2.6 Kinematics

2.6.1 Position, Velocity, Acceleration

Let P be a particle moving relative to Cartesian axes $Oxyz$ with position $(x(t), y(t), z(t))$, where t denotes time, and therefore position vector $\underline{r} = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$. As the particle moves, the vector $\underline{r}$ changes. The *velocity* of P is the rate of change of position, defined as $\underline{v} = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j} + \dot{z}(t)\underline{k}$, with the $\dot{}$ denoting differentiation with respect to time. So the components of the velocity are $\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)$ and the magnitude of the velocity is called the speed, $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. In the same way, the acceleration of P is $\underline{a} = \ddot{x}(t)\underline{i} + \ddot{y}(t)\underline{j} + \ddot{z}(t)\underline{k}$.

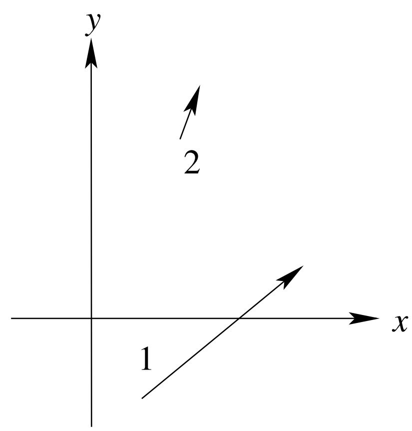
Example 2.30. *The position vector of a particle P is given by*

$$\underline{r} = k(1 + \cos t)\underline{i} + k(1 + \sin t)\underline{j} + 0\underline{k}$$

where t is time and $k \in \mathbb{R}$. Find the magnitudes of P's velocity and acceleration.

Example 2.31. *A particle P has velocity $3\underline{i}+2\underline{j}$ m/s. If P is initially at (3,-4)m, find its position after 3 seconds.*

Example 2.32. *Two ships are observed (at the same time) to have the following displacement (or position) vectors ($\underline{s}$) and velocity vectors ($\underline{v}$):*
$\underline{s}_1 = 3\underline{i} - 2\underline{j}$ *km,* $\underline{v}_1 = 10\underline{i} + 10\underline{j}$ *km/h;*
$\underline{s}_2 = 4\underline{i} + 6\underline{j}$ *km,* $\underline{v}_2 = \underline{i} + 3\underline{j}$ *km/h.*
If the velocities remain constant, find the time at which the two ships will be closest together, and their distance apart at that time.



Example 2.33. *Repeat example 2.32 with $\underline{v}_1$ replaced by $\underline{v}_1 = 2\underline{i} + 11\underline{j}$ km/h.*

2.6.2 Relative velocity

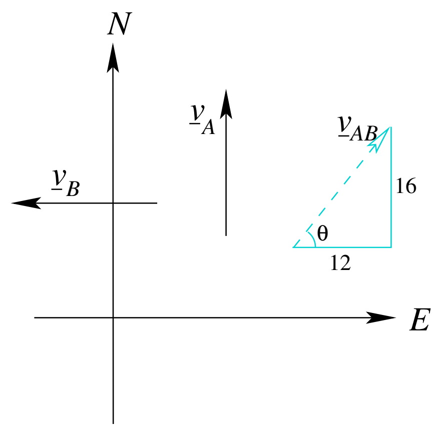
Consider two aircraft P and Q flying on the same straight line but with different speeds. Assuming that P is ahead of Q and flying faster, it will appear to an observer in Q that P is flying away from her with an apparent speed equal to the rate at which the distance between them increases, that is, the difference in their velocities.

In general, the two aircraft might also have different directions. An observer in Q will estimate the apparent velocity of P, called the *velocity of P relative to Q*, by the rate at which the displacement between them changes in magnitude and direction.

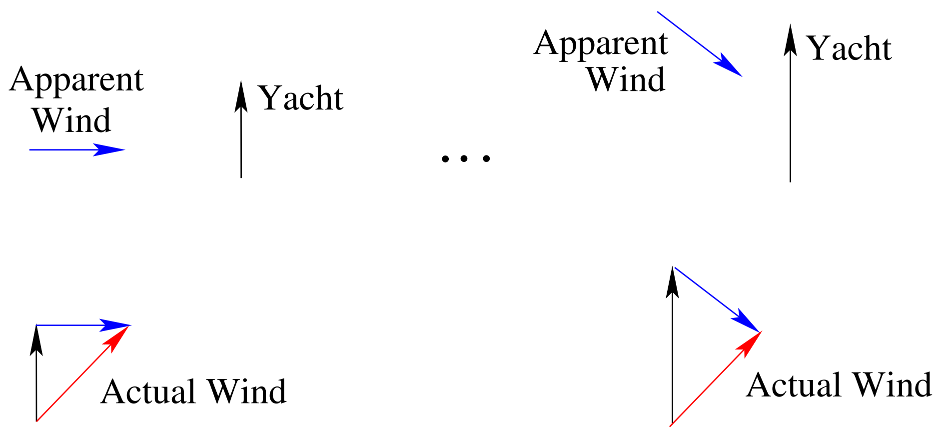
Definition 2.9. *The velocity of P relative to Q is the rate of change of the displacement $\overrightarrow{QP}$, i.e.*

$$\underline{v}_{\mathrm{PQ}} = \frac{\mathrm{d}}{\mathrm{d}t}(\overrightarrow{QP}) = \frac{\mathrm{d}}{\mathrm{d}t}(\underline{r}_{\mathrm{P}} - \underline{r}_{\mathrm{Q}}) = \frac{\mathrm{d}}{\mathrm{d}t}(\underline{r}_{\mathrm{P}}) - \frac{\mathrm{d}}{\mathrm{d}t}(\underline{r}_{\mathrm{Q}}) = \underline{v}_{\mathrm{P}} - \underline{v}_{\mathrm{Q}}.$$

Example 2.34. *A ship A is steaming due north at 16 km/h and a ship B is steaming due west at 12 km/h. Find the relative velocity of A with respect to B.*



Example 2.35. *On a yacht travelling north at 14 km/h it feels as if the wind is blowing from the west. The sails are adjusted and the yacht doubles its speed; the wind then appears to come from the north-west. Find the velocity of the wind.*



Example 2.36. *A stream of width* d *metres has a uniform current with speed 3 m/s. A pond skater sets off perpendicular to the bank with constant speed 4 m/s. After how long does it reach the far bank? How far has it travelled? How far downstream does it land?*

How much longer does the crossing take if the stream flows at 7 m/s?