A category of completely positive maps on $B(H)$

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Plan of the talk

- In the (rather technical) paper

  R.Gohm, Weak Markov Processes as Linear Systems, Mathematics of Control, Signals, and Systems (MCSS), 27, 375-413 (2015)

  we studied an abstract notion of processes with tools from multi-variable operator theory. We claimed that this is relevant for processes in quantum theory but didn't include much to substantiate this claim.
- In this talk we start by quoting and explaining briefly one of the results of this paper. We then study a very basic quantum process and see what it means in this case.
- I hope that motivates (me and others?!) to have another look at it ...

A category of processes

- A **process** is a tuple $(\mathcal{H}, V, \mathfrak{h})$ where $\mathcal{H}$ is a Hilbert space, $V = (V_1, \ldots, V_d) : \bigoplus_1^d \mathcal{H} \to \mathcal{H}$ is a row isometry and $\mathfrak{h}$ is a subspace of $\mathcal{H}$ which is co-invariant for V.
(Minimality assumption for $\mathcal{H}$ included.)
- We say that $(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g})$ is a **subprocess** of the process $(\mathcal{H}, V, \mathfrak{h})$ if $\mathfrak{g}$ is a closed subspace of $\mathfrak{h}$ which is co-invariant for V and $V^{\mathcal{G}} = V|_{\mathcal{G}}$ where $\mathcal{G} = \overline{span}\{V_\alpha \mathfrak{g} \colon \alpha \in F_d^+\}$. Note that $\mathfrak{g}$ is also co-invariant for $V^{\mathcal{G}}$ and $(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g})$ is a process in its own right.
- Given a subprocess $(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g})$ of a process $(\mathcal{H}, V, \mathfrak{h})$ we can form the **quotient process**

$$(\mathcal{H}, V, \mathfrak{h})/(\mathcal{G}, V^{\mathcal{G}}, \mathfrak{g}) := (\mathcal{K}, V^{\mathcal{K}}, \mathfrak{k})$$

where $\mathfrak{k} := \mathfrak{h} \ominus \mathfrak{g}$, $\mathcal{K} := \overline{span}\{V_\alpha \mathfrak{k} \colon \alpha \in F_d^+\}$, $V^{\mathcal{K}} := V|_{\mathcal{K}}$.

A category of processes

- It is convenient to reformulate these concepts within a **category of processes** which we define now. The objects of the category are the processes with a common multiplicity d. A **morphism** from $(\mathcal{R}, V^{\mathcal{R}}, \mathfrak{r})$ to $(\mathcal{S}, V^{\mathcal{S}}, \mathfrak{s})$ is a contraction $t : \mathfrak{r} \to \mathfrak{s}$ which intertwines the adjoints of the row isometries, i.e. $A_i^{\mathcal{S}}\, t = t\, A_i^{\mathcal{R}}$ for $i = 1, \ldots, d$, where

$$A_i^{\mathcal{S}} = V_i^{\mathcal{S}*}|_{\mathfrak{s}}, \quad A_i^{\mathcal{R}} = V_i^{\mathcal{R}*}|_{\mathfrak{r}}$$

- Composition of morphisms is given by composition of operators, the identity morphism is given by the identity operator.
- The tuples $(A_1^*, \ldots, A_d^*)$ are row contractions and the maps $\rho \mapsto \sum_{i=1}^{d} A_i\, \rho\, A_i^*$ are (contractive) completely positive maps. This is the connection to quantum processes and the way of looking at this category we want to focus on.

Extensions

- Given two processes, say a $\mathfrak{g}$-process and a $\mathfrak{k}$-process, it makes sense to ask in which way they can be put together to form a joint process, in such a way that the $\mathfrak{g}$-process is a subprocess and the $\mathfrak{k}$-process is a quotient process.
- **Theorem:** There is a one-to-one correspondence between such extensions (modulo the natural notion of equivalence) and contractions $\gamma : \epsilon_*^{\mathfrak{k}} \to \epsilon^{\mathfrak{g}}$, where $\epsilon_*^{\mathfrak{k}}$ and $\epsilon^{\mathfrak{g}}$ are the defect spaces of the two processes.
  The processes built in this way are called γ-extensions.
- $\gamma = 0$ is the direct sum. If there are nontrivial defects then there exist other possibilities.
- What does γ represent if we consider quantum processes, i.e., completely positive maps? We approach this question by an example.

Example

- Example: spontaneous emission (amplitude-damping channel)
- We want to describe a basic quantum mechanical process where a ground state $|1\rangle$ is stable forever and an excited state $|2\rangle$ falls back into the ground state with a probability p.
- In itself this is not a unitary process (as reversible quantum mechanics would require), it can be embedded into a reversible process, in a bigger universe. If we only have the information above then physicists would describe it by an open system dynamics, the so called amplitude-damping channel:

$$\rho \mapsto A_1 \, \rho \, A_1^* + A_2 \, \rho \, A_2^*$$

with density matrices ρ and

$$A_1 := \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}. \qquad A_2 := \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$

Example

- In fact

$$
\begin{aligned}
|1\rangle\langle 1| \mapsto A_1\,|1\rangle\langle 1|\,A_1^* + A_2\,|1\rangle\langle 1|\,A_2^* &= |1\rangle\langle 1|, \\
|2\rangle\langle 2| \mapsto A_1\,|2\rangle\langle 2|\,A_1^* + A_2\,|1\rangle\langle 2|\,A_2^* &= (1-p)\,|2\rangle\langle 2| + p\,|1\rangle\langle 1|,
\end{aligned}
$$

so the amplitude damping channel does what we want.

- In our notation we have a process specified by $\mathfrak{h} = \mathbb{C}^2$ together with the row contraction (A_1^*, A_2^*). We see immediately that with $\mathfrak{g} = \mathbb{C}|1\rangle$ we have

$$A_1\,\mathfrak{g} \subset \mathfrak{g},\ A_2\,\mathfrak{g} \subset \mathfrak{g},$$

so this gives a subprocess. The corresponding quotient process is generated by $\mathfrak{k} = \mathbb{C}|2\rangle$.

- This always happens if we have invariant states for a quantum process (here $|1\rangle$).

Example

- Reading the diagonal entries of A_1 and A_2 we find

$$A_1^{\mathfrak{g}} = 1,\ A_2^{\mathfrak{g}} = 0,\ A_1^{\mathfrak{k}} = \sqrt{1-p},\ A_2^{\mathfrak{k}} = 0.$$

  The two processes are easy to interpret: The $\mathfrak{g}$-process says that the $|1\rangle$-state is stable while the $\mathfrak{k}$-process says that the $|2\rangle$-state is only left unchanged with probability $1-p$.

- Suppose we only know the $\mathfrak{g}$- and $\mathfrak{k}$-processes. Our theory classifies how we can put them together. Let us compute the relevant defect operators:

$$D^{\mathfrak{g}} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . - \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \begin{pmatrix} 1 & 0 \end{pmatrix} .\right]^{\frac{1}{2}} \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$D_*^{\mathfrak{k}} = \left[1 - \begin{pmatrix} \sqrt{1-p} & 0 \end{pmatrix} . \begin{pmatrix} \sqrt{1-p} \\ 0 \end{pmatrix} .\right]^{\frac{1}{2}} \;=\; \sqrt{p}.$$

Example

- We find that the relevant defect spaces $\epsilon^{\mathfrak{g}}$ and $\epsilon_*^{\mathfrak{k}}$ are both one -dimensional and the classifying contraction $\gamma : \epsilon_*^{\mathfrak{k}} \to \epsilon^{\mathfrak{g}}$ is just a number in the complex unit disk: $\gamma \in \mathbb{C}, |\gamma| \leq 1$.
- What does it mean?
- There is a formula for the right upper corners of the matrices in the γ-extension which we can evaluate here:

$$(D_*^{\mathfrak{k}})^* \, \gamma^* D^{\mathfrak{g}} = \left(0, \ \sqrt{p} \, \bar{\gamma}\right)$$

For $\gamma = 1$ we recover the amplitude-damping channel we started from. If $|\gamma| = 1$ we have a channel without sinks (trace-preserving). If $|\gamma| < 1$ then it is possible to include further terms without destroying the property of being a row contraction.

Conclusion

- Interpretation: $|\gamma| < 1$ means that the probability of falling back into the state $|1\rangle$ is less than p and there may be something else happening: decay into another state $|3\rangle$ etc. This is clearly not excluded if we only know the $\mathfrak{g}$- and $\mathfrak{k}$-processes!
- **Conclusion:** The classification of extensions for processes is relevant for the study of quantum models. Further properties of the processes are encoded in the contraction γ (for example observability of the process by the subprocess corresponds to γ being injective).

  More complicated examples should be studied in this respect.

- **Thank you!**