Semi-Cosimplicial Objects and Spreadability

Rolf Gohm
Institute of Mathematics, Physics and Computer Science
Aberystwyth University
.
Joint work with D.G. Evans and C. Köstler

George Boole Mathematical Sciences Conference
Cork, August 17-28, 2015

Introduction

Main observation:

Spreadability
(a distributional symmetry considered by probabilists)

is more or less the same thing as

a **semi-cosimplicial object (SCO)**
(as considered in homological algebra)
in the category of (noncommutative) probability spaces.

This shows that spreadability has a homological flavour, unlike other distributional symmetries like exchangeability which come from symmetries given by group actions.

Noncommutative Probability Theory I

Basics and terminology of noncommutative probability theory.

A **noncommutative probability space** $(\mathcal{A}, \varphi)$ is a unital (associative complex) algebra $\mathcal{A}$ together with a unital linear functional φ on $\mathcal{A}$.

If we have $*$-algebras and states (normalized positive linear functionals) then we call it a **nc $*$-probability space**.

With commutative $*$-algebras we include classical probability.

Let's give the following arguments without the $*$.

Obvious modifications in the $*$-case.

Noncommutative Probability Theory II

We can think of nc probability spaces as a **category** with morphisms from $(\mathcal{B}, \psi)$ to $(\mathcal{A}, \varphi)$ given by unital algebra homomorphisms $\iota\colon \mathcal{B} \to \mathcal{A}$ such that $\varphi \circ \iota = \psi$.
We write $\iota : (\mathcal{B}, \psi) \to (\mathcal{A}, \varphi)$.

If $\mathcal{B}$ is a unital algebra then we call any unital algebra homomorphism $\iota : \mathcal{B} \to \mathcal{A}$, where $(\mathcal{A}, \varphi)$ is a nc probability space, a **nc random variable**.

Defining $\psi := \varphi \circ \iota$ we get a nc probability space $(\mathcal{B}, \psi)$ and $\iota : (\mathcal{B}, \psi) \to (\mathcal{A}, \varphi)$ is a morphism.

Spreadability - Definition.

A sequence $(\iota_n)_{n\in\mathbb{N}_0}$ of random variables is called **spreadable** if its replacement by a subsequence does not change the joint distributions (joint moments).

Explicitly:

A sequence $(\iota_n)_{n\in\mathbb{N}_0}$ of (nc) random variables is called **spreadable** if

$$\varphi(\iota_{n_1}(b_1)\ldots\iota_{n_k}(b_k)) = \varphi(\iota_{i(n_1)}(b_1)\ldots\iota_{i(n_k)}(b_k))$$

for any strictly increasing function $i : \mathbb{N}_0 \to \mathbb{N}_0$ and all $n_\ell \in \mathbb{N}_0,\ b_\ell \in \mathcal{B},\ k \in \mathbb{N}$.

$*$-**spreadable** in the category of $*$-probability spaces

Spreadability - Background and Applications I

Classical probability: It was recognized that spreadability allows a very transparent proof of de Finetti's theorem, via the mean ergodic theorem.

For sequences of classical random variables spreadability is equivalent to **exchangeability** (invariance of the distribution under permutations of the random variables).
So this is really the classical de Finetti theorem (which de Finetti had in mind).

de Finetti theorem: Exchangeable sequences are mixed i.i.d. (i.e., the distributions are convex combinations of product measures).

Spreadability - Background and Applications II

Also a lot of work on noncommutative de Finetti theorems.

In the spirit of the mean ergodic ideas mentioned above a very general **nc de Finetti theorem** (von Neumann algebras, Heisenberg picture!) was proved by Claus Köstler (JFA 2010). It works with a generalized notion of nc independence of random variables based on nc conditional expectations.

De Finetti had **philosophical motivations** for his theorem wrt the interpretation of probability.
(Not everybody is interested in that.)

I learned from Claus' result that in the nc domain the de Finetti theorem also does something else:

It is a **source for nc independence** (i.e., factorization properties) derived from certain symmetry properties.

This is highly significant!

Some Homological Algebra: Semi-Cosimplicial Objects I

Consider the category $\mathbf{\Delta}_S$ with

objects: finite ordered sets $[n] := \{0, \ldots, n\}, \quad n \in \mathbb{N}_0$,

morphisms: strictly increasing maps.

Generated by the **face maps**: $\quad (k = 0, \ldots, n$ and $n \in \mathbb{N})$

$$\delta^k \colon [n-1] \to [n], \quad m \mapsto m \text{ if } m < k, \; m \mapsto m+1 \text{ if } m \geq k.$$

They satisfy the **cosimplicial identities**:

$$\delta^j \delta^i = \delta^i \delta^{j-1} \quad \text{if } i < j$$

Δ_S is called the **semi-simplicial category**.

It is a subcategory of the simplicial category Δ
(which has all non-decreasing maps as morphisms).
$\to$ topology

Some Homological Algebra: Semi-Cosimplicial Objects II

A (covariant) **functor** from the category Δ_S into another category $\mathcal{C}$ is called a **semi-cosimplicial object (SCO)** in the category $\mathcal{C}$.

Or more explicitly:

A **semi-cosimplicial object (SCO)** in the category $\mathcal{C}$ is a **sequence** $(F^n)_{n\in\mathbb{N}_0}$ **of objects** in $\mathcal{C}$
together with morphisms (**coface operators**)

$$\delta^k : F^{n-1} \to F^n \qquad (k = 0, \ldots, n)$$

satisfying the **cosimplicial identities**

$$\delta^j \delta^i = \delta^i \delta^{j-1} \quad \text{if } i < j\,.$$

SCOs and Spreadability I

Given an **SCO in the category of nc probability spaces**, i.e.,
a sequence $(\mathcal{A}_n, \varphi_n)_{n\in\mathbb{N}_0}$ of nc probabilty spaces,
morphisms δ^k

$$\delta^k : (\mathcal{A}^{n-1}, \varphi_{n-1}) \to (\mathcal{A}^n, \varphi_n) \qquad (k = 0, \ldots, n)$$

satisfying the cosimplicial identities $\delta^j \delta^i = \delta^i \delta^{j-1} \quad \text{if } i < j$.
Form the **inductive limit** $(\mathcal{A}_\infty, \varphi_\infty)$ from

$$(\mathcal{A}^0, \varphi_0) \xrightarrow{\delta^1} (\mathcal{A}^1, \varphi_1) \xrightarrow{\delta^2} \ldots \xrightarrow{\delta^{n-1}} (\mathcal{A}^{n-1}, \varphi_{n-1}) \xrightarrow{\delta^n} \ldots$$

Let μ_n be the embedding of $\mathcal{A}_n$ into $\mathcal{A}_\infty$ and

$$\iota_0 := \mu_0, \quad \iota_n := \mu_n \circ \overbrace{\delta^0 \circ \ldots \circ \delta^0}^{n \text{ times}} : \quad \mathcal{A}_0 \to \mathcal{A}_\infty \,.$$

So the ι_n are **nc random variables (arising from the SCO)**.

SCOs and Spreadability II

Theorem

$(\iota_n)_{n\in\mathbb{N}_0}$ (as above) is a **spreadable** sequence of nc random variables.

Conversely: For any spreadable sequence we can always produce a stochastically equivalent version (i.e., with the same moments) which arises in such a way from an SCO.

Modifications:
The same holds in the category of $*$-probability spaces.
(Or even with non-unital algebras ...
The use of category theory helps here in many respects.)

There is a way to produce a system of endomorphisms (which we call "**partial shifts**") on the inductive limit which can be used for proving the de Finetti theorem by mean ergodic techniques.

Sketch of Proof I

We start with $\iota_n\colon \mathcal{A}_0 \to \mathcal{A}_\infty$ constructed as above:

$$\iota_0 := \mu_0, \quad \iota_n := \mu_n \circ \overbrace{\delta^0 \circ \ldots \circ \delta^0}^{n \text{ times}} \colon \quad \mathcal{A}_0 \to \mathcal{A}_\infty \,.$$

For $k > n$ we identify δ^k on $\mathcal{A}_{n-1}$ with δ^n
and think of it as the embedding of $\mathcal{A}_{n-1}$ into $\mathcal{A}_n$
(in the inductive limit, suppressing all μ_n).

In this sense, on $\mathcal{A}_0$, from the cosimplicial identities:

$$\delta^k \circ \overbrace{\delta^0 \circ \ldots \circ \delta^0}^{N \text{ times}} = \begin{cases} \overbrace{\delta^0 \circ \ldots \circ \delta^0}^{N \text{ times}} & \text{if } N < k \\ \\ \overbrace{\delta^0 \circ \ldots \circ \delta^0}^{N+1 \text{ times}} & \text{if } N \geq k. \end{cases}$$

So the k-th power does not appear on the right hand side.

Sketch of Proof II

Embedded in the inductive limit $\mathcal{A}_\infty$, we can think of δ^k (with k fixed) on all $\mathcal{A}_n$ simultaneously, as an endomorphism α_k of $\mathcal{A}_\infty$.

This is what we call a partial shift.

Applying α_k replaces the original sequence $(\iota_n)_n$ by the specific subsequence which just omits the k-th position.

Iterating this procedure yields any subsequence.

Distribution is preserved because the δ^k preserve it (by assumption: SCO in the category of nc probability spaces).

So $(\iota_n)_n$ is spreadable.

Sketch of Proof III

Conversely, given a spreadable sequence $(\iota_n)_n$ with $\iota_n : (\mathcal{B}, \psi) \to (\mathcal{A}, \varphi)$.

Form a free product $*_{n\in\mathbb{N}_0}\mathcal{B} =: \mathcal{A}^f$.

λ_n embedding $\mathcal{B}$ into the n-th position

$\pi : \mathcal{A}^f \to \mathcal{A},\ \pi \circ \lambda_n = \iota_n$ (universal property)

$\varphi^f := \varphi \circ \pi$ on $\mathcal{A}^f$

The sequence $(\lambda_n)_n$ has the same distribution as a sequence of random variables into $(\mathcal{A}^f, \varphi^f)$ as the original sequence $(\iota_n)_n$ into $(\mathcal{A}, \varphi)$.

On the free product we can easily realize the partial shift idea and, going in the other direction, produce an SCO.

Remark: Other filtrations often do it too, the free product always works.

Tensor Product Example

Let $\mathcal{B}$ be a unital algebra. Then we can form an SCO (for algebras) $(F^n)_{n\in\mathbb{N}_0}$ with tensor products $F^n := \bigotimes_0^n \mathcal{B}$ together with coface operators

$$\delta^k : \bigotimes_0^{n-1} \mathcal{B} \to \bigotimes_0^{n} \mathcal{B},\ x_0 \otimes \ldots \otimes x_{n-1} \mapsto x_0 \otimes \ldots \otimes x_{k-1} \otimes \mathbb{1} \otimes x_k \otimes \ldots \otimes x_{n-1}$$

If we choose unital linear functionals (or states in the $*$-case) invariant under these operations then we have an SCO in the category of $(*)$-probability spaces.

Products $\bigotimes \psi$ on $\bigotimes \mathcal{B}$ (for any ψ on $\mathcal{B}$) do it.

Convex combinations of such products do it.

Our construction applied here yields the basic example of an exchangeable sequence of random variables:

the embeddings of $\mathcal{B}$ into the different positions of the tensor product (with the functionals above).

How to find other SCOs ?

How to find other SCOs for nc probability spaces?

Recall that for classical sequences of random variables spreadability and exchangeability is the same thing.

In joint work with Claus (CMP09) we found that in the nc setting spreadability is much more general than exchangeability.

We did this by introducing "**braidability**", a way to replace the symmetric group (coming into exchangeability by permuting the variables) by the **braid group**.

This becomes more transparent with the SCO-approach.

Artin's Braid Groups

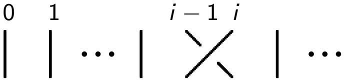
Definition (Artin 1925)

The **braid group** $\mathbb{B}_n$ is presented by $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \qquad \text{if } \mid i - j \mid = 1$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } \mid i - j \mid > 1$$

$\mathbb{B}_1 \subset \mathbb{B}_2 \subset \mathbb{B}_3 \subset \ldots \subset \mathbb{B}_\infty$ (inductive limit)
In fact we don't need inverses of **Artin's generators** and work with the **braid monoid** $\mathbb{B}_\infty^+$ generated by the σ_i.

Braid diagram for σ_i



SCOs from Braids

Suppose that $\mathbb{B}_\infty^+$ **acts on a set** X, we simply write $gx \in X$ for the result of $g \in \mathbb{B}_\infty^+$ acting on $x \in X$.

We define for $n \in \mathbb{N}_0$ the **fixed point spaces**

$$X^n := \{x \in X \colon \sigma_k\, x = x \ \text{if} \ k \geq n+2\}$$

increasing sequence $X^0 \subset X^1 \subset \ldots$ of subsets of the set X.

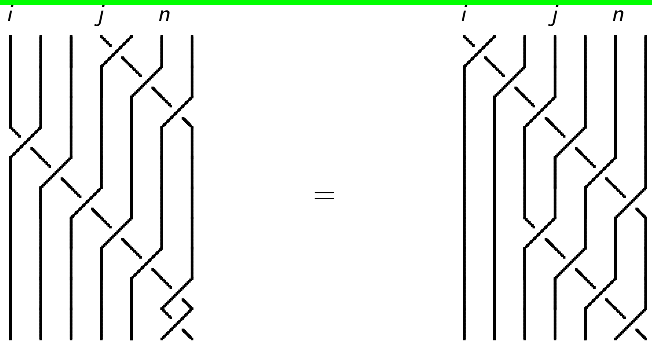
Theorem

$(X^n)_{n \in \mathbb{N}_0}$ is a semi-cosimplicial set (a SCO in the category of sets), with the **coface operators** δ^k **given by**

$$\begin{array}{rccc} \delta^k \colon & X^{n-1} & \to & X^n \qquad (k = 0, \ldots, n, \quad n \in \mathbb{N}_0) \\ & x & \mapsto & \sigma_{k+1} \ldots \sigma_{n+1}\, x \,. \end{array}$$

Of course, we can use this for other categories described by sets and mappings with additional structure.

Proof



The figure shows that

$(\sigma_{j+1} \ldots \sigma_{n+1})(\sigma_{i+1} \ldots \sigma_{n+1})\sigma_{n+1} = (\sigma_{i+1} \ldots \sigma_{n+1})(\sigma_j \ldots \sigma_{n+1})$

(for $0 \leq i < j \leq n$) in $\mathbb{B}_\infty^+$.
Together with

$$\sigma_{n+1}\, x = x \quad \text{for} \quad x \in X^{n-1}$$

the cosimplicial identities follow.

Tensor Product Example Revisited

Instructive to note that our tensor product example is also of this type:

Here we have **braid group representations which factor through representations of symmetric groups** (permuting the tensor products in this case).

This is typical for **exchangeability**.

On the other hand this shows a way to produce examples of sequences which are **spreadable but not exchangeable**:

just work with braid group representations which do not factor in this way!

An interesting Example from von Neumann Algebras I

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of finite factors with **finite Jones index** $\beta = [\mathcal{M} : \mathcal{N}]$.

With $\mathcal{M}_{-1} := \mathcal{N}$, $\mathcal{M}_0 := \mathcal{M}$, **Jones' basic construction** yields a tower

$$\mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \dots$$

We have the **Temperley-Lieb projections** $e_n \in \mathcal{M}_n$ and

$$e_n e_{n\pm 1} e_n = \beta^{-1} e_n, \quad e_n e_m = e_m e_n \text{ if } |n-m| \geq 2$$

(for all n, m). Then with $\beta = 2 + q + q^{-1}$ and defining

$$g_n := q e_n - (\mathbb{1} - e_n)$$

the g_n satisfy the braid relations.

An interesting Example from von Neumann Algebras II

With this representation $\sigma \mapsto g$,
define an **action** of $\mathbb{B}_\infty$ on the inductive limit $\mathcal{M}_\infty$ by
$\sigma x := gxg^{-1}$ for $\sigma \in \mathbb{B}_\infty$ and $x \in \mathcal{M}_\infty$.

The **trace** is invariant under this action.

$\mathcal{M} = \mathcal{M}_0$ **commutes** with e_n and hence with g_n for $n \geq 2$
(hence providing a 0-object in an SCO).

Conclusion from our Theorems (both!)
(after some additional computation):

The sequence $(\iota_n)_{n \in \mathbb{N}_0}$ of nc random variables $\iota_n \colon \mathcal{M} \to \mathcal{M}_\infty$
given by $\iota_0 :=$ id and for $n \geq 1$

$$\iota_n := Ad(g_n \dots g_1)$$

is **spreadable**.

An interesting Example from von Neumann Algebras III

Interesting subtlety:

For $\beta \leq 4$ (**small index**) this braid group representation is unitary and hence, with the conjugation action, we are in the category of $*$-probability spaces and we get $*$-**spreadability**.

The nc de Finetti theorem applies and yields for example some (known) properties of the trace ("Markov trace").

For $\beta > 4$ (**big index**) this braid group representation is not unitary and we only get **spreadability** (**without** "$*$").
No de Finetti available (as far as I know).

From there?

We have seen that SCOs provide a clear explanation of probabilists' spreadability to an algebraist.

Are more advanced tools from homological algebra useful for the study of spreadability?

For example the (simplicial) cohomology theory which is associated to SCOs ?

What does the latter mean in the case of the braid construction?

Via the braid construction we can obtain
a wealth of examples for spreadability:
Just use your favourite braid group representation.

SCOs in nc probability spaces beyond the braid construction?