3 Euclidean (axiomatic) geometry - an historical postscript

Probably the greatest achievement of ancient geometry, even mathematics, is the book "Elements" by Euclid. By assuming a few things that everybody could agree must be true, known as axioms, many results could then be proven.

First, many geometric objects are defined, including:

D1: A point is that which has no parts or magnitude.
D2: A line is length without breadth.
D3: The extremities of a line are points.
D4: When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle.
D5: An acute/obtuse angle is that which is less/greater than a right angle.
D6: An equilateral triangle is that three-sided figure which has three equal sides.
D7: An isosceles triangle is that three-sided figure which has two equal sides.
D8: A rhombus is that four-sided figure (quadrilateral) which has all its sides equal but its angles are not right angles.
D9: A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a centre point (the "centre") are equal to one another.

The *axioms* include:

A1: Things which are equal to the same thing are equal to one another.
A2: If equals are added to equals, then the wholes are equal.
A3: The whole is greater than the part.

Then the postulates are given:

P1: given any two distinct points P & Q, there exists a unique (straight) line containing both points. We denote the line PQ or QP.
P2: any finite line segment can be extended indefinitely in either direction.
P3: given a point P and a length r, a circle with centre P and radius r can be drawn.
P4: all right angles are equal to one another.
The only contentious postulate is the fifth one, the "parallel postulate":
P5: If a straight line meets two straight lines so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which the angles are less than two right angles.
It would appear that this could be proven from the other axioms, definitions and postulates, but no such proof could be found.

Written another way, we have: given a point P and line l not containing P, there exists a unique line l_p through P parallel to l. These five postulates give rise to what is known as *Euclidean geometry*. Without the fifth one, we can have different versions of *non-Euclidean* geometry, for example:
- elliptic geometry: there are no parallels to l through P (e.g. on the surface of a sphere);
- hyperbolic geometry: there are infinitely many parallels to l through P.

Within the formalism of Euclidean geometry, we can prove Euclid's **Proposition 1**: to describe (i.e. draw) an equilateral triangle on a given finite straight line segment.

Proof. Let AB be the given straight line; it is required to construct an equilateral triangle with AB as one of its sides.

Step 1: with centre A, describe the circle BCD with radius AB (we can do this by **P3**).

Step 2: with centre B, describe the circle ACE with radius AB (again, **P3** allows us to do this).

Step 3: from the point C where the circles intersect, draw the straight lines AC and AB (we know that each is unique by **P1**).

Now we have to show that the triangle ABC is equilateral:

- since the point A is the centre of the circle BCD, we know that AC=AB (by **D9**);
- since the point B is the centre of the circle ACE, we know that BC=AB.

So CA and CB are both equal to AB. By axiom **A1** they are therefore equal, CA = CB. Thus CA=CB=AB and the triangle is equilateral. □