Asymptotic Completeness for Weak Markov Processes

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Noncommutative Markov Processes

Noncommutative Markov processes are models for quantum systems. In the $1990's$ Bhat and Parthasarathy introduced socalled 'weak' processes as a simple common core of existing theories. Roughly:

The dynamics is a semigroup $(\theta_t)_{t\geq 0}$
of $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$ (or a C^*-algebra),

weak filtration: increasing sequence of subspaces $(\mathfrak{h}_t)_{t\geq 0}$,

weak Markov property: The transition operators

$$Z_t := P_0\,\theta_t(x)\,P_0$$

$\big[P_t$ projects onto $\mathfrak{h}_t$ and $x \in \mathcal{B}(\mathfrak{h}_0) = P_0\,\mathcal{B}(\mathcal{H})\,P_0\ \big]$
form a semigroup and

$$P_s\,\theta_{s+t}(x)\,P_s = \theta_s(Z_t(x))$$

Processes with discrete time parameter $t = n \in \mathbb{N}_0$

discrete (weak Markov) **process:** $(\mathcal{H}, V, \mathfrak{h})$ with

- $\mathcal{H}$ Hilbert space
- V row isometry $\left[\ V : \mathcal{H} \otimes \mathcal{P} \to \mathcal{H}\right.$ isometry, equivalently: $V = (V_1, \ldots, V_d)$ with V_k isometries with orthogonal ranges $\left.\right]$
- $\mathfrak{h} \subset \mathcal{H}$ co-invariant subspace $\left[\ V_k^* \mathfrak{h} \subset \mathfrak{h}\right.$ for all $\left.k\ \right]$
- minimality: $\mathcal{H} = \overline{span}\{V_\alpha \mathfrak{h} : \alpha \in F_d^+\}$ $\left[\ F_d^+\right.$ free semigroup $\left.\right]$

An operator theorist can look at it as the minimal isometric dilation of the compression of V to $\mathfrak{h}$.

related to our previous notion of weak process by

$$\theta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}),\ X \mapsto V\, X \otimes \mathbb{1}\, V^* = \sum V_k X V_k^*$$

$$\mathfrak{h}_0 = \mathfrak{h},\ \mathfrak{h}_n = P_n \mathfrak{h} \text{ with } P_n = sup(P_0, \theta(P_0), \ldots, \theta^n(P_0))$$

Subprocesses and Quotient Processes

$(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g})$ is called a **subprocess** of the process $(\mathcal{H}, V, \mathfrak{h})$ if $\mathfrak{g}$ is a closed subspace of $\mathfrak{h}$ which is co-invariant for V and $V^{\mathcal{G}} = V|_{\mathcal{G}}$ where $\mathcal{G} = \overline{span}\{V_\alpha \mathfrak{g} \colon \alpha \in F_d^+\}$.

$\mathfrak{g}$ is also co-invariant for $V^{\mathcal{G}}$ and $(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g})$ is a process.

Given a subprocess $(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g})$ of a process $(\mathcal{H}, V, \mathfrak{h})$ we can form the **quotient process**

$$(\mathcal{H}, V, \mathfrak{h})/(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g}) := (\mathcal{K}, V^{\mathcal{K}}, \mathfrak{k})$$

where $\mathfrak{k} := \mathfrak{h} \ominus \mathfrak{g},\ \mathcal{K} := \overline{span}\{V_\alpha \mathfrak{k} \colon \alpha \in F_d^+\},\ V^{\mathcal{K}} := V|_{\mathcal{K}}$.

$(\mathcal{K}, V^{\mathcal{K}}, \mathfrak{k})$ is a process. In general $\mathfrak{k}$ is not co-invariant for V.

short exact sequence of processes:

$$0 \longrightarrow (\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g}) \xrightarrow{\mathbf{1}_{\mathfrak{g}}} (\mathcal{H}, V, \mathfrak{h}) \xrightarrow{P_{\mathfrak{k}}} (\mathcal{K}, V^{\mathcal{K}}, \mathfrak{k}) \longrightarrow 0$$

γ-cascades 1

Given processes $(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g})$ and $(\mathcal{K}, V^{\mathcal{K}}, \mathfrak{k})$ and any contraction $\gamma : \mathcal{E}_*^{\mathfrak{k}} \to \mathcal{E}^{\mathfrak{g}}$ we can define a combined process: the γ-**cascade**

$$(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g}) \lhd_\gamma (\mathcal{K}, V^{\mathcal{K}}, \mathfrak{k}) := (\mathcal{H}, V, \mathfrak{h})$$

$$\mathcal{E}^{\mathfrak{g}} := \overline{span}(\mathfrak{g}, V^{\mathcal{G}}(\mathfrak{g} \otimes \mathcal{P})) \ominus \mathfrak{g}, \ \mathcal{E}_*^{\mathfrak{k}} := \overline{span}(\mathfrak{k}, V^{\mathcal{K}}(\mathfrak{k} \otimes \mathcal{P})) \ominus V^{\mathcal{K}}(\mathfrak{k} \otimes \mathcal{P})$$

(wandering subspaces!)

$$\mathfrak{h} \ := \ \mathfrak{g} \oplus \mathfrak{k}, \quad \mathcal{H} := \mathfrak{g} \oplus \mathcal{K} \oplus \bigoplus_{\alpha \in F_d^+} (\mathcal{D}_{\gamma^*})_\alpha \, .$$

$$V \ := \ \begin{cases} \left(\mathbb{1}_{\mathfrak{g}} \oplus \begin{pmatrix} \gamma^* \\ D_{\gamma^*} \end{pmatrix} \right) V^{\mathcal{G}} & \text{on } \mathfrak{g} \\ V^{\mathcal{K}} & \text{on } \mathcal{K} \\ \textit{canonical row shift} & \text{on } \bigoplus_{\alpha \in F_d^+} (\mathcal{D}_{\gamma^*})_\alpha \end{cases}$$

γ-cascades 2

If $\gamma = 0$ then the γ-cascade is nothing but the direct sum of the processes $(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g})$ and $(\mathcal{K}, V^{\mathcal{K}}, \mathfrak{k})$.

In the dilation picture we deal here with dilations of row contractions of the form $\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$.

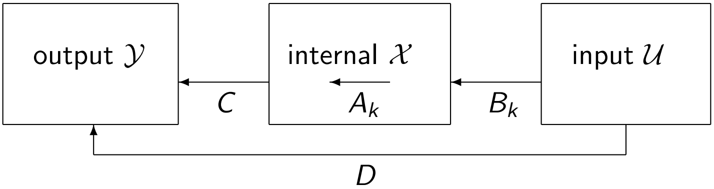
THEOREM: There is a one-to-one correspondence between equivalence classes of extensions of the process $(\mathcal{K}, V^{\mathcal{K}}, \mathfrak{k})$ by the process $(\mathcal{G}, V^{\mathcal{G}} \mathfrak{g})$ as in

$$0 \longrightarrow (\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g}) \longrightarrow (\mathcal{H}, V, \mathfrak{h}) \longrightarrow (\mathcal{K}, V^{\mathcal{K}}, \mathfrak{k}) \longrightarrow 0$$

and contractions γ from $\mathcal{E}_*^{\mathfrak{k}}$ to $\mathcal{E}^{\mathfrak{g}}$.

This correspondence is given by the γ-cascade construction.

Representation of Structure Maps



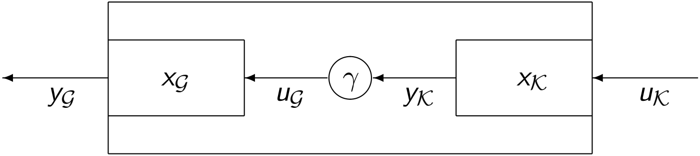
Noncommutative Fornasini-Marchesini system
via representation of structure maps (A, B, C, D)
within a process $(\mathcal{H}, V, \mathfrak{h})$: $\mathcal{X} = \mathfrak{h}$ internal space,
$\mathcal{U} \subset \mathcal{E}$ input space, $\mathcal{Y} \subset \mathfrak{h} \oplus \mathcal{E}$ (a wandering) output space.

$$
\begin{aligned}
A = (A_k) &:= V^*|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \otimes \mathcal{P} \\
B = (B_k) &:= V^*|_{\mathcal{U}} : \mathcal{U} \to \mathcal{X} \otimes \mathcal{P} \\
C &:= P_{\mathcal{Y}}|_{\mathcal{X}} : \mathcal{X} \to \mathcal{Y}, \quad D := P_{\mathcal{Y}}|_{\mathcal{U}} : \mathcal{U} \to \mathcal{Y}
\end{aligned}
$$

Cascading

It can be checked that in a γ-cascade of processes,
choosing $\mathcal{Y}^{\mathcal{K}} := \mathcal{E}_*^{\mathfrak{k}}$ and $\mathcal{U}^{\mathcal{G}} := \mathcal{E}^{\mathfrak{g}}$,
the corresponding structure maps cascade as linear systems.

This motivates the terminology.



Observability

Given an output pair (A, C) for an internal space $\mathcal{X}$ and an output space $\mathcal{Y}$, a subset $\mathcal{X}' \subset \mathcal{X}$ is called **observable** if $(CA^{\alpha}|_{\mathcal{X}'})_{\alpha \in F_d^+}$, the observability map restricted to $\mathcal{X}'$, is injective (as a map from $\mathcal{X}'$ to the $\mathcal{Y}$-valued functions on F_d^+).

The interpretation of observability is that every $\xi \in \mathcal{X}'$ can be reconstructed from the outputs $CA^{\alpha}\xi$.

Asymptotic Completeness

THEOREM:
Consider the γ-cascade $(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g}) \lhd_\gamma (\mathcal{K}, V^{\mathcal{K}}, \mathfrak{k}) = (\mathcal{H}, V, \mathfrak{h})$ with the output space $\mathcal{Y} := \mathcal{E}^{\mathfrak{g}}$ (automatically wandering as input space for the subprocess!). TFAE:

(1) $\mathfrak{k}$ is observable in $(\mathcal{H}, V, \mathfrak{h})$.

(2) $\overline{span}\{(A^\alpha)^* \, \mathfrak{g} \colon \alpha \in F_d^+\} = \mathfrak{h}$

(3) $\mathcal{G} = \mathcal{H}$

(4) $V^{\mathcal{K}}$ is a row shift and $\gamma \colon \mathcal{E}_*^{\mathfrak{k}} \to \mathcal{E}^{\mathfrak{g}}$ is injective (isometric).

If the transition operator Z of $(\mathcal{H}, V, \mathfrak{h})$ is unital then we also have the following equivalent condition:

(5) $\lim_{n\to\infty} Z^n(P_{\mathfrak{g}}) = \mathbb{1}_{\mathfrak{h}}$ (in the strong operator topology)

In this case we say that

$$0 \longrightarrow (\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g}) \longrightarrow (\mathcal{H}, V, \mathfrak{h}) \longrightarrow (\mathcal{K}, V^{\mathcal{K}}, \mathfrak{k}) \longrightarrow 0$$

is **asymptotically complete**.

Subprocesses from invariant states

Given a process $(\mathcal{H}, V, \mathfrak{h})$, suppose that ϕ is a **normal state** of $\mathcal{B}(\mathfrak{h})$ which is **invariant** for the transition operator Z, i.e.,

$$\phi(Z(x)) = \phi(x) \quad \text{for all } x \in \mathcal{B}(\mathfrak{h})\,.$$

Then with $P_{\mathfrak{g}} := s(\phi)$, the **support** projection for the state ϕ, the subspace $\mathfrak{g}$ is co-invariant for V.

Hence we can always find a subprocess from a normal invariant state and this subprocess is nontrivial, in the sense that $\mathfrak{g} \neq \mathfrak{h}$, if and only if the state is not faithful.

rich probabilistic source for subprocesses!

Operator algebraic processes 1

Our theory can also be applied for noncommutative Markov processes in an operator algebraic setting:

Let $\mathcal{A}$ and $\mathcal{C}$ be C^*-algebras and let

$$j : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$$

be a non-zero $*$-homomorphism. By iteration we find $*$-homomorphisms

$$j_n : \mathcal{A} \to \mathcal{A} \otimes \bigotimes_1^n \mathcal{C} \ .$$

We can interpret the (j_n) as noncommutative random variables and together they form an **(operator-algebraic) Markov chain**.

Operator algebraic processes 2

To get probabilistic statements we need to consider states on these algebras.

If ϕ resp. ψ are states on $\mathcal{A}$ respectively $\mathcal{C}$ and we impose the stationarity condition

$$(\phi \otimes \psi) \circ j = \phi$$

we get an (operator-algebraic) **stationary Markov chain**.

from *GNS*-construction: Hilbert spaces and vector states
The whole structure extends to a weak process.

Even better:
The invariant vector state (from ϕ) gives rise to a subprocess (as explained above).

associated γ-cascade (automatically!)

Scattering Theory and Asymptotic Completeness

What does asymptotic completeness mean here?

In 2000 Kümmerer and Maassen introduced a **scattering theory** for noncommutative Markov chains (in analogy to Lax-Phillips scattering theory).

It turns out that asymptotic completeness in the sense of scattering theory is **equivalent** to our notion of asymptotic completeness for the associated weak process.

Conceptually this is a nice way to understand some of the criteria developed for checking asymptotic completeness.

Summary of Strategy

constructions from (nc) probability worked out via (nc) operator and system theory, in a conceptually clear way.

and from there:

application of (nc) operator and system theory to obtain a better understanding of the corresponding quantum models.

guidance for the development of (nc) operator and system theory in relevant directions.

details for the constructions I rushed over to be found in

R.G.: Weak Markov Processes as Linear Systems (arxiv)

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