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# Non-commutative Symbolic Coding

ROLF GOHM†, BURKHARD KÜMMERER‡ and TATJANA LANG§

† *Department of Mathematics, University of Reading, Whiteknights, PO Box 220, Berkshire, Reading RG6 6AX, United Kingdom (e-mail: r.gohm@reading.ac.uk)*
‡ *Fachbereich Mathematik, Technische Universität Darmstadt, Schloßgartenstr. 7, 64289 Darmstadt, Germany (e-mail: kuemmerer@mathematik.tu-darmstadt.de)*
§ *Universität Stuttgart, Mathematisches Institut A, Pfaffenwaldring 57, 70569 Stuttgart, Germany (e-mail: langta@mathematik.uni-stuttgart.de)*

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*Abstract.* We give a non-commutative generalization of classical symbolic coding in the presence of a synchronizing word. This is done by a scattering theoretical approach. Classically, the existence of a synchronizing word turns out to be equivalent to asymptotic completeness of the corresponding Markov process. A criterion for asymptotic completeness in general is provided by the regularity of an associated extended transition operator. Commutative and non-commutative examples are analyzed.

*Introduction*

In the growing field of quantum information theory there is a part called quantum coding, in which concepts of classical coding theory are transferred to a quantum setting. The ideas in this paper are motivated by a subarea of classical coding which so far has not received much attention in this respect, namely coding in symbolic dynamics. A good introduction is [**LiMa**]. Symbolic coding proved to be important both theoretically, for example by providing models for dynamical systems, as well as practically, for example in computer science. Our non-commutative generalization of symbolic coding is concerned with non-commutative dynamical systems in an operator algebraic setting. An example applicable to physics is given in the end.

This paper is situated on the borderline between symbolic dynamics, operator algebras, and open quantum systems. We begin this introduction with some remarks showing how ideas from these three areas come together in the present paper. We then describe some of the mathematical background and give a more detailed motivation of our investigations. Finally, we introduce the notation used in the following and close with a brief outline of the contents.

*Synchronizing words and scattering.* A basic idea in this paper is relating two apparently distant fields of mathematics and thereby opening new perspectives for both of them. On the one side there is symbolic dynamics, in particular, the investigation of certain topological Markov chains constructed from road coloured graphs. On the other side there are certain open quantum systems interacting with a heat bath, where the free dynamics of the heat bath is given by a discrete time quantum white noise. The link between both fields is established by the operator algebraic description of a quantum Markov process in coupling form. It was introduced in [**Kü1**] and is described below.

The relation between quantum Markov processes in coupling form and open quantum systems has been known for some time and is since the subject of investigations (cf., e.g., [**Kü3**], [**Kü4**] and the references therein). On the other hand the interpretation of such Markov processes as an operator algebraic version of a dynamical system which is constructed from a road coloured graph is new and established in the first part of the present paper.

As one of our basic observations we show in the second part of this paper (cf. Proposition 2.3) that the existence of a synchronizing word of a road coloured graph — a notion stemming from symbolic dynamics (cf. [**LiMa**]) — and asymptotic completeness of an open system coupled to a white noise — a notion from the operator algebraic description of open quantum systems (cf. [**KüMa**]) — amount to the same thing.

Both of these notions have been studied before within their respective contexts: On the one hand it is well known in symbolic dynamics that a synchronizing word of a road coloured graph induces a code which establishes a conjugacy between the Markovian dynamical system associated with this graph and the Bernoulli system on the colours ([**LiMa**]). On the other hand the Markovian dynamics of an open quantum system coupled to a white noise can be viewed as a local perturbation of the free white noise dynamics. Here, ideas from Lax-Phillips-scattering theory ([**LaPh**]) can be brought into play as was done in [**KüMa**]. If such a system is asymptotically complete then one of the Møller operators establishes a conjugacy between the Markovian dynamics of the coupled system and the white noise dynamics of the 'heat bath'. In this paper we discuss such a situation in discrete time where the white noise dynamics is given by a Bernoulli system. The conjugacy is established in Theorem 2.1. It turns out that in the commutative situation of a Markov process constructed from a road coloured graph, this conjugacy is identical to the one induced by the synchronizing word (cf. our discussion following 2.2).

From a physical point of view asymptotic completeness is an important feature of an open system coupled to white noise: It allows to prepare the quantum state of the open system by preparing suitable states on the 'incoming' white noise, which is easier to access by a physical preparation in some cases of physical interest as, e.g., in the micro-maser. This application of asymptotic completeness has been analyzed in some detail in [**WBKM**], cf. the discussions in [**Kü3**], [**Kü4**]. The fact, that such a preparation is possible even without knowing the initial state of the open system, is the quantum analogue of the feature of a synchronizing word:

when applied, it synchronizes all possible initial states to one and the same state. Thus asymptotic completeness can be understood as an extension of the notion of a synchronizing word to the quantum context. It still preserves essential features of a synchronizing word.

Our treatment thus sheds new light on both areas: Synchronizing words and the codings induced by them appear in the light of scattering theory, which conversely earnes an interpretation in terms of coding theory. In particular, this connection opens the way for carrying ideas from symbolic dynamics into the framework of non-commutative operator algebras and quantum coding. As an example we mention our discussion of the label product at the beginning of Section 5.

While in many cases proving the existence of a synchonizing word for a given road coloured graph is just a matter of 'looking closely' at the graph, it can be considerably more difficult to prove asymptotic completeness in the general situation. An important part of the present paper is concerned with establishing a more tractable condition, which is equivalent to asymptotic completeness. Indeed, our characterization in terms of the extended transition operator in Theorem 4.3 turns out to be helpful in proving asymptotic completeness in the non-commutative framework. As an application we prove asymptotic completeness for the cut-off-version of the micro-maser in Section 6. As already mentioned it is used in [**WBKM**] to propose a scheme for preparing quantum states in this system. The underlying technique of extending completely positive operators is of interest on its own. For more information about this topic and about its relevance for scattering theory we refer to [**Go3**].

We use this opportunity to draw the reader's attention to an interesting feature of the present discussion: Although the existence of a synchronizing word is of a purely graph theoretical or topological nature, the conjugacy induced by it is neither continuous nor well-defined on the associated topological spaces. Instead one has to introduce stationary measures. Only then, the conjugacy can be formulated in a measure theoretical way. Nevertheless, this conjugacy does not really depend on the particular choice of the stationary measure. Under the heading 'almost conjugacy' such phenomena are discussed in Chapter 9 of [**LiMa**]. In our non-commutative setting this feature is reflected by the fact that our discussion of conjugacies requires weak closures and takes place within the framework of von Neumann algebras, standing for 'non-commutative measure theory', while C*-algebras would represent a topological framework. Again the conjugacy does not really depend on the particular choice of the stationary state, cf. Proposition 4.4 and the remarks following 1.4 and 5.1.

*Non-commutative stationary Markov chains.* In order to cover all cases of interest within a unified treatment we consistently discuss the case of one-sided Markov processes, cf. [**Go3**] for a further development of this setting. In this formulation the underlying ideas from scattering theory are not so easily visible. Therefore, we briefly sketch the two-sided situation here, where the relation to scattering can be seen more directly (cf. [**KüMa**]).

Given a classical finite state space $A$ and a transition matrix $T$. From an initial distribution $\mu$ on $A$ a Markov measure $\hat{\mu}$ on $A^{\mathbb{N}}$ is constructed to obtain the corresponding Markov process. If $\mu$ is stationary under $T$ then $\hat{\mu}$ is stationary under time translation. In this case $T$ is regained by applying the conditional expectation from $L^\infty(A^{\mathbb{N}}, \hat{\mu})$ onto the functions depending only on the first coordinate. When trying to extend this construction of a stationary Markov process to the non-commutative operator algebraic situation, functions on $A$ should be replaced by a non-commutative operator algebra, e.g., the algebra of all $n \times n$-matrices. Now one is faced with the problem that in this case a conditional expectation onto the first coordinate exists only, if the state on the whole algebra factorizes into a tensor product of a state on the first factor and a state on the other factors (cf., e.g., [**Kü2**]). Since a Markov measure $\hat{\mu}$ is not a product measure, whenever the process is not a Bernoulli process, such an extension to the non-commutative situation seems to be impossible.

However, a different approach turned out to be possible ([**Kü1**]): Suppose $T$ is a 'quantum transition matrix', i.e., a completely positive identity preserving operator on an operator algebra $\mathcal{A}$, which, for simplicity, is assumed to be finite-dimensional. As a first step try to find a further algebra $\mathcal{C}$, a pair of states $\varphi$ on $\mathcal{A}$ and $\psi$ on $\mathcal{C}$, and an automorphism $T_1$ of $\mathcal{A} \otimes \mathcal{C}$ leaving the product state $\varphi \otimes \psi$ invariant, such that the conditional expectation $P_\psi$ from $\mathcal{A} \otimes \mathcal{C}$ onto $\mathcal{A} \otimes 1\!\!1$ with respect to $\varphi \otimes \psi$ — it exists in this case — satisfies $T(x) = P_\psi(T_1(x \otimes 1\!\!1))$ (here we identified $\mathcal{A}$ with $\mathcal{A} \otimes 1\!\!1$). It is easy to check that this construction is always possible if the algebra $\mathcal{A}$ is commutative. In general, however, there are some restrictions (cf. [**Kü1**]). From these ingredients a stationary dynamical system can be constructed as follows ([**Kü1**], cf. also [**Kü3**], [**Kü4**]): Define a von Neumann algebra $\hat{\mathcal{A}}$ with normal state $\hat{\varphi}$ as $(\hat{\mathcal{A}}, \hat{\varphi}) := (\mathcal{A}, \varphi) \otimes \bigotimes_{\mathbb{Z}} (\mathcal{C}, \psi)$. The tensor right shift on $\bigotimes_{\mathbb{Z}} (\mathcal{C}, \psi)$ is trivially extended to a stationary automorphism $S$ of $(\hat{\mathcal{A}}, \hat{\varphi})$. Similarly, by identifying $\mathcal{C}$ with the zeroth component in the infinite tensor product $\bigotimes_{\mathbb{Z}} (\mathcal{C}, \psi)$, $T_1$ on $(\mathcal{A} \otimes \mathcal{C})$ is trivially extended to an automorphism of $(\hat{\mathcal{A}}, \hat{\varphi})$, still denoted by $T_1$. Now the evolution $\hat{T}$ can be defined as $\hat{T} := T_1 \circ S$. This stationary dynamical system has a natural interpretation as a stationary Markov process ([**Kü1**]). Now it becomes evident in which way the Markovian evolution appears as a coupling of $\mathcal{A}$ via $T_1$ to $\bigotimes_{\mathbb{Z}} (\mathcal{C}, \psi)$, the latter being equipped with the free evolution $S$. The definition of asymptotic completeness as formulated in [**KüMa**] essentially means that for all $x \in \hat{\mathcal{A}}$ the limit $\lim_{n \to \infty} S^{-n} \circ \hat{T}^n(x)$ exists (strongly) and lies in $\bigotimes_{\mathbb{Z}} (\mathcal{C}, \psi)$. Then the limit defines one of the Møller operators of scattering theory.

In our one-sided treatment we use from $T_1$ only the embedding $J : \mathcal{A} \ni x \mapsto T_1(x \otimes 1\!\!1)$. We shall call it a 'transition' in 1.1. Similarly, an iterated version $J_n$ can be defined as $J_n : \mathcal{A} \ni x \mapsto \hat{T}^n(x \otimes 1\!\!1)$. In 1.4 we define asymptotic completeness for the one-sided version of $J_n$ which is defined in 1.1. If $J$ comes from a two-sided Markov process then the one-sided version of asymptotic completeness in 1.4 is equivalent to the two-sided version described above.

*Road coloured graphs and Markov chains.* Let us now describe in more detail the classical version of the situation treated in this paper. We consider a finite directed graph $G$ with edges $E$ and vertices $A$ together with two maps $s : E \to A$ and $t : E \to A$ associating with each edge $e$ its starting vertex $s(e)$ and its target $t(e)$.

In the following we concentrate on a special class of graphs: Let $C$ be any finite set (of *colours* or *labels*) and $c : E \to C$ a map (*colouring*). We call $G$ a *C-graph* if for each vertex $a \in A$ the map $c$ induces a bijection between the set $s^{-1}(a)$ of edges starting in $a \in A$ and the set $C$. In particular, there is the same number of edges starting in each vertex $a \in A$. Such graphs are also called *road coloured graphs* (cf. [**LiMa**]). Note that more than one edge between two vertices are allowed.

To any directed graph $G$ there is canonically associated a shift space of finite type

$$E^- := \{(\ldots, e_{-n}, \ldots, e_{-1}) \in E^{-\mathbb{N}} : t(e_{k-1}) = s(e_k) \text{ for all } k\}.$$

In our considerations of one-sided shifts it seems convenient to deal with left infinite spaces (cf. [**AMT**]).

For a $C$-graph there is a canonical surjection $\eta$ from $E^-$ to $C^- := C^{-\mathbb{N}}$ given by

$$\eta(\ldots, e_{-n}, \ldots, e_{-1}) := (\ldots, c(e_{-n}), \ldots, c(e_{-1})) \ .$$

A finite sequence $c_1 c_2 \ldots c_n \in C^n$ of colours is called a *synchronizing word* if there exists a vertex $a \in A$ such that for any allowed sequence $e_1 e_2 \ldots e_n$ with $c(e_1) = c_1, c(e_2) = c_2, \ldots, c(e_n) = c_n$ the target $t(e_n)$ is always the same vertex $a$. It follows that if in a sequence $(\ldots, c_{-n}, \ldots, c_{-1}) \in C^{-\mathbb{N}}$ this synchronizing word occurs infinitely often then $\eta^{-1}(\ldots, c_{-n}, \ldots, c_{-1})$ has only one point.

In order to turn $\eta$ into an isomorphism one needs to consider measures: Consider a strictly positive probability distribution $\nu$ on $C$ given by $\nu : \mathcal{C} \to \mathbb{R}$ with $\nu(c) > 0$ for $c \in C$ and $\sum_{c \in C} \nu(c) = 1$. It associates a probability for each edge and thus induces transition probabilities from a point $a \in A$ to another point $b \in A$ given by

$$t_{a,b} := \sum_{\substack{s(e)=a \\ t(e)=b}} \nu(c(e)) \ .$$

These transition probabilities form a stochastic matrix $T$ on $A$. If the graph is irreducible then so is $T$ and by Perron-Frobenius theory there is a unique strictly positive stationary probability distribution $\mu$ on $A$. From now on we will consider only irreducible graphs.

For a C-graph the set $E$ of edges can be canonically identified with the set $A \times C$. Therefore, the product distribution $\mu \otimes \nu$ on $A \times C$ can be viewed as a probability distribution on the edges $E$. It can be extended to a shift invariant Markov measure $\mu^-$ on $E^-$: on the cylinder set $\{(\omega_n)_{n \leq -1} \in E^- : \omega_{-n} = e_{-n}, \ldots, \omega_{-1} = e_{-1}\}$ it is given by the product $\mu(s(e_{-n}))\, \nu(c(e_{-n}))\, \nu(c(e_{-n+1})) \ldots \nu(c(e_{-1}))$. On $C^-$ we consider the infinite product measure $\nu^- := \bigotimes_{-\mathbb{N}} \nu$. If the C-graph admits a synchronizing word then $\eta$ extends to a measure-theoretical isomorphism between the one-sided Markov chain on $E^-$ and the Bernoulli shift on $C^-$ (cf. [**AMT**], [**LiMa**], [**Kit**], [**Sch**]).

*Non-commutative road coloured graphs: transitions.* As our basic objects we need to extend the notion of road coloured graphs or $C$-graphs to the non-commutative setting.

The basic idea is the following: Given a $C$-graph with vertices $A$ and colours $C$, we already noticed that the set $E$ of its edges can be identified with $A \times C$ such that $s(a,c) = a$ and $c(a,c) = c$ for all $(a,c) \in A \times C = E$. Thus a $C$-graph is completely characterized by its (surjective) target map $t : A \times C \to A$. Conversely, any surjective map $t : A \times C \to A$ gives rise to such a $C$-graph.

This can be translated into an algebraic language: If $\mathcal{A}$, resp. $\mathcal{C}$, denotes the finite dimensional algebra of complex valued functions on $A$, resp. $C$, (under pointwise multiplication) then the algebra of functions on $A \times C$ can be identified with the algebra $\mathcal{A} \otimes \mathcal{C}$. Hence there is a biunique correspondence between surjections $t : A \times C \to A$ and identity preserving injective *-homomorphisms $J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ such that $(Jf)((a,c)) = f(t(a,c))$ for $f \in \mathcal{A}$, $(a,c) \in A \times C$. Such a map will be called a *transition* in the following.

Now we can allow the algebras to be non-commutative thus leading to a non-commutative version of a $C$-graph. It is the purpose of the present paper to develop the ideas sketched above also for the non-commutative context, thus creating a piece of a non-commutative coding theory.

*Contents.* In Section 1 we define transitions and construct the associated non-commutative Markov chains. Fundamental notions for transitions are irreducibility and asymptotic completeness. The latter notion is borrowed from scattering theory and in Section 2 we explain how an asymptotically complete transition leads to an asymptotically complete scattering theory for Markov chains. The Møller operator provides a conjugacy between the given Markov chain and a Bernoulli shift. In the rest of Section 2 we show that in the commutative case this conjugacy is exactly the map $\eta^{-1}$ obtained above and thus asymptotic completeness corresponds to the presence of a synchronizing word. In this sense we interpret the Møller operator as a procedure of non-commutative symbolic coding.

On the way to get criteria also in the non-commutative setting we discuss in the preparatory Section 3 the notion of regularity for positive maps. In Section 4 we introduce the (dual) extended transition operator associated to a transition. We show that a transition is asymptotically complete if and only if the associated extended transition operator is regular. This criterion turns out to be quite useful and it yields some new aspects even in the commutative case. This is discussed in Section 5 where we explain that on the level of C-graphs extended transition corresponds to the consideration of label products. In Section 6 we study in detail a non-commutative example which is derived from the Jaynes-Cummings model well known in quantum optics. We determine for which parameters it is asymptotically complete.

1. *Transitions*
For convenience we assume that $\mathcal{A}$ and $\mathcal{C}$ are finite dimensional C*-algebras.

1.1 DEFINITION *A transition is given by an identity preserving injective *-homomorphism* $J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$.

For example, a target map $t : A \times C \to A$ as in the introduction gives rise to a transition on the corresponding algebra of functions, given by $J(f) := f \circ t$. We shall analyze this example in more detail in Sections 2 and 5.

For another example, take a unitary $u \in \mathcal{A} \otimes \mathcal{C}$ and then define $J(a) := u^* \, (a \otimes \mathbb{1}) \, u$ for all $a \in \mathcal{A}$. An example of this type is discussed in Section 6.

Let us show how a transition $J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ gives rise to a *non-commutative topological Markov chain.* We construct a family $(J_n)_{n \in \mathbb{N}_0}$ of identity preserving injective *-homomorphisms by the following recursion

$$\begin{aligned}
J_0 \quad &= Id : \mathcal{A} \to \mathcal{A}, \quad a \mapsto a \\
J_1 \quad &= J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}, \quad a \mapsto \sum_i a_i \otimes c_i \\
\ldots \\
J_n \quad &: \mathcal{A} \to \mathcal{A} \otimes \bigotimes_1^n \mathcal{C}, \quad a \mapsto \sum_i J_{n-1}(a_i) \otimes c_i \in \left( \mathcal{A} \otimes \bigotimes_1^{n-1} \mathcal{C} \right) \otimes \mathcal{C}.
\end{aligned}$$

To handle these homomorphisms simultaneously, we form the infinite tensor product $\mathcal{C}^+ := \bigotimes_1^\infty \mathcal{C}$ and $\mathcal{A}^+ := \mathcal{A} \otimes \mathcal{C}^+$. By $x \mapsto x \otimes \mathbb{1}$ we have many natural inclusions and we may write $J_n : \mathcal{A} \to \mathcal{A}^+$ (for all $n$). On $\mathcal{C}^+$ we have a right tensor shift $S^+$, i.e.,

$$S^+(c_1 \otimes c_2 \otimes \ldots) = \mathbf{1} \otimes c_1 \otimes c_2 \otimes \ldots$$

It is not difficult to check that $J_n(a) = (J^+)^n (a \otimes \mathbb{1})$, where $J^+ : \mathcal{A}^+ \to \mathcal{A}^+$ is an identity preserving injective *-homomorphism given for $a \otimes c \in \mathcal{A} \otimes \mathcal{C}^+$ by

$$J^+(a \otimes c) = J(a) \otimes c \in (\mathcal{A} \otimes \mathcal{C}) \otimes \bigotimes_2^\infty \mathcal{C},$$

in shorthand notation: $J^+ = J \circ (Id_{\mathcal{A}} \otimes S^+)$. We may think of $J^+$ as a time evolution producing the sequence $(J_n)_{n \in \mathbb{N}_0}$ from $J_0$ by $J$. This structure is typical for non-commutative Markov chains and it is called a *coupling to a shift*, see [**Kü1, Kü2, KüMa, Go3**] for variations of this theme.

In fact, to understand why it is called a Markov chain one must add a probabilistic content to this structure by considering states. A state $\psi$ on $\mathcal{C}$ gives rise to a *conditional expectation* $P_\psi : \mathcal{A} \otimes \mathcal{C} \to \mathcal{A}$ determined by $a \otimes c \mapsto a \cdot \psi(c)$. The operator $P_\psi$ is completely positive and identity preserving (cf. [**Tak**], IV.4.25). Now we can define the corresponding *transition operator*

$$T_\psi : \mathcal{A} \to \mathcal{A} : a \mapsto P_\psi(J(a)).$$

By its definition $T_\psi$ is an identity preserving completely positive map which generalizes the stochastic matrix of transition probabilities in the introduction.

Let $\phi$ be a state on $\mathcal{A}$. Since $\phi(T_\psi(a)) = \phi(P_\psi(J(a))) = \phi \otimes \psi(J(a))$ $(a \in \mathcal{A})$ the following *observation* is evident: $\phi$ is invariant under $T_\psi$, i.e. $\phi = \phi \circ T_\psi$, if and only if $\phi = (\phi \otimes \psi) \circ J$. We will use the notation $J : (\mathcal{A}, \phi) \to (\mathcal{A}, \phi) \otimes (\mathcal{C}, \psi)$ for that. It follows from the Markov-Kakutani fixed point theorem that for given $\psi$ there is at least one such state $\phi$. In this paper we will always consider $\phi$ and $\psi$ related in this way which gives rise to *stationary Markov chains.* In fact, on $\mathcal{C}^+$ and $\mathcal{A}^+ = \mathcal{A} \otimes \mathcal{C}^+$ we consider the product states $\psi^+ := \bigotimes_1^\infty \psi$ and $\phi^+ := \phi \otimes \psi^+$. Then it is easy to check that $\phi^+ \circ J^+ = \phi^+$, i.e., $\phi^+$ is invariant for the time evolution.

The usual language of non-commutative probability theory applies. The $J_n$ may be regarded as non-commutative random variables in the sense of [**AFL**]. Similarly as above we have a conditional expectation

$$P_{\psi^+} : \mathcal{A}^+ \to \mathcal{A}, \quad a \otimes c \mapsto a \cdot \psi^+(c)$$

and it is also not difficult to check that $P_{\psi^+} J_n = T_\psi^n$ for all $n$ which is an analogue of the classical Chapman-Kolmogorov equations for Markov chains. It is possible to define explicitly a non-commutative Markov property which is valid in this setting. Because we shall not need it here we refer to [**Kü2**] for a discussion of this topic.

Now we describe some properties of transitions which will be important for the coding procedures to be defined later. Recall that a completely positive identity preserving operator $T : \mathcal{A} \to \mathcal{A}$ is called *irreducible* if for a projection $p \in \mathcal{A}$, $T(p) \le p$ implies $p = 0$ or $p = \mathbb{1}$ (cf. e.g., [**EnWa**], [**EHK**], [**Gro**]). Here and in the following a projection is always self-adjoint. Similarly we say *the transition $J$ is irreducible* if for a projection $p \in \mathcal{A}$, $J(p) \le p \otimes \mathbb{1}$ implies $p = 0$ or $p = \mathbb{1}$.

1.2 PROPOSITION *Let $\psi$ be a faithful state on $\mathcal{C}$. The following conditions are equivalent:*

*(a) $J$ is irreducible.*

*(b) $T_\psi$ is irreducible.*

*Proof.* If $J$ is not irreducible then there is a projection $0 \neq p \neq \mathbb{1}$ with $J(p) \le p \otimes \mathbb{1}$, hence $T_\psi(p) = P_\psi(J(p)) \le P_\psi(p \otimes \mathbb{1}) = p$. Conversely, suppose $J$ is irreducible, $0 \neq p \neq \mathbb{1}$ a projection. Because $\psi$ is faithful (i.e., $\psi(y^*y) = 0 \Rightarrow y = 0$) also $P_\psi$ is faithful. This follows easily from ([**Tak**], IV.5.12). Putting $p^\perp := \mathbb{1} - p$ we obtain $0 \neq P_\psi(p^\perp \otimes \mathbb{1} \cdot J(p) \cdot p^\perp \otimes \mathbb{1}) = p^\perp \cdot P_\psi(J(p)) \cdot p^\perp = p^\perp \cdot T_\psi(p) \cdot p^\perp$. Here we used the module property of a conditional expectation (cf. [**Tak**], III.3.4). □

The following argument will be used repeatedly: If $T : \mathcal{A} \to \mathcal{A}$ is positive and identity preserving and $\phi$ is a $T$-invariant state with support projection $p_\phi$, then $T(p_\phi^\perp) \le \mathbb{1}$ and $\phi(T(p_\phi^\perp)) = \phi(p_\phi^\perp) = 0$. Considering the spectral projections of $T(p_\phi^\perp)$, this implies $T(p_\phi^\perp) \le p_\phi^\perp$. In particular, if $T$ is irreducible, an invariant state is necessarily faithful.

1.3 PROPOSITION *If $J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ is an irreducible transition and $\psi$ is a faithful state on $\mathcal{C}$ then $T_\psi$ is irreducible and has a unique invariant state $\phi$ which is faithful.*

*Proof.* All assertions are already proved except uniqueness. By ([**KüNa**], 2.4), the fixed points of $T$ form a subalgebra. Therefore, if $T_\psi$ is irreducible, the fixed space of $T_\psi$ is one-dimensional. Since $\mathcal{A}$ is finite-dimensional, it follows that there is at most one invariant state. □

From now on we shall always assume that the states $\phi$ and $\psi$ are faithful. Then the product states $\phi^+$ and $\psi^+$ are also faithful. On $\mathcal{A}^+$ we have a norm $\|\cdot\|_{\phi^+}$ associated to the inner product $\langle x, y\rangle := \phi^+(y^* x)$. Similarly, on $\mathcal{C}^+$ we have the norm $\|\cdot\|_{\psi^+}$ which is a restriction of the former. Now we define a fundamental property of transitions whose study will occupy us for the rest of this paper.

1.4 DEFINITION *Let $\phi$ and $\psi$ be faithful states and let $Q_\phi : \mathcal{A}^+ \to \mathcal{C}^+ \subset \mathcal{A}^+$ be the conditional expectation determined by $a \otimes c \mapsto \phi(a) \cdot c$. A transition $J : (\mathcal{A}, \phi) \to (\mathcal{A}, \phi) \otimes (\mathcal{C}, \psi)$ is called asymptotically complete if for all $a \in \mathcal{A}$*

$$\|J_n(a) - Q_\phi\, J_n(a)\|_{\phi^+} \to 0 \quad (n \to \infty).$$

*Remarks.* It turns out that asymptotic completeness does not depend on the choice of (faithful) states but only on the transition $J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$. It is thus legitimate to say that $J$ is asymptotically complete. We postpone the proof of this fact to Proposition 4.4 when we have better tools for it.
Asymptotic completeness is a concept from scattering theory. It will become clear later (see Theorem 2.1) how a kind of scattering theory can be constructed from a transition.

The following observation sometimes simplifies the check for asymptotic completeness.

1.5 LEMMA *The following properties are equivalent:*

*(a) $J$ is asymptotically complete.*

*(b) For all $a \in \mathcal{A}$*

$$\|Q_\phi\, J_n(a)\|_{\psi^+} \to \|a\|_\phi \quad (n \to \infty).$$

*Proof.* With respect to the inner product, $Q_\phi$ may be considered as an orthogonal projection. Therefore

$$\|a\|_\phi^2 - \|Q_\phi\, J_n(a)\|_{\psi^+}^2 = \|J_n(a)\|_{\phi^+}^2 - \|Q_\phi\, J_n(a)\|_{\psi^+}^2 = \|J_n(a) - Q_\phi\, J_n(a)\|_{\phi^+}^2 \,,$$

from which the result follows. □

1.6 PROPOSITION *If $J : (\mathcal{A}, \phi) \to (\mathcal{A}, \phi) \otimes (\mathcal{C}, \psi)$ is asymptotically complete then for all $a \in \mathcal{A}$*

$$T_\psi^n(a) \to \phi(a)\mathbb{1} \quad (n \to \infty)\,.$$

*We say that the transition operator $T_\psi$ is regular (see Definition 3.1). In particular $T_\psi$ and $J$ are irreducible.*

*Proof.* From $J_n(a) - Q_\phi J_n(a) \to 0$ we conclude that also

$$P_{\psi^+} J_n(a) - P_{\psi^+} Q_\phi J_n(a) \to 0 \quad (in\ \mathcal{A}).$$

The left part is $T_\psi^n(a)$ while the right part is a multiple of $\mathbb{1}$ which by stationarity must be $\phi(a)\mathbb{1}$. It is evident that regularity implies irreducibility. □

But it turns out that neither irreducibility nor regularity of the transition operator $T_\psi$ is enough to imply asymptotic completeness for the transition $J$. This will be clear as soon as the connection with synchronizing words is established, see Section 2. The introduction of extended transition operators in Section 4 may be considered as an approach to cure this shortcoming.

For the purpose of this paper it is convenient to consider finite dimensional algebras $\mathcal{A}$ and $\mathcal{C}$ and we restricted ourselves to this case. When turning to infinite dimensional algebras one should assume that $\mathcal{A}$ and $\mathcal{C}$ are von Neumann algebras and $\mathcal{A} \otimes \mathcal{C}$ is their spatial tensor product. All states considered should be normal as well as the transition $J$. Then $P_\psi$ and $T_\psi$ are automatically normal. Now all definitions can be kept and all arguments remain true with the one exception, that a state $\phi$ invariant under $T_\psi$ constructed by a fixed point theorem doesn't need to be normal. This should be an additional requirement. One can also give a C*-algebraic version. In this case, irreducibility has to be formulated in terms of invariant order ideals. We restrain ourselves to do so.

## 2. *Møller Operators for Markov Chains*

In this section we construct a non-commutative version of the conjugacy $\eta^{-1}$ from colours to edges which was mentioned in the introduction. As in Section 1, let $J : (\mathcal{A}, \phi) \to (\mathcal{A}, \phi) \otimes (\mathcal{C}, \psi)$ be a transition (where $\phi$ and $\psi$ are faithful states). We have a homomorphism $J^+ : (\mathcal{A}^+, \phi^+) \to (\mathcal{A}^+, \phi^+)$. There is a standard procedure to get an automorphism from that. First, construct the $C^*$-inductive limit given by the inclusions

$$\ldots \longleftarrow \mathcal{A} \otimes \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}^+ \overset{J\otimes\mathbb{1}\otimes\mathbb{1}}{\longleftarrow} \mathcal{A} \otimes \mathcal{C} \otimes \mathcal{C}^+ \overset{J\otimes\mathbb{1}}{\longleftarrow} \mathcal{A} \otimes \mathcal{C}^+ = \mathcal{A}^+.$$

Using the natural identifications of $\ldots \mathcal{A} \otimes (\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}^+)$, $\mathcal{A} \otimes (\mathcal{C} \otimes \mathcal{C}^+)$ with $\mathcal{A} \otimes \mathcal{C}^+$, we can use $J^+$ and $\phi^+$ to define homomorphisms and states on the larger algebras. The norm closure then yields a $C^\star-$algebra with an automorphism and an invariant state, extending the data we started with. The GNS-representation is faithful and we can form weak closures to get a von Neumann algebra $\hat{\mathcal{A}}$ with an automorphism $\hat{\mathcal{J}}$ and an invariant normal state $\hat{\phi}$, all represented on the GNS-Hilbert space $\hat{\mathcal{H}}$. See [**Sak**], 1.23, for more details on $C^*$-inductive limits.

We can proceed similarly for $\mathcal{C}^+$ with the tensor shift $S^+$ and the state $\psi^+$. We get a second $C^*$-inductive limit and its weak closure $\hat{\mathcal{C}}$ which is simply the two-sided infinite tensor product with the two-sided tensor shift $S$ and the two-sided infinite product state $\hat{\psi}$, all represented on the GNS-Hilbert space $\hat{\mathcal{K}}$. As a product state

$\hat{\psi}$ is a normal faithful state on $\hat{\mathcal{C}}$. This is not a priori clear for $\hat{\phi}$ on $\hat{\mathcal{A}}$ but we shall prove faithfulness in the case of interest (see Theorem 2.1 below).

Observe that, because $\mathcal{A}^+ = \mathcal{A} \otimes \mathcal{C}^+$, we have $(\mathcal{C}^+, \psi^+)$ as a common subobject of $(\hat{\mathcal{A}}, \hat{\phi})$ and $(\hat{\mathcal{C}}, \hat{\psi})$ with a common (one-sided shift-)dynamics. We want to use this intersection to compare the two dynamical systems with each other. The following theorem shows that when the transition $J$ is asymptotically complete then we can construct a very specific conjugacy between them.

2.1 Theorem *Let $J : (\mathcal{A}, \phi) \to (\mathcal{A}, \phi) \otimes (\mathcal{C}, \psi)$ be a transition. Then the following assertions are equivalent:*

*(a) $J$ is asymptotically complete.*

*(b) There exists an isomorphism $\Phi : (\hat{\mathcal{A}}, \hat{\phi}) \to (\hat{\mathcal{C}}, \hat{\psi})$, which for $x \in \hat{\mathcal{J}}^{-N}(\mathcal{A}^+)$ ($N \in \mathbb{N}$) is given by*

$$\Phi(x) := \lim_{n\to\infty} S^{-n}\, Q_\phi\, \hat{\mathcal{J}}^n(x) \quad (\textit{ultraweakly or w.r.t. } \|\cdot\|_{\hat{\phi}}).$$

*(Recall that $Q_\phi$ is the conditional expectation from $\mathcal{A}^+$ onto $\mathcal{C}^+$.)*
*In this case the state $\hat{\phi}$ is faithful on $\hat{\mathcal{A}}$ and we have the intertwining property $S\Phi = \Phi\hat{\mathcal{J}}$, i.e., $\hat{\mathcal{J}}$ and $S$ are conjugate.*

*Proof:* Let us first assume (b). Because $\hat{\psi}$ is faithful on $\hat{\mathcal{C}}$ the isomorphism $\Phi$ forces $\hat{\phi}$ to be faithful on $\hat{\mathcal{A}}$. Further it is easy to check that on uniformly bounded sets the ultraweak topology and the topology given by the norm $\|\cdot\|_{\hat{\phi}}$ coincide and therefore it does not matter which of these topologies we use in the limit formula. Suppose that $x \in \hat{\mathcal{J}}^{-N}(\mathcal{A}^+)$. Note that $S^{-n}\, Q_\phi\, \hat{\mathcal{J}}^n(x)$ is defined for $n \geq N$, so the limit $n \to \infty$ in (b) makes sense. The intertwining property $S\Phi = \Phi\hat{\mathcal{J}}$ follows from the existence of the limit, just replace $n$ by $n+1$. Thus from (b) we immediately get the additional properties stated in Theorem 2.1. Further for all $a \in \mathcal{A}$ we get

$$\|Q_\phi\, \hat{\mathcal{J}}^n(a)\|^2_{\psi^+} = \|S^{-n} Q_\phi\, \hat{\mathcal{J}}^n(a)\|^2_{\hat{\psi}} \stackrel{n\to\infty}{\longrightarrow} \|\Phi(a)\|^2_{\hat{\psi}} = \|a\|^2_\phi$$

which by Lemma 1.5 implies that $J$ is asymptotically complete. Thus (b) implies (a).

Now we assume (a), i.e., $J$ is asymptotically complete. This means that $\hat{\mathcal{J}}^n(x) - Q_\phi\, \hat{\mathcal{J}}^n(x) \to 0$ w.r.t. the norm $\|\cdot\|_{\phi^+}$ or, equivalently, w.r.t. the ultraweak topology. Because $\hat{\mathcal{J}}|_{\mathcal{C}^+} = S|_{\mathcal{C}^+}$, we conclude that $\left(S^{-n}\, Q_\phi\, \hat{\mathcal{J}}^n(x)\right)$ is a (ultraweak) Cauchy sequence and thus converges to an element $\Phi(x) \in \hat{\mathcal{C}}$. Here we use that $\hat{\mathcal{C}}$ is a von Neumann algebra. ¿From $\hat{\mathcal{J}}^n(x) - Q_\phi\, \hat{\mathcal{J}}^n(x) \to 0$ and the homomorphism property of $\hat{\mathcal{J}}$ and $S$ we infer that $\Phi$ is a homomorphism on $\hat{\mathcal{J}}^{-N}(\mathcal{A}^+)$. Note also that $\hat{\psi} \circ \Phi(x) = \hat{\phi}(x)$, since $\Phi$ is a limit of maps which respect the states. If $0 \neq x \in \hat{\mathcal{J}}^{-N}(\mathcal{A}^+)$ then, because the GNS-representation is faithful, there exists $y \in \hat{\mathcal{J}}^{-N}(\mathcal{A}^+)$ such that

$$0 \neq \hat{\phi}(y^\star x^\star x y) = \hat{\psi}(\Phi(y^\star)\Phi(x^\star)\Phi(x)\Phi(y)),$$

in particular $\Phi(x) \neq 0$. Thus $\Phi$ is injective on $\hat{\mathcal{J}}^{-N}(\mathcal{A}^+)$. By extension we obtain $\Phi$ on the norm closure of the $\hat{\mathcal{J}}^{-N}(\mathcal{A}^+)$ for all $N$ as an injective homomorphism, i.e., a $C^*$-isomorphism from the first $C^*$-inductive limit onto its range which is contained in $\hat{\mathcal{C}}$. If $y \in S^{-N}\mathcal{C}^+$ then

$$\Phi(\hat{\mathcal{J}}^{-N} S^N y) = \lim_{n\to\infty} S^{-n} J^{n-N} S^N y = y.$$

We conclude that the range of $\Phi$ contains $S^{-N}\mathcal{C}^+$ for all $N \in \mathbb{N}$ and thus the second $C^*$-inductive limit. Because $\hat{\psi} \circ \Phi(x) = \hat{\phi}(x)$, the $C^*$-algebraic isomorphism $\Phi$ can be unitarily implemented on the GNS-spaces and extends to the weak closures, i.e., we have a normal isomorphism from $\hat{\mathcal{A}}$ onto $\hat{\mathcal{C}}$. This proves (b). $\square$

*Remarks.* As described in the introduction the idea of constructing intertwiners in this way appears in the work of B. Kümmerer and H. Maassen, cf. [**KüMa**], and has its roots in a structural analogy between certain two-sided Markov processes and the situation analyzed in the scattering theory developed by P.D. Lax and R.S. Phillips, cf. [**LaPh**]. Motivated by this point of view we call the intertwiner $\Phi$ a Møller operator. For further modifications of Theorem 2.1. see [**Lan**].

It is easy to check that the Møller operator satisfies $\Phi|_{\mathcal{C}^+} = Id|_{\mathcal{C}^+}$. This leads to the result that the conjugacy considered here is essentially one-sided. In fact, with

$$\mathcal{A}^- := \bigvee_{n=-\infty}^{0} \hat{\mathcal{J}}^n \mathcal{A}, \quad \mathcal{C}^- := \bigvee_{n=-\infty}^{-1} S^n \mathcal{C},$$

$$J^- := \hat{\mathcal{J}}^{-1}|_{\mathcal{A}^-}, \quad S^- := S^{-1}|_{\mathcal{C}^-}, \quad \Phi^- := \Phi|_{\mathcal{A}^-}$$

2.2 Corollary *$\Phi^-$ is a one-sided conjugacy, i.e.,*

$$\begin{aligned} \Phi^-(\mathcal{A}^-) &= \mathcal{C}^- \\ S^-\Phi^- &= \Phi^- J^- \end{aligned}$$

*Proof.* It is enough to check the first assertion. From the limit formula clearly $\Phi(\mathcal{A}) \subset \mathcal{C}^-$. Then the intertwining property yields $\Phi^-(\mathcal{A}^-) \subset \mathcal{C}^-$. Further

$$\hat{\mathcal{C}} = \Phi(\hat{\mathcal{A}}) = \Phi(\mathcal{A}^- \vee \mathcal{C}^+) = \Phi^-(\mathcal{A}^-) \vee \mathcal{C}^+.$$

Because $\hat{\mathcal{C}}$ is the tensor product of $\mathcal{C}^-$ and $\mathcal{C}^+$, we conclude that $\Phi^-(\mathcal{A}^-) = \mathcal{C}^-$. $\square$

Now we want to explain how in the commutative case the one-sided conjugacy $\Phi^-$ corresponds to the conjugacy $\eta^{-1}$ mentioned in the introduction. Assume that $J = J_\gamma$ for a target map $\gamma : A \times C \to A$. The results above can be applied to the von Neumann algebras of essentially bounded functions on the corresponding probability spaces, for example $\mathcal{A} = L^\infty(A)$, $\mathcal{C} = L^\infty(C)$,

$$C^+ = C^{\mathbb{N}_0},\ \mathcal{C}^+ = L^\infty(C^+), \quad C^- = C^{-\mathbb{N}},\ \mathcal{C}^- = L^\infty(C^-),$$
$$A^+ = A \times C^+,\ \mathcal{A}^+ = L^\infty(A^+), \quad \text{etc.}$$

The measures are always faithful product measures. For $f \in \mathcal{A} = L^\infty(A)$ we have $Jf = f \circ \gamma$. Let us write $J \sim \gamma$ for such a correspondence. Then for $J^+ : \mathcal{A}^+ \to \mathcal{A}^+$ we get $J^+ \sim \gamma^+$ with $\gamma^+ : A^+ \to A^+$ given by

$$\gamma^+ : (a, c_0, c_1, c_2, \ldots) \mapsto (\gamma(a, c_0), c_1, c_2, \ldots),$$

which may be interpreted as a one-sided edge shift. The inductive limit above corresponds here to a so-called natural extension, which is a well known device to produce invertible measure preserving transformations from not necessarily invertible ones, cf. [**Pet**]. In our situation this is nothing but the two-sided edge shift. Similarly, we have $S \sim \sigma$ where $\sigma$ is the two-sided left shift on $\hat{C} := C^{\mathbb{Z}}$.

To establish a connection we assume that there is a synchronizing word with target $a \in A$ for $\gamma$. Almost all $c \in C^-$ contain this synchronizing word infinitely often. In the following we shall always neglect the complement which has measure zero. Then if $c_{-(M+m)}, \ldots, c_{-(M+1)}$ is an occurrence of the synchronizing word inside $c$ we can compute

$$a_0 \;:=\; \gamma(\ldots\gamma(\gamma(a, c_{-M}), c_{-M+1}), \ldots, c_{-1}) \in A,$$

and the map

$$\eta^{-1} : C^- \to C^- \times A, \quad c \mapsto (c, a_0)$$

is well defined. Intuitively, $a_0$ is the final target after having passed the colour sequence $c \in C^-$. We can identify $\eta^{-1}(C^-) \subset C^- \times A$ with the shift space $E^-$ in the introduction and clearly $\eta^{-1}$ coincides with $\eta^{-1} : C^- \to E^-$ described there. We define $\mathcal{A}^- := L^\infty(E^-)$ and check that this is consistent with the notation used earlier in this section. In fact, define further $\hat{A} := \eta^{-1}(C^-) \times C^+ \subset C^- \times A \times C^+$ and $\hat{\gamma} : \hat{A} \to \hat{A}$ by

$$\hat{\gamma}(\ldots, c_{-n}, \ldots, c_{-1}, a_0, c_0, \ldots, c_n, \ldots) = (\ldots, c_{-n}, \ldots, c_{-1}, c_0, \gamma(a_0, c_0), c_1, \ldots, c_n, \ldots).$$

Then $\hat{\gamma}$ is the natural extension of $\gamma^+$ and thus $\hat{\mathcal{J}} \sim \hat{\gamma}$ and our notation is consistent.

Denote by $i_b : \hat{C} \to C^- \times A \times C^+$ the operation which inserts $b \in A$ between the two halves. Then if $f \in \hat{\mathcal{A}} = L^\infty(\hat{A})$ is a function not depending on $c_{-n}$ for all $n > N$ (and some $N \in \mathbb{N}$), the limit

$$(\Phi f)(c) = \lim_{n\to\infty} (S^{-n} Q_\phi \hat{\mathcal{J}}^n f)(c) = \lim_{n\to\infty} f(\hat{\gamma}^n i_b\, \sigma^{-n}(c))$$

exists for (almost) all $c \in \hat{C}$. In fact, note that the conditional expectation $Q_\phi$ computes an average of values of $f$ indexed by $b \in A$, but in the limit the value is the same for all $b$ and thus the result depends neither on $b$ nor on $\phi$. We do not really need a limit $n \to \infty$ in this commutative case, we only have to choose $n$ large enough (dependent on $c$) to find an occurrence of the synchronizing word. Explicitly, for all $n$ large enough,

$$\hat{\gamma}^n i_b\, \sigma^{-n}(c) \;=\; \hat{\gamma}^n(\ldots, c_{-(n+1)}, b, c_{-n}, \ldots) \;=\; (\ldots, c_{-1}, a_0, c_0, \ldots),$$

independent of $b$.

If $f \in \mathcal{A}^-$, i.e., $f$ does not depend on $c_0, c_1, \ldots$, then we see that $\Phi f \in \mathcal{C}^-$ and $(\Phi^- f)(c) = f(\eta^{-1}(\ldots, c_{-n}, \ldots, c_{-1}))$. But this means that $\Phi^- \sim \eta^{-1}$, as claimed.

Using dominated convergence the defining limit for the Møller operator $\Phi$ also exists in the topologies considered in Theorem 2.1. Thus we have proved one direction of

2.3 Proposition *The following assertions are equivalent:*

*(a)* $J_\gamma$ *is asymptotically complete.*

*(b)* $\gamma$ *admits a synchronizing word.*

The other direction may also be explained in the setting above. We omit the details because we will obtain a complete proof for Proposition 2.3 in Section 5 (see Theorem 5.1), by other means.

Here we emphasize the fact that our scattering approach gives a non-commutative generalization of the type of conjugacies considered in [**LiMa**], Chapter 9, or [**AMT**], which are based on synchronizing words. It may thus be interpreted as non-commutative coding. See also the corresponding discussion in the introduction. Looking for criteria for asymptotic completeness thus means looking for non-commutative analogues of synchronizing words. The following sections take this direction.

## 3. *Regularity*

This is a preparatory section. We discuss a property of positive maps which we call regularity. Terminology is not standardized here, sometimes such maps are called asymptotically stable or they are said to have an absorbing state. Our terminology is borrowed from the case of stochastic matrices, see Definition 3.4. We will need this concept when we look for criteria for asymptotic completeness in the following sections.

3.1 Definition *Let* $M$ *be a von Neumann algebra,* $S : \mathcal{M} \to \mathcal{M}$ *positive and identity preserving. Then* $S$ *is called regular if it is ultraweakly continuous and if there exists a normal state* $\omega$ *such that for all* $x \in \mathcal{M}$

$$S^n(x) \to \omega(x)\mathbb{1} \quad \textit{ultraweakly} \quad \textit{if} \quad n \to \infty.$$

If $S_* : \mathcal{M}_* \to \mathcal{M}_*$ denotes the preadjoint of $S$, then regularity means that for all normal states $\rho$:

$$S_*^n(\rho) \to \omega \quad \text{weakly} \quad \text{if} \quad n \to \infty.$$

It follows that $\omega$ is the unique $S$-invariant state. We deliberately do not assume that $S$ is irreducible, so $\omega$ need not be faithful.

We are especially interested in $\mathcal{M} = \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space.

3.2 PROPOSITION *Let $S : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a positive identity preserving operator, $\Omega \in \mathcal{H}$ a unit vector such that the induced vector state $\phi_\Omega : \mathcal{B}(\mathcal{H}) \ni x \mapsto \langle x\Omega, \Omega\rangle \in \mathbb{C}$ is $S-$invariant, i.e., $\phi_\Omega \circ S = \phi_\Omega$.*
*The following assertions are equivalent:*

*(a) $S$ is regular with invariant state $\phi_\Omega$.*

*(b) The fixed space of $S$ is one-dimensional, hence consists of multiples of $\mathbb{1}$.*

*(c) $\lim_{n\to\infty} S^n(q) = 0$ strongly (or weakly), where $q$ is the orthogonal projection onto $\Omega^\perp$.*

*Proof.* (a)⇒(b) is immediate from the definition of regularity.
(b)⇒(c). Since $S(q) \leq q$ (compare the argument preceding Proposition 1.3), the limit $\lim_{n\to\infty} S^n(q)$ exists strongly, hence is 0 by (b). Note also that for monotone convergence of positive operators strong and weak convergence are equivalent.
(c)⇒(a). With $p := q^\perp$ (so that $p$ is the orthogonal projection onto $\mathbb{C}\Omega$), decompose any $x \in \mathcal{B}(\mathcal{H})$ as

$$x = pxp + pxq + qxp + qxq.$$

If $x \in \mathcal{B}(\mathcal{H})$ is positive then $0 \leq qxq \leq \lambda q$ for some number $\lambda$. Thus assumption (c) implies $\lim_{n\to\infty} S^n(qxq) = 0$ strongly for positive and hence for arbitrary $x \in \mathcal{B}(\mathcal{H})$. By the Kadison-Schwarz inequality (cf. [**Tak**], IV.3.8), we see that

$$S^n(pxq)^\star\, S^n(pxq) \leq S^n(qxpxq) \to 0 \quad \text{strongly for } n \to \infty.$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} S^n(x) &= \lim_{n\to\infty} S^n(pxp) \\ &= \lim_{n\to\infty} S^n(p\,\phi_\Omega(x)) \\ &= \phi_\Omega(x) \lim_{n\to\infty} S^n(p) = \phi_\Omega(x)\mathbb{1} \quad \text{strongly.} \end{aligned}$$

□

3.3 PROPOSITION *If $\mathcal{H}$ is finite-dimensional and $S$ and $\Omega$ are as in 3.2 with $S(x) = \sum_{i=1}^k a_i^\star\, x\, a_i$ for some elements $a_i \in \mathcal{B}(\mathcal{H})$, then $S$ is regular if and only if $\Omega$ is cyclic for $\{a_i^\star : 1 \leq i \leq k\}$, i.e., the vectors $\{a_{i_1}^\star \dots a_{i_n}^\star \Omega : n \in \mathbb{N}, 1 \leq i_1, \dots, i_n \leq k\}$ span $\mathcal{H}$.*

*Proof.* Again let $p$ be the orthogonal projection onto $\mathbb{C}\Omega$ and $q := p^\perp$. If $a^\star\Omega \neq 0$ for some $a \in \mathcal{B}(\mathcal{H})$ then the orthogonal projection onto $\mathbb{C}a^\star\Omega$ is the support projection of $a^\star p\, a$. Therefore, the support projection of $S^n(p) = \sum_{i_1,\dots,i_n=1}^k a_{i_1}^\star \dots a_{i_n}^\star p\, a_{i_n} \dots a_{i_1}$ is the orthogonal projection onto the linear span of $\{a_{i_1}^\star \dots a_{i_n}^\star \Omega : 1 \leq i_1, \dots, i_n \leq k\}$.

Since $\mathcal{H}$ is finite dimensional, $\Omega$ is cyclic for $\{a_i^\star : 1 \leq i \leq k\}$ if and only if $S^n(p)$ is strictly positive for some $n \in \mathbb{N}$, hence if and only if $S^n(q) \leq \lambda q$ for some number $\lambda < 1$. Since $S^n(q)$ is monotonically decreasing in $n$ this is in turn equivalent to $\lim_{n\to\infty} S^n(q) = 0$, hence to the regularity of $S$. □

*Remarks:* The linear span of $\{a_{i_1}^\star \dots a_{i_n}^\star \Omega : 1 \leq i_1, \dots, i_n \leq k\}$ is increasing in $n$. This follows from $S(p) \geq p$. More directly, since $\phi_\Omega \circ S = \phi_\Omega$, $\Omega$ is a common eigenvector for the $a_i$, i.e., $a_i\Omega = \lambda_i\Omega$ for some $\lambda_i \in \mathbb{C}$, hence

$$\Omega = \mathbb{1}\,\Omega = \sum_i a_i^\star a_i \Omega = \sum_i a_i^\star \lambda_i\,\Omega = \sum_i \lambda_i a_i^\star \Omega \quad \text{etc.}$$

Note that $S(x) = \sum_{i=1}^k a_i^\star\, x\, a_i$ means that $S$ is a completely positive map, given in a so-called Kraus decomposition. In this case, Proposition 3.3 provides a useful tool for checking regularity, compare Section 6.

If $\mathcal{M}$ is finite dimensional and commutative, i.e., $\mathcal{M} = \mathbb{C}^n$ for some $n$, then a positive identity preserving map $S$ is nothing but a (row) stochastic matrix. For stochastic matrices there is the following definition of regularity by Seneta ([**Sen**], Def. 4.7):

3.4 DEFINITION *[**Sen**]*
*A stochastic matrix is regular if its essential indices form a single essential class which is aperiodic.*

This refers to a well known classification of indices, i.e., of states of the associated Markov chain: An index is called essential if there is at least one path to another one and for all such paths there also exist paths in the backward direction (with nonvanishing probability). Indices related in this way form an essential class. See [**Sen**], Chapter 1.2, for more details. Note that regularity of a stochastic matrix only depends on its pattern of zeros, or, in other words, on the associated directed graph.

The definitions of regularity are compatible:

3.5 PROPOSITION *For a stochastic matrix $L$ the following assertions are equivalent:*

*(a) $L$ is regular in the sense of Seneta (Definition 3.4).*

*(b) There is a unique invariant probability vector $\mu$ for $L$, and the rows of $L^k$ converge to $\mu$ for $k \to \infty$.*

*(c) $L$ is regular according to Definition 3.1.*

*Proof.* (a)$\Rightarrow$(b) is the content of Theorem 4.7 in [**Sen**].
The converse is also true: If there is more than one essential class for $L$ then the uniqueness of $\mu$ is lost, and if there is a nontrivial period then the convergence of $L^k$ fails. Compare the discussion in [**Sen**], Chapter 4.2 and 1.2.
Regularity according to Definition 3.1 means that for any probability vector $\nu$ we have $\lim_{k\to\infty} \nu L^k = \mu$. Choosing point measures for $\nu$ shows that this is equivalent to (b). $\square$

4. *Extended Transition Operators*
We look for a more direct criterion for asymptotic completeness of a transition $J$, as provided in the commutative case by synchronizing words. This goal can be achieved by considering extended transition operators.

Let $J : (\mathcal{A}, \phi) \to (\mathcal{A}, \phi) \otimes (\mathcal{C}, \psi)$ be a transition. As usual we assume that $\mathcal{A}$ and $\mathcal{C}$ are finite dimensional and that the states $\phi$ and $\psi$ are faithful. Then by the GNS-construction we obtain Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{C} \subset \mathcal{B}(\mathcal{K})$ and cyclic unit vectors $\Omega_\varphi \in \mathcal{H}$ and $\Omega_\psi \in \mathcal{K}$ such that $\phi(a) = \langle a\,\Omega_\varphi, \Omega_\varphi\rangle$ for $a \in \mathcal{A}$ and $\psi(c) = \langle c\,\Omega_\psi, \Omega_\psi\rangle$ for $c \in \mathcal{C}$. The following formula defines an isometry:

$$v : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}, \quad va\,\Omega_\varphi := J(a)\,\Omega_\varphi \otimes \Omega_\psi \quad (a \in \mathcal{A}).$$

In fact, for $a, b \in \mathcal{A}$

$$\begin{aligned}\langle v\,a\,\Omega_\varphi, v\,b\,\Omega_\varphi\rangle &= \langle J(a)\,\Omega_\varphi \otimes \Omega_\psi, J(b)\,\Omega_\varphi \otimes \Omega_\psi\rangle = \langle \Omega_\varphi \otimes \Omega_\psi, J(a^*b)\Omega_\varphi \otimes \Omega_\psi\rangle \\ &= \phi \otimes \psi\,(J(a^*b)) = \phi(a^*b) = \langle a\,\Omega_\varphi, b\,\Omega_\varphi\rangle.\end{aligned}$$

4.1 Definition *The operator*

$$Z' : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \quad x \mapsto v^\star\,(x \otimes \mathbb{1}_\mathcal{K})\,v$$

*is called the dual extended transition operator associated to the transition* $J : (\mathcal{A}, \phi) \to (\mathcal{A}, \phi) \otimes (\mathcal{C}, \psi)$.

*Remarks.* It is easy to check that $Z'$ is a completely positive identity preserving map with invariant vector state $\langle \cdot\,\Omega_\varphi, \Omega_\varphi\rangle$. The defining formula provides a Stinespring representation, cf. [**Tak**], IV.3.6. If we choose an orthonormal basis $(\epsilon_i)$ of $\mathcal{K}$, then there are linear maps $a_i \in \mathcal{B}(\mathcal{H})$ with

$$v(\xi) = \sum_i a_i(\xi) \otimes \epsilon_i \text{ for all } \xi \in \mathcal{H}.$$

From this we find

$$Z'(x) = \sum_i a_i^\star\,x\,a_i \text{ for all } x \in \mathcal{B}(\mathcal{H}).$$

This is often called a Kraus decomposition.

The following result 4.2(a) shows that $Z'$ extends the dual $T'$ of the transition operator $T$ corresponding to $J$. This explains our terminology. By duality we mean here the consideration of commutants. Extended transition operators have been introduced by R. Gohm in [**Go1, Go2, Go3**] where much additional information about them can be found.

4.2 Proposition *(Extension properties of $Z'$)*

*(a) Let $\mathcal{A}'$ be the commutant of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$. Then $Z'(\mathcal{A}') \subset \mathcal{A}'$ and $Z'|_{\mathcal{A}'} = T'$, where*

$$T' : \mathcal{A}' \to \mathcal{A}', \quad \langle T'(a')\,a\,\Omega_\varphi, \Omega_\varphi\rangle = \langle a'\,T(a)\,\Omega_\varphi, \Omega_\varphi\rangle \quad (a \in \mathcal{A},\, a' \in \mathcal{A}').$$

(b) *Suppose that there is a conditional expectation* $P : \mathcal{A} \otimes \mathcal{C} \to J(\mathcal{A})$, *which leaves the state* $\phi \otimes \psi$ *invariant. Then* $Z'(\mathcal{A}) \subset \mathcal{A}$ *and* $Z'|_{\mathcal{A}} = T^+$, *where*

$$T^+ : \mathcal{A} \to \mathcal{A}, \quad \langle T^+(a)\,\Omega_\varphi, b\,\Omega_\varphi\rangle \;=\; \langle a\,\Omega_\varphi, T(b)\,\Omega_\varphi\rangle, \quad (a, b \in \mathcal{A}).$$

*Proof:* If $a' \in \mathcal{A}'$ then $a' \otimes \mathbb{1} \in (\mathcal{A} \otimes \mathcal{C})'$. With $a, b, c \in \mathcal{A}$ we get

$$\begin{aligned}
\langle\, Z'(a')\, ab\,\Omega_\varphi, c\,\Omega_\varphi\rangle &= \langle\, v^*\,(a' \otimes \mathbb{1})\, v\, ab\,\Omega_\varphi, c\,\Omega_\varphi\rangle \\
&= \langle\, (a' \otimes \mathbb{1})\, J(ab)\,\Omega_\varphi \otimes \Omega_\psi, J(c)\,\Omega_\varphi \otimes \Omega_\psi\rangle \\
&= \langle\, J(a)\,(a' \otimes \mathbb{1})\, J(b)\,\Omega_\varphi \otimes \Omega_\psi, J(c)\,\Omega_\varphi \otimes \Omega_\psi\rangle \\
&= \langle\, (a' \otimes \mathbb{1})\, vb\,\Omega_\varphi, va^*c\,\Omega_\varphi\rangle \\
&= \langle\, av^*\,(a' \otimes \mathbb{1})\, vb\,\Omega_\varphi, c\,\Omega_\varphi\rangle \\
&= \langle\, a\, Z'(a')\, b\,\Omega_\varphi, c\,\Omega_\varphi\rangle, \ \text{ i.e. } \ Z'(a') \in \mathcal{A}'.
\end{aligned}$$

With $b = c = \mathbb{1}$ it follows that

$$\begin{aligned}
\langle Z'(a')\, a\,\Omega_\varphi, \Omega_\varphi\rangle &= \langle\, (a' \otimes \mathbb{1})\,\Omega_\varphi \otimes \Omega_\psi, J(a^*)\,\Omega_\varphi \otimes \Omega_\psi\rangle \\
&= \langle\, a'\,\Omega_\varphi, T(a^*)\,\Omega_\varphi\rangle = \langle a'\, T(a)\,\Omega_\varphi, \Omega_\varphi\rangle.
\end{aligned}$$

This proves (a).

To prove (b) we associate to each $a \in \mathcal{A}$ the element $\tilde{a} \in \mathcal{A}$ so that $P(a \otimes \mathbb{1}) = J(\tilde{a})$. Then

$$\begin{aligned}
\langle Z'(a)\, b\,\Omega_\varphi, c\,\Omega_\varphi\rangle &= \langle\, v^*\,(a \otimes \mathbb{1})\, v\, b\,\Omega_\varphi, c\,\Omega_\varphi\rangle \\
&= \langle\, (a \otimes \mathbb{1})\, J(b)\,\Omega_\varphi \otimes \Omega_\psi, J(c)\,\Omega_\varphi \otimes \Omega_\psi\rangle \\
&= \langle\, P(a \otimes \mathbb{1})\, J(b)\,\Omega_\varphi \otimes \Omega_\psi, J(c)\,\Omega_\varphi \otimes \Omega_\psi\rangle \\
&= \langle\, J(\tilde{a}b)\,\Omega_\varphi \otimes \Omega_\psi, J(c)\,\Omega_\varphi \otimes \Omega_\psi\rangle \\
&= \langle\, \tilde{a}b\,\Omega_\varphi, c\,\Omega_\varphi\rangle, \text{ i.e. } Z'(a) = \tilde{a} \in \mathcal{A}.
\end{aligned}$$

In the step where $P$ is inserted we used the module property of conditional expectations (cf. [**Tak**], III.3.4). With $b = \mathbb{1}$ it follows that

$$\begin{aligned}
\langle\, Z'(a)\,\Omega_\varphi, c\,\Omega_\varphi\rangle &= \langle\, (a \otimes \mathbb{1})\,\Omega_\varphi \otimes \Omega_\psi, J(c)\,\Omega_\varphi \otimes \Omega_\psi\rangle \\
&= \langle\, a\,\Omega_\varphi, T(c)\,\Omega_\varphi\rangle.
\end{aligned}$$

This proves (b). □

Our main result about the dual extended transition operator $Z'$ is the following

4.3 Theorem *The following assertions are equivalent:*

(a) *The transition* $J : (\mathcal{A}, \phi) \to (\mathcal{A}, \phi) \otimes (\mathcal{C}, \psi)$ *is asymptotically complete.*

(b) *The dual extended transition operator* $Z'$ *(associated to* $J$*) is regular.*

*Proof.* The GNS-construction for $(\mathcal{A}^+ = \mathcal{A} \otimes \mathcal{C}^+, \phi^+ = \phi \otimes \psi^+)$ and for the Markovian time evolution $J^+ = J \circ (Id_{\mathcal{A}} \otimes S^+)$ (using shorthand notation as in

Section 1) gives us a Hilbert space $\mathcal{H} \otimes \mathcal{K}^+$ and an isometry $v_+ = v \circ (\mathbb{1}_{\mathcal{H}} \otimes s)$. Note that $\mathcal{K}^+$ is the infinite tensor product of the GNS-spaces arising from the copies of $(\mathcal{C}, \psi)$. (Cf. [**KaRi**], 11.5.29, for the definition of an infinite tensor product of Hilbert spaces along a given sequence of unit vectors.) Thus $s$ is a tensor shift on $\mathcal{K}^+$ which maps a vector $\eta^+ \in \mathcal{K}^+$ to the vector $\Omega_\psi \otimes \eta^+$. If $\xi$ is a unit vector in $\mathcal{H} \otimes \mathcal{K}^+$ and $x \in \mathcal{B}(\mathcal{H})$, then we get

$$\langle\, (x \otimes \mathbb{1})\, v_+\xi, v_+\xi\,\rangle = \langle\, (x \otimes \mathbb{1})\; v \circ (\mathbb{1} \otimes s)\xi,\; v \circ (\mathbb{1} \otimes s)\,\xi\rangle = \langle\, (Z'(x) \otimes \mathbb{1})\; \xi, \xi\,\rangle.$$

Iterating this formula shows that

$$\forall n \geq 0 \quad \langle\, (x \otimes \mathbb{1})\; v_+^n\, \xi,\; v_+^n\, \xi\,\rangle = \langle\, ({Z'}^n(x) \otimes \mathbb{1})\; \xi, \xi\,\rangle.$$

The conditional expectation $Q_\phi$ from $\mathcal{A} \otimes \mathcal{C}^+$ onto $\mathbb{1} \otimes \mathcal{C}^+ \simeq \mathcal{C}^+$ determined by $x \otimes y \mapsto \phi(x)\, y$ induces an orthogonal projection $p \otimes \mathbb{1}$ from $\mathcal{H} \otimes \mathcal{K}^+$ onto $\Omega_\varphi \otimes \mathcal{K}^+ \simeq \mathcal{K}^+$. Inserting $x = p \in \mathcal{B}(\mathcal{H})$ into the previous formula gives

$$\langle\, (p \otimes \mathbb{1})\, v_+^n\, \xi,\; v_+^n\, \xi\,\rangle = \langle\, ({Z'}^n(p) \otimes \mathbb{1})\; \xi, \xi\,\rangle.$$

The state $\langle \cdot\, \Omega_\varphi, \Omega_\varphi\rangle$ is invariant for $Z'$. Assuming (b), i.e. $Z'$ is regular, we find

$${Z'}^n(p) \to \langle\, p\, \Omega_\varphi, \Omega_\varphi\,\rangle \mathbb{1} = \mathbb{1} \quad (n \to \infty).$$

Because $\xi$ and $v_+^n \xi$ are unit vectors we conclude that $\|(p \otimes \mathbb{1}) v_+^n\, \xi\| \to 1$ and $\|v_+^n \xi - (p \otimes \mathbb{1}) v_+^n\, \xi\| \to 0$ which is a way to state asymptotic completeness for $J$.

In the other direction, i.e. assuming (a), we may reverse the arguments above to get

$$\langle\, ({Z'}^n(p) \otimes \mathbb{1})\; \xi, \xi\,\rangle = \langle\, (p \otimes \mathbb{1}) v_+^n\, \xi,\; v_+^n\, \xi\,\rangle \to 1.$$

Choose now $\xi = \eta \otimes \eta^+$ with unit vectors $\eta \in \mathcal{H}$ and $\eta^+ \in \mathcal{K}^+$. Then we get

$$\langle\, {Z'}^n(p)\eta, \eta\,\rangle = \langle\, ({Z'}^n(p) \otimes \mathbb{1})\; \xi, \xi\,\rangle \to 1 \quad (n \to \infty).$$

We conclude that ${Z'}^n(p^\perp) \to 0$ weakly. Thus $Z'$ is regular by Proposition 3.2(c).

□

We refer to [**Go2**] for some modifications of this result which on the one hand use more directly the setting of [**KüMa**] and on the other hand reveal some interesting connections with the physical concept of (state) entanglement. Further related results are in [**Go3**], Chapter 2. For us Theorem 4.3 first of all provides an interesting criterion for asymptotic completeness. The problem is reduced to checking regularity for an operator on a finite dimensional space (if $\mathcal{A}$ is finite dimensional). In fact, the reader may check that 4.2 and 4.3 are also valid for infinite dimensional von Neumann algebras.

We can now give a proof of an assertion already stated in Section 1, namely, independence of asymptotic completeness from the choice of (faithful) states.

4.4 Proposition *Let $J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ be a transition. If $J : (\mathcal{A}, \phi_1) \to (\mathcal{A}, \phi_1) \otimes (\mathcal{C}, \psi_1)$ is asymptotically complete then also $J : (\mathcal{A}, \phi_2) \to (\mathcal{A}, \phi_2) \otimes (\mathcal{C}, \psi_2)$ is asymptotically complete. Here $\phi_1, \phi_2, \psi_1, \psi_2$ are faithful states.*

*Proof.* Let us use subscripts 1 and 2 also for associated objects and define

$$\begin{aligned} \Gamma: \quad & \mathcal{H}_1 \quad \to \mathcal{H}_2, \quad a\,\Omega_{\phi_1} \mapsto a\,\Omega_{\phi_2} \quad (a \in \mathcal{A}), \\ \Lambda: \quad & \mathcal{K}_1 \quad \to \mathcal{K}_2, \quad c\,\Omega_{\psi_1} \mapsto c\,\Omega_{\psi_2} \quad (c \in \mathcal{C}). \end{aligned}$$

The maps $\Gamma$ and $\Lambda$ are bijections (recall our assumption of finite dimensionality). However, if the states are different they are not unitary. We first show that

$$v_2\,\Gamma = (\Gamma \otimes \Lambda)\, v_1.$$

In fact, if $J(a) = \sum_i a_i \otimes c_i$ then

$$\begin{aligned} v_2\,\Gamma\,a\,\Omega_{\phi_1} = v_2\,a\,\Omega_{\phi_2} = J(a)\;\Omega_{\phi_2} \otimes \Omega_{\psi_2} &= \sum_i a_i\,\Omega_{\phi_2} \otimes c_i\,\Omega_{\psi_2} \\ = (\Gamma \otimes \Lambda)\;J(a)\;\Omega_{\phi_1} \otimes \Omega_{\psi_1} &= (\Gamma \otimes \Lambda)\;v_1\,a\,\Omega_{\phi_1}. \end{aligned}$$

From that we infer that for all $x \in \mathcal{B}(\mathcal{H}_1)$

$$Z_2'\,(\Gamma\,x\,\Gamma^{-1}) = \Gamma\,Z_1'(x)\,\Gamma^{-1}.$$

In fact, for all $a, b \in \mathcal{A}$

$$\begin{aligned} \langle\, Z_2'\,(\Gamma\,x\,\Gamma^{-1})\,a\,\Omega_{\phi_2}, b\,\Omega_{\phi_2}\,\rangle &= \langle\, (\Gamma\,x\,\Gamma^{-1} \otimes \mathbb{1})\; v_2\,a\,\Omega_{\phi_2}, v_2\,b\,\Omega_{\phi_2}\,\rangle \\ &= \langle\, (\Gamma\,x\,\Gamma^{-1} \otimes \mathbb{1})\; v_2\,\Gamma a\,\Omega_{\phi_1}, v_2\,\Gamma b\,\Omega_{\phi_1}\,\rangle \\ = \langle\, (\Gamma \otimes \Lambda)^{-1}\;(\Gamma\,x\,\Gamma^{-1} \otimes \mathbb{1})\;&(\Gamma \otimes \Lambda) v_1\,a\,\Omega_{\phi_1}, v_1\,b\,\Omega_{\phi_1}\,\rangle \\ = \langle\, (x \otimes \mathbb{1})\;v_1\,a\,\Omega_{\phi_1}, v_1\,b\,\Omega_{\phi_1}\,\rangle &= \langle\, Z_1'(x)\;a\,\Omega_{\phi_1}, b\,\Omega_{\phi_1}\,\rangle \\ &= \langle\, \Gamma\,Z_1'(x)\,\Gamma^{-1}\;a\,\Omega_{\phi_2}, b\,\Omega_{\phi_2}\,\rangle. \end{aligned}$$

From this formula it is evident that $Z_1'$ is regular if and only if $Z_2'$ is regular: use for example Proposition 3.2(b). Now the assertion in Proposition 4.4 follows by applying Theorem 4.3. □

5. *Extended Transition in the Commutative Case*

In this section we shall examine the dual extended transition operator for a transition acting between commutative algebras. This leads to a very transparent method for checking asymptotic completeness in this case.

Let $\gamma : \{1, ..., n\} \times \{1, ..., m\} \to \{1, ..., n\}$ be a surjective map and let $J_\gamma$ be the associated transition. We may assume that $J_\gamma$ is irreducible. The Hilbert spaces $\mathcal{H}$ resp. $\mathcal{K}$ are identified with $\mathbb{C}^n$ resp. $\mathbb{C}^m$ with the usual scalar product and with canonical ONB's $(\delta_i)_{i=1}^n$ resp. $(\epsilon_k)_{k=1}^m$. We have cyclic vectors $\Omega_\varphi = \sum_{i=1}^n \sqrt{\phi_i}\,\delta_i$ resp. $\Omega_\psi = \sum_{k=1}^m \sqrt{\psi_k}\,\epsilon_k$, $\sum_{i=1}^n \phi_i = \sum_{k=1}^m \psi_k = 1$. Then we find

$$\begin{aligned} J_\gamma: \quad \mathbb{C}^n \;&\longrightarrow \mathbb{C}^n \otimes \mathbb{C}^m \\ \delta_j \;&\mapsto \textstyle\sum_{\gamma(i,k)=j} \delta_i \otimes \epsilon_k\ . \end{aligned}$$

For $\gamma(i,k) = j$ we also write $i \xrightarrow{k} j$. For the isometry $v$ extending $J_\gamma$ we get $v(\sqrt{\varphi_j}\,\delta_j) = \sum_{i \xrightarrow{k} j} \sqrt{\varphi_i}\,\delta_i \otimes \sqrt{\psi_k}\,\epsilon_k$, i.e.,

$$v(\delta_j) = \sum_{i \xrightarrow{k} j} \sqrt{\frac{\varphi_i \psi_k}{\varphi_j}}\,\delta_i \otimes \epsilon_k \;.$$

In the representation $v(\xi) = \sum_{i \xrightarrow{k} j} a_k(\xi) \otimes \epsilon_k$ (compare the remarks after Definition 4.1) the operators $a_k$ therefore take the form

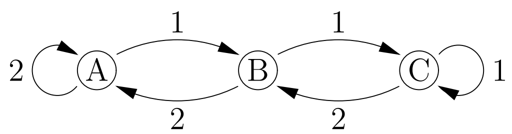
$$a_k(\delta_j) = \sum_{i \xrightarrow{k} j} \sqrt{\frac{\varphi_i \psi_k}{\varphi_j}}\,\delta_i$$

or as a $n{\times}n$-matrix with respect to the canonical base

$$(a_k)_{ij} = \begin{cases} \sqrt{\frac{\varphi_i \psi_k}{\varphi_j}} & \text{if } i \xrightarrow{k} j \\ 0 & \text{else .} \end{cases}$$

Remember that the transition $J_\gamma$ may be represented by a $C$-graph $G_\gamma$, where $C = \{1, ..., m\}$. We call the matrix above the *normalized adjacency matrix* for the label $k \in C$ of the graph $G_\gamma$. This is consistent with the definitions in [**LiMa**], §2.2, §3.1, up to our normalizing factors $\sqrt{\frac{\varphi_i \psi_k}{\varphi_j}}$ .

Example: Consider the following C-graph $G_\gamma$:




With $\phi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\psi = (\frac{1}{2}, \frac{1}{2})$ we get

$$a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&1 \end{pmatrix} \;,\quad a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1&0&0 \\ 1&0&0 \\ 0&1&0 \end{pmatrix} \;.$$

Back to the general case, note that we already have a Kraus decomposition $\sum_{k=1}^m a_k^* \bullet a_k$ of the dual extended transition operator $Z' : M_n \to M_n$, where $M_n = \mathcal{B}(C^n)$ are the $n \times n-$matrices. There is an interesting way to avoid non-commutativity in the regularity check for $Z'$. Consider the preadjoint $Z'_*$ which is given by $Z'_*(\rho) = \sum_{k=1}^m a_k\,\rho\,a_k^*$. Let $(e_{ij})_{i,j=1}^n$ be the $n{\times}n$-matrix units and $\tilde{e}_{ij} := \sqrt{\varphi_i \varphi_j}\,e_{ij}$. Now

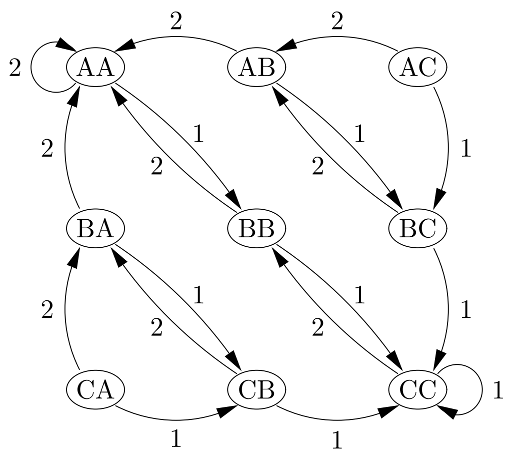
$$\begin{aligned} a_k\,\tilde{e}_{rs}\,a_k^* &= \sum_{\substack{i \xrightarrow{k} r \\ j \xrightarrow{k} s}} \sqrt{\frac{\varphi_i \psi_k}{\varphi_r}}\,\sqrt{\varphi_r \varphi_s}\,\sqrt{\frac{\varphi_j \psi_k}{\varphi_s}}\,e_{ij} \\ &= \sum_{\substack{i \xrightarrow{k} r \\ j \xrightarrow{k} s}} \psi_k\,\tilde{e}_{ij} \;. \end{aligned}$$

We also write $(i,j) \xrightarrow{k} (r,s)$ for $i \xrightarrow{k} r$ and $j \xrightarrow{k} s$. Using $(\tilde{e}_{ij})$ as a basis we get a $n^2{\times}n^2$-matrix $L_k$ with

$$(L_k)_{ij,rs} = \begin{cases} \psi_k & \text{if } (i,j) \xrightarrow{k} (r,s) \\ 0 & \text{else.} \end{cases}$$

For this matrix we can also give an interpretation in terms of $C$-graphs. If we are given two $C$-graphs $G$ and $H$, then we may form their label product $G * H$ ([**LiMa**], Def.3.4.8): the set of vertices is given by the cartesian product of the sets of vertices of $G$ and $H$, and there is a $k$-labeled edge from one pair to another if and only if there are $k$-labeled edges for both components. Obviously $G * H$ is a $C$-graph. We now see that $L_k$ may be interpreted as the normalized adjacency matrix for the label $k \in C$ of the graph $G_\gamma * G_\gamma$, the label product of $G_\gamma$ with itself.

The label product of irreducible graphs may be reducible in general. In our example, $G_\gamma * G_\gamma$ looks as follows:




and

$$L_1 = \frac{1}{2}\begin{pmatrix} 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \end{pmatrix} \quad , \quad L_2 = \frac{1}{2}\begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0 \end{pmatrix} \quad ,$$

where the ordering of basis elements is AA, BB, CC, AB, BC, AC, BA, CB, CA.

For the preadjoint $Z'$ of the extended transition operator we get the $n^2 \times n^2$-matrix $L = \sum_{k=1}^m L_k$, i.e.,

$$L_{ij,rs} = \sum_{(i,j) \xrightarrow{k} (r,s)} \psi_k \ .$$

$L$ may be called the (full) *normalized adjacency matrix* of the label product $G_\gamma * G_\gamma$. The normalization is done in such a way that $L$ is a (row) stochastic matrix, and the equation $L\mathbb{1} = \mathbb{1}$ corresponds to $Z'_* p = p$, where $p := \sum_{i,j=1}^n \tilde{e}_{ij}$ is the orthogonal projection onto $\mathbb{C}\Omega_\varphi$.

In our example we have

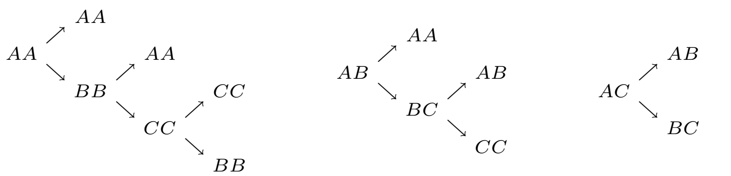
$$L = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

In accordance with Proposition 4.2 we have in the left upper corner a stochastic 3×3-matrix representing the transition operator $T$ associated with the transition $J$. This phenomenon can also be seen in the label product $G_\gamma * G_\gamma$ where the graph $G_\gamma$ is reproduced in the diagonal and thus there can be no edge leaving the diagonal. Note in addition that $G_\gamma * G_\gamma$ is symmetric with respect to the diagonal.

Let us now turn to regularity. Having represented $Z'_*$ by a stochastic matrix $L$, we can now use regularity for stochastic matrices, as discussed in Section 3.

We call a C-graph *regular* if its (suitably normalized) adjacency matrix is regular. Recall that this only depends on the position of zeros in the matrix and that there is a direct description in terms of the graph: A C-graph is regular if there is a subset of vertices which together with its connecting edges form an irreducible and aperiodic C-graph (the essential class), while every vertex not in this subset is inessential but connected to the essential class by a path.

We want to emphasize that a regularity check for C-graphs is quite elementary. We illustrate the (self-explaining) algorithm given in [**Sen**], Chapter 1.2, for the graph $G_\gamma * G_\gamma$ of our example (depicted above):




This shows that in our example the label product $G_\gamma * G_\gamma$, while not being irreducible any more, is still regular.

We can prove a general statement in this direction which relates these considerations to previous sections.

5.1 Theorem *Let the (irreducible) transition $J_\gamma$ be given with C-graph $G_\gamma$, extended transition operator $Z'$ and with the stochastic matrix $L$ associated to $Z'_*$. The following assertions are equivalent:*

*(a) $J_\gamma$ is asymptotically complete.*

*(b) $Z'$ is regular.*

*(c) $L$ is regular.*

*(d) $G_\gamma * G_\gamma$ is regular.*

*(e) There is a synchronizing word for $G_\gamma * G_\gamma$.*

*(f) There is a synchronizing word for $G_\gamma$.*

*Remarks:*

By (a)⇔(c), asymptotic completeness of $J_\gamma$ depends only on the zero pattern of $L$. This is a commutative version of Proposition 4.4.

If $L$ is regular then the unique essential class must be the diagonal of $G_\gamma * G_\gamma$. The restriction of $L$ is the transition operator $T$ associated to $J_\gamma$. Thus the invariant probability distribution $\tilde{\mu}$ is given by $\tilde{\mu} = (\mu, 0)$, where $\mu$ is the invariant probability distribution for $T$ (compare our introduction).

An irreducible C-graph with a synchronizing word is aperiodic, cf. [**LiMa**], Chapter 5.

*Proof.* (a)⇔(b) follows from Theorem 4.3.
(b)⇔(c) is not quite immediate because the matrix $L$ corresponds to $Z'_*$ (not to $Z'$ itself). Consider an element $\rho \in \mathbb{M}_n$ on which $Z'_*$ acts. With respect to the basis $(\tilde{e}_{ij})$ we get a vector of length $n^2$ on which $L$ acts and which we also denote by $\rho$. We have already noted above that $p$, the orthogonal projection onto $\mathbb{C}\Omega_\varphi$, corresponds to the constant vector $\mathbb{1}$. Now regularity of $L$ means that for any $\rho$ the sequence $L^k \rho$ converges to a scalar multiple of $\mathbb{1}$. This means that $(Z'_*)^k \rho$ converges to a scalar multiple of $p_{\Omega_\varphi}$. But this is equivalent to the regularity of $Z'$. (b)⇔(c) is proved.
(c)⇔(d) follows because $L$ is the adjacency matrix of $G_\gamma * G_\gamma$.

The remaining equivalences are given by the following

5.2 Lemma *Let $G$ be any irreducible C-graph. The following assertions are equivalent:*
*(a) There is a synchronizing word for $G$.*
*(b) There is a synchronizing word for $G * G$.*
*(c) $G * G$ is regular.*

*Proof.* (a)⇒(b): A synchronizing word $w$ for $G$ also does the job for $G * G$.

(b)⇒(c): Any C-graph has at least one essential class (cf. [**Sen**], Lemma 1.1). The existence of the synchronizing word $w$ implies that there is at most one essential class, and this class contains the target of $w$. It is aperiodic and all other vertices are connected to it by a path (e.g. the one labeled by $w$).

(c)⇒(a): First note that the C-graph $G$ is reproduced in the diagonal of the C-graph $G * G$. Thus the essential class of the regular graph $G * G$ must be a subset of the diagonal, indeed the whole diagonal because of the irreducibility of $G$. Because any vertex of $G * G$ is connected to the essential class by a path, for any pair $(x, y)$ of vertices of $G$ there is a word $w_{x,y}$ labeling a path from $(x, y)$ to $(x_0, x_0)$, where $x_0$ is a fixed vertex of $G$ destined to be the target of the synchronizing word to be constructed.

We proceed by induction and show that for $k$ vertices $x_1, ..., x_k$ of $G$ there is a word $w_k$ labeling paths from $x_l$ to $x_0$ for all $1 \leqslant l \leqslant k$ simultaneously: For $k = 1$ take $w_1 = w_{x_0,x_1}$. For $k+1$ take $w_{k+1} = w_k w_{x_0,x'_{k+1}}$, where $x'_{k+1}$ is the target of the path labeled $w_k$ starting at $x_{k+1}$. Since our graphs are finite we are done. □

## 6. *A Non-Commutative Example*

The following class of examples is motivated from the Jaynes-Cummings model in physics where a quantum harmonic oscillator interacts with a two level atom [**WBKM, MeSa**].

Denote by $M_d$ the algebra of $d \times d-$matrices with complex entries. A transition $J : M_d \to M_d \otimes M_2$ $(d \geq 2)$ will be implemented by a unitary $u \in M_d \otimes M_2$ such that $J(x) = u^\star\,(x \otimes \mathbb{1})\,u$.

It is convenient to identify $M_d \otimes M_2$ with $M_2(M_d)$, the algebra of $2\times 2-$matrices with entries from $M_d$, such that $x \otimes \mathbb{1}_2$ is identified with $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ $(x \in M_d)$, while $\mathbb{1}_d \otimes \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$ corresponds to $\begin{pmatrix} y_{11}\mathbb{1}_d & y_{12}\mathbb{1}_d \\ y_{21}\mathbb{1}_d & y_{22}\mathbb{1}_d \end{pmatrix}$ $(y_{ij} \in \mathbb{C})$.

Denote by $\{e_i : 1 \leq i \leq d\}$ the canonical orthonormal basis of $\mathbb{C}^d$ and by $\{e_{ij} : 1 \leq i, j \leq d\}$ the corresponding canonical matrix units of $M_d$ characterized by $e_{ij}e_k = \delta_{jk}e_i$.

Let $\alpha_1, \ldots, \alpha_d$ be real numbers with $0 \leq \alpha_k \leq 1$, $\alpha_1 := 1$, and put $\beta_k := i\sqrt{1-\alpha_k^2}$ so that $\begin{pmatrix} \alpha_k & \beta_k \\ \beta_k & \alpha_k \end{pmatrix}$ is a unitary $2\times 2-$matrix $(1 \leq k \leq d)$. On the cost of additional notation the following considerations can easily be extended to the case where $\begin{pmatrix} \alpha_k & \beta_k \\ \beta_k & \alpha_k \end{pmatrix}$ is replaced by any unitary $2 \times 2-$matrix $\begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix}$ for $2 \leq k \leq d$.

Now define the $d \times d-$matrices

$$a := \begin{pmatrix} 1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_d \end{pmatrix}, \quad a_+ := \begin{pmatrix} \alpha_2 & & & \\ & \ddots & & \\ & & \alpha_d & \\ & & & 1 \end{pmatrix},$$

$$b := \begin{pmatrix} 0 & & & \\ & \beta_2 & & \\ & & \ddots & \\ & & & \beta_d \end{pmatrix}, \quad s := \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Finally, define

$$u := \begin{pmatrix} a_+ & s^\star b \\ bs & a \end{pmatrix} \in M_2(M_d).$$

For the computations to come the following elementary relations are useful:

$$ab = ba, \quad a^\star = a, \quad b^\star = -b$$
$$a^2 + b^\star b \;=\; \mathbb{1}_d \;=\; a_+^2 + s^\star b^\star b s$$
$$as = sa_+, \quad a_+ s^\star = s^\star a, \quad b = ss^\star b = bss^\star.$$

In particular, it follows that $u$ is unitary, hence a transition $J$ is defined by

$$J : M_d \to M_d \otimes M_2, \quad x \mapsto u^\star\,(x \otimes \mathbb{1})\,u.$$

Explicitly, one gets

$$\begin{aligned} J(x) &= \begin{pmatrix} a_+ & s^\star b^\star \\ b^\star s & a \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} a_+ & s^\star b \\ bs & a \end{pmatrix} \\ &= \begin{pmatrix} T_1(x) & T_2(x) \\ T_3(x) & T_4(x) \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} T_1(x) &= a_+ x\, a_+ \,+\, s^\star\, b^\star\, x\, b\, s \\ T_2(x) &= a_+\, x\, s^\star\, b \,+\, s^\star\, b^\star\, x\, a \\ T_3(x) &= b^\star\, s\, x\, a_+ \,+\, a\, x\, b\, s \\ T_4(x) &= b^\star\, s\, x\, s^\star\, b \,+\, a\, x\, a. \end{aligned}$$

For the following considerations it is convenient to equip $M_d$ and $M_2$ with tracial states denoted by $\tau$. In view of Proposition 4.4 this is no loss of generality.

The set $\{\sqrt{2}\, e_{ij} : 1 \le i, j \le 2\}$ forms an orthonormal basis of the corresponding Hilbert space $M_2$, further $\Omega := \mathbb{1}_d$ is a unit vector in the Hilbert space $M_d$, and the dual extended transition operator $Z' : \mathcal{B}(M_d) \to \mathcal{B}(M_d)$ according to Definition 4.1 is given by

$$Z' = \sum_{i=1}^{4} a_i^\star \bullet a_i \quad \text{with } a_i := \frac{1}{\sqrt{2}} T_i.$$

6.1 THEOREM *The following conditions are equivalent:*

*(a)* $\alpha_i \neq 1$ *for* $2 \le i \le d$

*(b)* $\Omega$ *is cyclic for* $\{a_1^\star, \ldots, a_4^\star\}$, *i.e.,* $J$ *is asymptotically complete.*

*Proof.* Assume (a). The adjoint of $T_1$ satisfies

$$\begin{aligned} \langle T_1^\star(x), y \rangle &= \langle x, T_1(y) \rangle = \tau(T_1(y)^\star x) \\ &= \tau((a_+ y^\star a_+ + s^\star b^\star y^\star bs)x) \\ &= \tau(y^\star(a_+ x a_+ + bsxs^\star b^\star)) \\ &= \langle a_+ x a_+ + bsxs^\star b^\star, y \rangle \end{aligned}$$

$(x, y \in M_d)$, hence

$$T_1^\star(x) \;=\; a_+ x a_+ + bsxs^\star b^\star$$

Similarly,

$$\begin{aligned} T_2^\star(x) &= s^\star b x a_+ + a x s^\star b^\star \\ T_3^\star(x) &= a_+ x b^\star s + b s x a \\ T_4^\star(x) &= s^\star b x b^\star s + a x a \end{aligned}$$

Obviously, $\Omega := \mathbb{1}_d$ is cyclic for $\{a_1^\star, \dots, a_4^\star\}$ iff $\mathbb{1}_d$ is cyclic for $\{T_1^\star, \dots, T_4^\star\}$. We denote by $M_{\mathbb{1}} \subset M_d$ the generated cyclic subspace. Hence we have to show that $M_{\mathbb{1}} = M_d$. Asymptotic completeness of $J$ is equivalent to that by Proposition 3.3.

*Step 1*
In a first step we show that there is a polynomial $P$ such that $P(T_4^\star)(\mathbb{1}) = e_{11}$ and hence $e_{11} \in M_{\mathbb{1}}$.

Denote by $\mathcal{D}$ the commutative subalgebra of $M_d$ of all diagonal matrices. Then $T_4^\star(\mathcal{D}) \subset \mathcal{D}$, and as a map from $\mathcal{D}$ to $\mathcal{D}$ $T_4^\star$ is given by the $d \times d-$matrix

$$t_4^\star := \begin{pmatrix} 1 & |\beta_2|^2 & & \\ & \alpha_2^2 & \ddots & \\ & & \ddots & |\beta_d|^2 \\ & & & \alpha_d^2 \end{pmatrix}$$

When denoting by $\mathcal{H}_k$ the linear span of the canonical basis vectors $\{e_1, \dots, e_k\}$, $1 \leq k \leq d$, then

$$(t_4^\star - \alpha_k^2)\,\mathcal{H}_k \subset \mathcal{H}_{k-1} \quad \text{for } k \geq 2.$$

Hence

$$r := (t_4^\star - \alpha_2^2) \dots (t_4^\star - \alpha_d^2)$$

maps $\mathcal{H} = \mathcal{H}_d$ inro $\mathcal{H}_1 = \mathbb{C}\, e_1$. In particular, $r(\mathbb{1}) = \lambda\, e_{11}$ for some $\lambda$.
Since $t_4^\star$ is the adjoint of a Markov matrix, i.e. $t_4(\mathbb{1}) = \mathbb{1}$, we obtain

$$\langle\, \mathbb{1}, r(\mathbb{1}) \rangle = \langle\, r^\star(\mathbb{1}), \mathbb{1}\,\rangle = \langle\, (1-\alpha_d^2) \dots (1-\alpha_2^2)\mathbb{1}, \mathbb{1}\,\rangle \neq 0,$$

since by assumption $\alpha_i^2 \neq 1$ for $2 \leq i \leq d$. Therefore, $\lambda \neq 0$ and we may put

$$P(x) := \frac{1}{\lambda}(1-\alpha_2^2) \dots (1-\alpha_d^2).$$

*Step 2* From

$$T_2^\star(e_{1i}) \;=\; s^\star b\, e_{1i}\, a_+ \;+\; a\, e_{1i}\, s^\star b^\star \;=\; 0 \;+\; \overline{\beta}_{i+1} e_{1,i+1} \quad (1 \leq i \leq d-1)$$

we conclude $e_{1i} \in M_{\mathbb{1}}$, too, for $2 \leq i \leq d$, since $\overline{\beta}_{i+1} \neq 0$ for $1 \leq i \leq d-1$.

Similarly,

$$T_3^\star(e_{i1}) \;=\; a_+ e_{i1}\, b^\star s \;+\; b\, s\, e_{i1}\, a \;=\; 0 \;+\; \beta_{i+1} e_{i+1,1} \quad (1 \leq i \leq d-1)$$

implies $e_{i1} \in M_{\mathbb{1}}$ for $2 \leq i \leq d$.

Finally, for $1 \leq i, j \leq d-1$,

$$T_1^\star(e_{ij}) \;=\; a_+ e_{ij}\, a_+ \;+\; b\, s\, e_{ij} s^\star b^\star \;=\; \alpha_{i+1}\, \alpha_{j+1}\, e_{ij} \;+\; \beta_{i+1}\, \overline{\beta}_{j+1}\, e_{i+1,j+1},$$

hence all the other matrix units, too, are in $M_{\mathbb{1}}$, i.e., $M_{\mathbb{1}} = M_d$ which proves (b).

Conversely, if $\alpha_i = 1$, hence $\beta_i = 0$ for some $i$, $2 \le i \le d$, then $b\,s\,e_{i-1} = 0$, hence $b\,s\,\mathcal{H}_{i-1} \subset \mathcal{H}_{i-1}$. Since $b\,s\,\mathcal{H}_{i-1}^{\perp} \subset \mathcal{H}_{i-1}^{\perp}$, $bs$ commutes with the orthogonal projection $p_{i-1}$ onto $\mathcal{H}_{i-1}$. It follows easily that the operators $T_1^\star, T_2^\star, T_3^\star, T_4^\star$ leave the subspaces

$$p_{i-1}\, M_d\, p_{i-1},\ p_{i-1}^{\perp}\, M_d\, p_{i-1},\ p_{i-1}\, M_d\, p_{i-1}^{\perp},\ p_{i-1}^{\perp}\, M_d\, p_{i-1}^{\perp}$$

all invariant. In particular,

$$\mathbb{1} \in p_{i-1}\, M_d\, p_{i-1} \oplus p_{i-1}^{\perp}\, M_d\, p_{i-1}^{\perp}$$

is not cyclic. □

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## References


**[AFL]** L. Accardi, A. Frigerio, and J.T. Lewis. Quantum stochastic processes. *Publ. RIMS* **18** (1982), 97–133.

**[AMT]** J. Ashley, B. Marcus and S. Tuncel. The classification of one-sided Markov chains. *Ergodic Theory Dyn. Syst.* **17** (1997), 269–295.

**[EnWa]** M. Enomoto and Y. Watatani. A Perron-Frobenius Type Theorem for Positive Linear Maps on C*-Algebras. *Math. Japonica* **24** (1979), 53–63.

**[EHK]** D. Evans, R. Hoegh-Krohn. Spectral Properties of Positive Maps on C*-Algebras. *J. London Math. Soc.* **17** (1978), 345–355.

**[Go1]** R. Gohm. A Duality between Extension and Dilation. *In: Advances in quantum dynamics, G. Price et al. (Eds.), Contemp. Math.* **335** (2003), 139–147

**[Go2]** R. Gohm. Kümmerer-Maassen scattering theory and entanglement. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, vol.**7(2)**, World Scientific (2004), 271-280

**[Go3]** R. Gohm. Noncommutative Stationary Processes. Springer Lecture Notes in Mathematics **1839** (2004)

**[Gro]** U. Groh. Some observations on the spectra of positive operators on finite-dimensional C*-algebras. *Linear Algebra Appl.* **42** (1982), 213–222.

**[KaRi]** R.V. Kadison, J.R. Ringrose. *Fundamentals of the theory of operator algebras I,II.* Academic Press (1983)

**[Kit]** B.P. Kitchens. *Symbolic Dynamics : One-Sided, Two-Sided and Countable State Markov Shifts* . Springer-Verlag, 1998.

**[Kü1]** B. Kümmerer, Markov Dilations on W*-algebras. J.Funct. Anal. 63 (1985), 139-177

**[Kü2]** B. Kümmerer, Survey on a theory of non-commutative stationary Markov processes. *In: Quantum Prob. and Appl. III*, Springer Lecture Notes in Mathematics **1303** (1988), 154-182

**[Kü3]** B. Kümmerer, Quantum Markov Processes. In A. Buchleitner and K. Hornberger (Eds.): Coherent Evolution in Noisy Enviroments, Springer Lecture Notes in Physics **611** (2002), 139 - 198

**[Kü4]** B. Kümmerer, Quantum Markov Processes and Applications in Physics. In O.E. Barndorff-Nielsen, U. Franz, R. Gohm, B. Kümmerer, S. Thorbjørnsen, Quantum Independent Increment Processes II, Springer Lecture Notes in Mathematics **1866** (2005), 259 - 330,

**[KüMa]** B. Kümmerer and H. Maassen. A Scattering Theory for Markov Chains. *Infinite Dimensional Analysis, Quantum Probability and Related Topics* vol.**3** (2000), 161-176

**[KüNa]** B. Kümmerer and R. Nagel. Mean Ergodic Semigroups on W*-algebras. *Acta Sci. Math.* **41** (1979), 151–159.

**[Lan]** T. Lang. *Ein streutheoretischer Zugang zu Kodierungsproblemen von klassischen und Quanten-Markoff-Prozessen*. Dissertation, Stuttgart (2003)

**[LaPh]** P.D Lax, R.S. Phillips. Scattering theory. Academic Press (1967)

**[LiMa]** D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, 1995.

**[MeSa]** P. Meystre, M. Sargent. *Elements of Quantum Optics*. Springer-Verlag, 1991

**[Pet]** K. Petersen, Ergodic theory. Cambridge University Press (1989)

**[Sch]** K. Schmidt. Coding of Markov Shifts. *Operator algebras and their connections with topology and ergodic theory*, Lect. Notes Math. 1132, (1983), 497–508.

**[Sen]** E. Seneta. *Non-negative Matrices and Markov Chains*. Springer-Verlag, 1981.

**[Sak]** S. Sakai. *$C^*$-algebras and $W^*$-algebras*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60, Springer (1971)

**[Tak]** M. Takesaki. *Theory of Operator Algebras I*. Springer-Verlag, 1979.

**[WBKM]** T. Wellens, A. Buchleitner, B. Kümmerer, H. Maassen: *Quantum state preparation via asymptotic completeness*. Phys. Rev. Lett. **85** (2000), 3361-3364