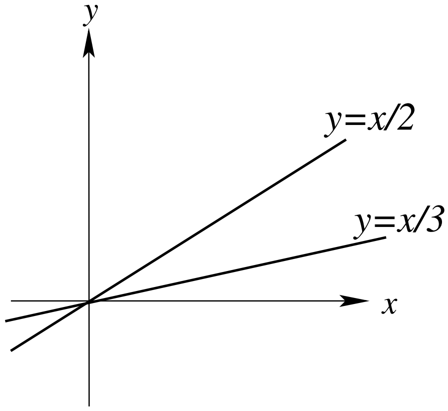
1.2 Curves

In general, the function $F(x, y)$ may not be linear, and therefore $F(x, y) = 0$ describes a curve.

Example 1.13.
(i) $F(x, y) = x^2 + y^2 - 4$. The curve described by $F(x, y) = 0$ is the circle centred at the origin with radius 2.
(ii) $F(x, y) = x^2 - y - 1$. $F(x, y) = 0$ describes a parabola.
(iii) $F(x, y) = x^2 + y^2 + 1$. The curve described by $F(x, y) = 0$ is empty*, i.e. there are no points in the plane that satisfy $x^2 + y^2 = -1$.*
(iv) $G(x, y) = x^2 + y^2$. $G(x, y) = 0$ consists of one point, the origin.
(v) $G(x, y) = x^2 - 5xy + 6y^2$. We can factorise this, giving $G(x, y) = (x - 2y)(x - 3y)$; then $G(x, y) = 0$ consists of all the points on the lines $y = x/2$ and $y = x/3$.

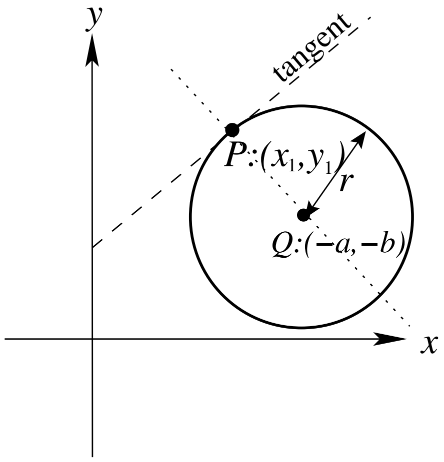


1.2.1 General Equation of a Circle

Consider the expression $S(x, y) = x^2 + y^2 + 2ax + 2by + c$, where $a, b, c \in \mathbb{R}$. We complete the square (twice) to give: $S(x, y) = (x + a)^2 + (y + b)^2 + c - a^2 - b^2$. Then $S(x, y) = 0$ is a circle, centre $(-a, -b)$ and radius $r^2 = a^2 + b^2 - c$.

Example 1.14.
(i) If $S(x, y) = x^2 + y^2 - 6x + 8y + 21$, then $S(x, y) = 0$ describes a circle (with $2a = -6$, $2b = 8$ & $c = 21$) centred at (3,-4) with radius $r = \sqrt{3^2 + 4^2 - 21} = 2$.
(ii) If $S(x, y) = x^2 + y^2 + 5x - 1$, then $S(x, y) = 0$ describes a circle centred at $(-\frac{5}{2}, 0)$ with radius $r = \sqrt{(5/2)^2 + 1} = \frac{1}{2}\sqrt{29}$.
(iii) If $S(x, y) = 5x^2 + 5y^2 + 10x - 12y - 2$, then we write $S(x, y) = 5(x^2 + y^2 + 2x - \frac{12}{5}y - \frac{2}{5})$ to show that $S(x, y) = 0$ describes a circle centred at $(-1, \frac{6}{5})$ with radius $r = \sqrt{1^2 + \left(\frac{6}{5}\right)^2 + \frac{2}{5}} = \frac{1}{5}\sqrt{71}$.
(iv) If $S(x, y) = x^2 + y^2 + 4x + 6y + 15$, then $S(x, y) = 0$ appears to be a circle centred at (-2,-3) but the radius is $r = \sqrt{4 + 9 - 15}$ which is not a real number, so this does not *describe a circle.*

1.2.2 Tangent to a circle



Consider the circle S with centre Q:$(-a,-b)$, radius $r=\sqrt{a^2+b^2-c}$ and equation $x^2+y^2+2ax+2by+c=0$. We define $S(x,y)=x^2+y^2+2ax+2by+c$. If $P:(x_1,y_1)$ lies on the circle, we have $S(x_1,y_1)=0$. The tangent to the circle at P is perpendicular to the line PQ and goes through P. The slope of PQ is $\frac{(y_1+b)}{(x_1+a)}$ so the tangent has slope $-\frac{(x_1+a)}{(y_1+b)}$ and thus equation $y-y_1=-\left(\frac{x_1+a}{y_1+b}\right)(x-x_1)$. We expand to give $yy_1+by-y_1^2-by_1+xx_1+ax-x_1^2-ax_1=0$. Now, since P satisfies $S(x_1,y_1)=0$, we can substitute $-(x_1^2+y_1^2)=c+2ax_1+2by_1$ to give $yy_1+xx_1+a(x+x_1)+b(y+y_1)+c=0$. As a sanity check, note that this is linear in both x and y, so it really is a line. Rearranging shows that:

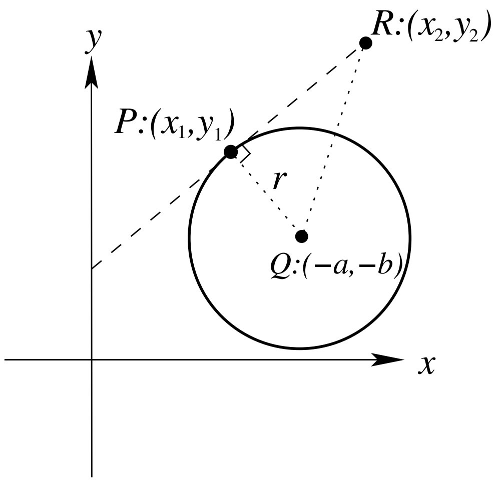
Definition 1.9. *In standard form the equation of the tangent to the circle $x^2+y^2+2ax+2by+c=0$ at the point (x_1,y_1) is*

$$(x_1+a)x+(y_1+b)y+ax_1+by_1+c=0.$$

Example 1.15. *Find the tangent to the circle $S:x^2+y^2-6x-3y-5=0$ at (1,-2).*

1.2.3 Tangential distance

Definition 1.10. *The distance of a point R:(x_2,y_2) from the point P at which the tangent through R touches the circle $S:x^2+y^2+2ax+2by+c=0$ with centre Q is called the* tangential distance *of R from S, and is given by $\sqrt{S(x_2,y_2)}$.*



To see this, note that

$$\begin{aligned}(\text{RP})^2 &= (\text{RQ})^2 - (\text{PQ})^2 \\ &= (x_2 + a)^2 + (y_2 + b)^2 - (a^2 + b^2 - c) \\ &= x_2^2 + 2ax_2 + y_2^2 + 2by_2 + c \\ &= S(x_2, y_2).\end{aligned}$$

This does *not* hold if R is inside the circle, although it is true that $S(x_2, y_2) < 0$ in this case.

Thus, given a circle $S(x, y) = 0$ and a point R:(x_2, y_2) we have:

$S(x_2, y_2) < 0$ if R is inside the circle;
$S(x_2, y_2) = 0$ if R is on the circle (circumference);
$S(x_2, y_2) > 0$ if R is outside the circle, with tangential distance of $\sqrt{S(x_2, y_2)}$.

Example 1.16. *Given the circle $S : x^2 + y^2 - 3x + 5y + 2 = 0$, determine if the point (2,1) is inside or outside the circle and, if outside, how far it is from the point where the tangent through it meets the circle.*

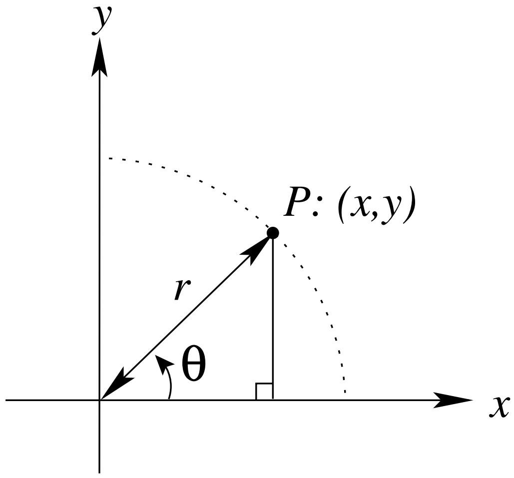
Example 1.17. *For which values of c is the point $(c, -1)$ outside the circle $S : x^2 + y^2 - 3x + 5y + 2 = 0$?*

1.2.4 Polar coordinates

Instead of Cartesian (x, y) coordinates, we can describe the position of a point in terms of polar coordinates (r, θ), where r is distance from the origin and θ is (anticlockwise) angular distance from the x-axis. The two coordinate systems are related by

$$\begin{aligned} x &= r\cos\theta \qquad & r &= \sqrt{x^2 + y^2} \\ y &= r\sin\theta \qquad & \theta &= \tan^{-1}\left(\tfrac{y}{x}\right), \end{aligned}$$

as can be seen from the right-angled triangle in the diagram below. The angle θ is usually restricted to the range $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$.



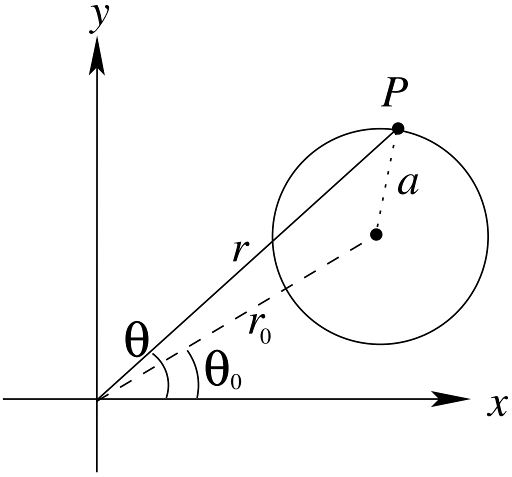
Then the circle $x^2 + y^2 - a^2 = 0$ can be written $r = a$; the line $y = x\tan\theta_0$ can be written $\theta = \theta_0$; and the line $x = 2$ can be written $r\cos\theta = 2$.

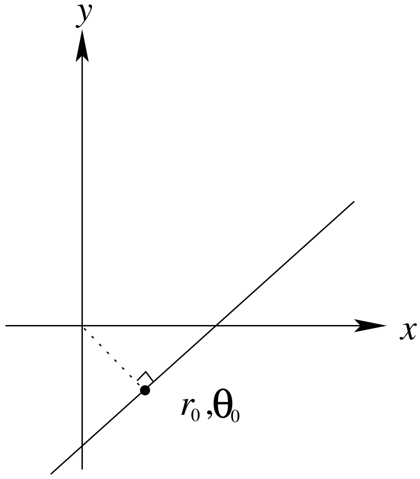
Example 1.18. *Find a polar equation for the circle $(x - 1)^2 + (y - 3)^2 = 16$.*

Definition 1.11. *The* **polar equation of a circle** *of radius a centred at $r = r_0, \theta = \theta_0$ is*

$$r^2 - 2r_0 r\cos(\theta - \theta_0) = a^2 - r_0^2.$$

To see this, consider a point P on the circumference of the circle, with distance r from the origin and subtending an angle θ, and use the cosine law.





Definition 1.12. *The* **polar equation of a line** *is*

$$r \cos(\theta - \theta_0) = r_0,$$

where the point $r = r_0, \theta = \theta_0$ lies on the line at the foot of the perpendicular through the origin. This can be derived using the sine law.

Example 1.19. *Find a polar equation for the line $x + \sqrt{3}y - 2 = 0$.*