Thoma's theorem about extremal characters as a noncommutative de Finetti theorem

Rolf Gohm
Institute of Mathematics and Physics
Aberystwyth University
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Joint work with C. Köstler

Gregynog
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Thoma's theorem

$\mathbb{S}_\infty$ is the inductive limit of the symmetric groups $\mathbb{S}_n$ as $n \to \infty$.
A function $\chi\colon \mathbb{S}_\infty \to \mathbb{C}$ is a **character** if it is constant on conjugacy classes, positive definite and normalized.

Theorem (Thoma 64, Kerov & Vershik 81, Okounkov 99)

An **extremal character** of the group $\mathbb{S}_\infty$ is of the form

$$\chi(\pi) = \prod_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} a_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} b_j^k \right)^{m_k(\pi)} .$$

Here $m_k(\pi)$ is the number of disjoint k-cycles in the permutation π and the two sequences $(a_i)_{i=1}^{\infty}, (b_j)_{j=1}^{\infty}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq 0, \qquad b_1 \geq b_2 \geq \cdots \geq 0, \qquad \sum_{i=1}^{\infty} a_i + \sum_{j=1}^{\infty} b_j \leq 1.$$

Note: **Thoma multiplicativity** for disjoint cycle decomposition.

Noncommutative random variables

From quantum mechanics we learn:

noncommutative **random variables**

$$=$$

selfadjoint operators + expectation functional ϕ (state)

(If they commute we are back in classical probability theory.)

A sequence $x_1, x_2, \dots$ of (noncommutative) random variables is

exchangeable

if joint moments are invariant under permutations of the variables:

$$\phi(x_{i_1} \dots x_{i_n}) = \phi(x_{\pi(i_1)} \dots x_{\pi(i_n)})$$

Claus Köstler's noncommutative de Finetti theorem

de Finetti: exchangeability $\Rightarrow$ conditional independence
Claus Köstler has shown (JFA 2010) that this idea also works for noncommutative random variables:

Theorem

Every exchangeable sequence x_1, x_2 of (noncommutative) random variables is **conditionally independent** over the **tail algebra**

$$\mathcal{A}^{\text{tail}} := \bigcap_n \text{vN}(x_n, x_{n+1}, \dots)\,.$$

- vN means the generated von Neumann algebra (weak closure in GNS-rep with respect to the state).
- Conditional independence: factorization of the state-preserving conditional expectation E onto the tail algebra $\mathcal{A}^{\text{tail}}$:

$$E(xx') = E(x)E(x') \text{ if } x \in vN(x_i, i \in I), x' \in vN(x_j, j \in J),\ I \cap J = \emptyset$$

Characters and traces

Every character χ of $\mathbb{S}_\infty$ gives rise to a **unitary representation**

$$\pi\colon \mathbb{S}_\infty \to \mathcal{U}(\mathcal{A}), \qquad \text{with } \mathcal{A} = \mathrm{vN}(\pi(\mathbb{S}_\infty))$$

such that there is a tracial state tr on $\mathcal{A}$.

The converse is also true.

Notation: omit π from now on $\quad \Rightarrow \mathrm{tr} = \chi$ on $\mathbb{S}_\infty$

The character is extremal iff $\mathcal{A}$ is a factor.

Reformulation of Thoma's problem:
Characterize **factorial tracial states** on the group algebra of $\mathbb{S}_\infty$.

Noncommutative probability approach

Find a natural exchangeable sequence of noncommutative random variables (=selfadjoint operators) which generate $\mathcal{A} = \mathrm{vN}(\mathbb{S}_\infty)$

We think of $\mathbb{S}_\infty$ as acting on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$
by permutations.

Observation:
Transpositions in $\mathbb{S}_\infty$ go via π into idempotent unitaries.
$\Rightarrow$ selfadjoint with spectrum $\{1, -1\}$, coin tosses.

A generating sequence must be noncommutative.
Classical probability is not enough.

First try: the **Coxeter generators** $\sigma_i = (i-1, i) \ (i \in \mathbb{N})$.
Not exchangeable (in general).

Star generators are exchangeable

Consider the **star generators** $\gamma_i = (0, i) \ \ (i \in \mathbb{N})$.

They are exchangeable:

$$\begin{aligned} & \chi(\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_n}) \\ = \ & \chi(\pi\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_n}\pi^{-1}) \\ = \ & \chi\big((\pi\gamma_{i_1}\pi^{-1})(\pi\gamma_{i_2}\pi^{-1})\ldots(\pi\gamma_{i_n}\pi^{-1})\big) \\ = \ & \chi\big(\gamma_{\pi(i_1)}\gamma_{\pi(i_2)}\ldots\gamma_{\pi(i_n)}\big) \end{aligned}$$

where π is any permutation with $\pi(0) = 0$.

So this is a nice example of a

noncommutative exchangeable sequence of coin tosses.

Star generators and cycles

Another useful elementary fact:

A k-**cycle** $\tau = (n_1\, n_2\, n_3 \ldots n_k) \in \mathbb{S}_\infty$ can be written in the star generators γ_i as

$$\tau = \gamma_{n_1}\gamma_{n_2}\gamma_{n_3}\cdots\gamma_{n_{k-1}}\gamma_{n_k}\gamma_{n_1},$$

where in the case $\tau(0) \neq 0$ we put $n_1 = 0$ and define γ_0 to be the unit element.

$\Rightarrow$

disjoint cycles are supported by disjoint sets of star generators

And now we get

from the noncommutative de Finetti theorem:

$$E(\tau\,\tau') = E(\tau)E(\tau')$$

τ and τ' are disjoint cycles

E is the conditional expectation onto the tail algebra of the sequence $\gamma_1, \gamma_2 \ldots$ of star generators.

Evaluation of the conditional expectation

What is $E(\tau)$ for a k-cycle τ ?

In general the tail algebra $\mathcal{A}^{\text{tail}}$ is nontrivial.

We need to introduce another key player:

$$A_0 := E(\gamma_1) \in \mathcal{A}^{\text{tail}}$$

which is a selfadjoint contraction.

Let us restrict to the case of an extremal character (factor trace) from now on.

Key observation: Let τ be a k-cycle. Then

$$E(\tau) = \begin{cases} A_0^{k-1} & \text{if } \tau(0) \neq 0 \\ \operatorname{tr}(A_0^{k-1})\,\mathbb{1} & \text{if } \tau(0) = 0 \end{cases}$$

Thoma multiplicativity

For $\pi \in \mathbb{S}_\infty$ let

$$\pi = \tau_1 \dots \tau_n$$

be the disjoint cycle decomposition.

At most one of the cycles, say τ_1, moves the point 0.

$$\begin{aligned} \Rightarrow \quad \chi(\pi) &= \chi(\tau_1 \dots \tau_n) \\ &= \mathrm{tr}(E(\tau_1 \dots \tau_n)) \\ &= \mathrm{tr}(E(\tau_1)E(\tau_2)\dots E(\tau_n)) \\ &= \mathrm{tr}(A_0^{k_1-1}\,\mathrm{tr}(A_0^{k_2-1})\dots \mathrm{tr}(A_0^{k_n-1})) \\ &= \mathrm{tr}(A_0^{k_1-1})\,\mathrm{tr}(A_0^{k_2-1})\dots \mathrm{tr}(A_0^{k_n-1}) \\ &= \chi(\tau_1)\chi(\tau_2)\dots\chi(\tau_n) \end{aligned}$$

which means we have proved **Thoma multiplicativity**

as a consequence of the noncommutative de Finetti theorem.

Spectral Theory

It also follows that the tail algebra for the sequence $\gamma_1, \gamma_2, \ldots$ is generated by the selfadjoint contraction A_0:

$$\mathcal{A}^{\text{tail}} = \text{vN}(A_0)$$

So $\mathcal{A}^{\text{tail}}$ is commutative. For a k-cycle τ

$$\text{tr}(\tau) = \text{tr}(A_0^{k-1}) = \int_{-1}^{1} t^{k-1}\, d\mu \,,$$

where μ is the spectral measure of A_0.

Spectral analysis completes the proof of Thoma's theorem.

It turns out that the spectrum is **discrete** and the (positive) Thoma parameters provide positive eigenvalues $a_i > 0$ and negative eigenvalues $-b_i < 0$.

Noncommutative independence also helps to prove that.

But that is another talk!

References

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