the (nearest) directrix ($e > 1$);
each point on a parabola is equidistant from the focus and the directrix ($e = 1$);

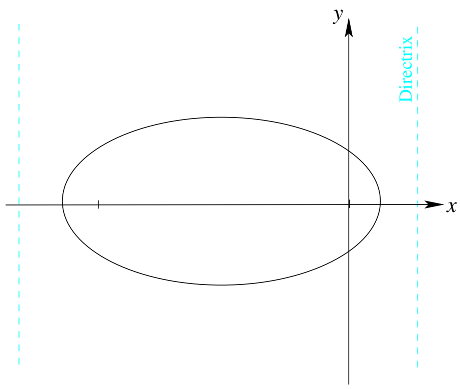
Definition 1.14. *If we place the focus at the origin and the directrix at $x = k$ then in polar form the equation of a conic is $r = \dfrac{ke}{1 + e\cos\theta}$.*

Remark: To see this note that $e = r/(k - r\cos\theta)$, since eccentricity is distance to focus divided by distance to directrix.



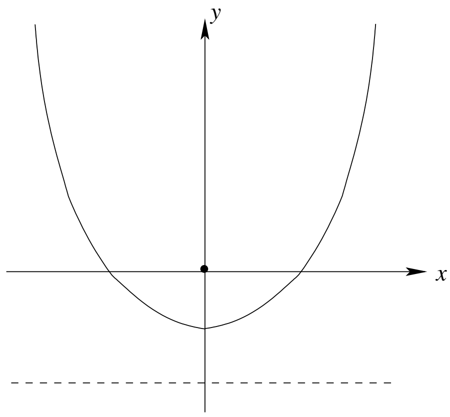
If there is a change of sign in the denominator, to $1 - e\cos\theta$, the directrix is at $x = -k$.
If $\pm\cos\theta$ is replaced by $\pm\sin\theta$, then the directrix is at $y = \pm k$ respectively.

Example 1.27. *Find a polar equation for the conic with eccentricity $e = \frac{1}{2}$, one focus at the origin and corresponding directrix at $x = 1$.*



Example 1.28. *Find the directrix of the conic with focus at the origin and polar equation* $r = \dfrac{8}{2 - 2\sin\theta}$.

Solution: First re-write in the simpler form $r = \dfrac{4}{1 - \sin\theta}$, and compare with $r = \dfrac{ke}{1 + e\cos\theta}$.
$\sin$ instead of $\cos$ means that the directrix is $y = cst$.
The coefficient of $\sin\theta$ is one, so this is a parabola ($e = 1$).
The minus sign indicates that the directrix is at negative y.
Therefore the directrix is at $y = -4$. □



1.4.3 Parametric representation of a conic

Circle: Any point (x, y) on the unit circle can be expressed in terms of a parameter t (or, equivalently, θ) in the range $[0, 2\pi]$ through $x = \cos(t)$, $y = \sin(t)$. Eliminating t by squaring and adding shows that this is indeed the unit circle $x^2 + y^2 = 1$. The parameter t labels points around the circle, starting and finishing at (1,0). We can find an expression for the slope $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at any point on the circle in terms of t, and hence the equation of the tangent to the circle using the expression after Definition 1.4, and similarly for the normal.

Parabola: any point P on a parabola can be written in terms of a parameter $t \in (-\infty, \infty)$, e.g. $x = at^2, y = 2at$: we can eliminate t from these two expressions to give $y^2 = 4ax$, the standard equation.

The gradient at any point is $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{a}{x}} = \frac{2a}{y} = \frac{1}{t}$. The equation of the tangent to the parabola is therefore $y - 2at = \frac{1}{t}(x - at^2)$, or

$$x - ty + at^2 = 0.$$

In the same way, the equation of the normal is $y - 2at = -t(x - at^2)$, or $tx + y = at^3 + 2at$.

Example 1.29. *Show that the normal to the parabola* $y^2 = 4ax$ *at the point P:*$(at^2, 2at)$ *meets the parabola again at the point* $(au^2, 2au)$ *with* $u = -\frac{1}{t}(2 + t^2)$.

Remark: The parametric representation of a conic, or any curve in general, is not unique. In its standard form, an ellipse has parameterisation $x = a\cos t, y = b\sin t$, with $t \in (-\pi, \pi]$ often replaced with θ. The standard form of the hyperbola can be parameterised by $x = a\sec t, y = b\tan t$, with $t \in (-\infty, \infty)$, or $x = a\cosh u, y = b\sinh u$, with $u \in (-\infty, \infty)$.