Stochastic Burgers equation with vorticity

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Outline

"Bog standard" Burgers equation

Viscous Burgers equation: $v^\mu(x,t)$,

$$\partial_t v^\mu + (v^\mu \cdot \nabla)v^\mu = \frac{\mu^2}{2}\Delta v^\mu, \qquad v^\mu(x,0) = \nabla S_0(x) + O(\mu^2).$$

Hamilton-Jacobi equation: $\nabla \mathcal{S}(x,t) = v^\mu(x,t)$,

$$\partial_t \mathcal{S}^\mu + \frac{1}{2}|\nabla \mathcal{S}^\mu|^2 = \frac{\mu^2}{2}\Delta \mathcal{S}^\mu, \qquad \mathcal{S}^\mu(x,0) = S_0(x) + O(\mu^2).$$

Heat equation: $\mathcal{S}^\mu(x,t) = -\mu^2 \ln u^\mu(x,t)$,

$$\partial_t u^\mu = \frac{\mu^2}{2}\Delta u^\mu, \qquad u^\mu(x,0) = T_0(x) \exp\left(-\mu^{-2} S_0(x)\right).$$

Stochastic Hamilton-Jacobi Equation

Hamiltonian Kernel

$a(q,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$, $\quad k(q,t), V(q,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$.

$\nabla \cdot a(q,t) = 0$,

$$H(p,q,\circ\partial t) = \frac{1}{2}|p - a(q,t)|^2 \, \mathrm{d}t + V(q,t) \, \mathrm{d}t + k(q,t) \circ \partial W_t.$$

Hamilton-Jacobi equation:

$$\mathrm{d}\mathcal{S}_t^\mu(x) + H(\nabla \mathcal{S}_t^\mu(x), x, \circ\partial t) = \frac{\mu^2}{2} \Delta \mathcal{S}_t^\mu(x) \, \mathrm{d}t,$$

with initial condition,

$$\mathcal{S}_0^\mu(x) = S_0(x) - \mu^2 \ln T_0(x).$$

Stochastic Burgers equation

Hopf-Cole transformation 1

$$v^{\mu}(x,t) = \nabla \mathcal{S}_t^{\mu}(x) - a(x,t).$$

$$\begin{aligned} \mathrm{d}v^{\mu} + \mathrm{d}a(x,t) + (v^{\mu}\cdot\nabla)v^{\mu}\,\mathrm{d}t + v^{\mu}\wedge(\nabla\wedge v^{\mu})\,\mathrm{d}t \\ = \frac{\mu^2}{2}\Delta(v^{\mu} + a(x,t))\,\mathrm{d}t - \nabla V(x,t)\,\mathrm{d}t - \nabla k(x,t)\,\mathrm{d}W_t, \end{aligned}$$

with initial condition,

$$v^{\mu}(x,0) = \nabla S_0(x) - \mu^2\nabla \ln T_0(x) - a(x,0).$$

Stratonovich heat equation

Hopf-Cole transformation 2

$$\mathcal{S}_t^{\mu}(x) = -\mu^2 \ln u^{\mu}(x,t).$$

$$\mu^2 \, \mathrm{d}u^{\mu} = \left(\frac{\mu^4}{2}\Delta + \mu^2 a(x,t)\cdot\nabla + \frac{1}{2}a(x,t)^2 + V(x,t) \right) u^{\mu}\, \mathrm{d}t$$
$$+ k(x,t)u^{\mu} \circ \partial W_t,$$

with the initial condition,

$$u^{\mu}(x,0) = T_0(x) \exp\left(-\mu^{-2} S_0(x) \right).$$

Feynman Kac formula

Theorem

$$u^\mu(x,t) = \mathbb{E}_X \Bigg\{ T_0(X_t^{x,t}) \times \exp\Bigg(-\frac{1}{\mu^2} S_0(X_t^{x,t}) + \frac{1}{\mu^2}\int_0^t k(X_s^{x,t}, t-s)\,\mathrm{d}W_{t-s} + \frac{1}{\mu^2}\int_0^t \left(\frac{1}{2}|a(X_s^{x,t}, t-s)|^2 + V(X_s^{x,t}) \right) \mathrm{d}s \Bigg) \Bigg\},$$

where $X_0^{x,t} = x$ *and,*

$$\mathrm{d}X_s^{x,t} = a(X_s^{x,t}, T_s^t)\,\mathrm{d}s + \mu\,\mathrm{d}B_s, \quad T_s^t = t-s, \quad s \in [0,t].$$

Stochastic Flow map

Stochastic flow map $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$,

$$\begin{aligned} \mathrm{d}\dot{\Phi}_s &= -\nabla V(\Phi_s)\,\mathrm{d}s - \nabla k(\Phi_s, s)\,\mathrm{d}W_s - \frac{\partial a}{\partial s}(\Phi_s, s)\,\mathrm{d}s \\ &\qquad - (\nabla \wedge a(\Phi_s, s)) \wedge \mathrm{d}\Phi_s, \\ \mathrm{d}\Phi_s &= \dot{\Phi}_s\,\mathrm{d}s, \end{aligned}$$

with initial conditions,

$$\Phi_0(x_0) = x_0, \qquad \dot{\Phi}_0(x_0) = \nabla S_0(x_0) - a(x_0, 0).$$

Diffeormorphism

$\Phi_t(x_0)$ random diffeomorphism up to caustic time $T(\omega)$.

Stochastic Action

Before caustic time $t < T(\omega)$:

$$\tilde{S}(x_0, t) = S_0(x_0) + \frac{1}{2}\int_0^t |\dot{\Phi}_s(x_0)|^2 \,\mathrm{d}s + \int_0^t \dot{\Phi}_s(x_0) \cdot a(\Phi_s(x_0), s) \,\mathrm{d}s - \int_0^t V(\Phi_s(x_0)) \,\mathrm{d}s - \int_0^t k(\Phi_s(x_0), s) \,\mathrm{d}W_s.$$

H-J function

Define,

$$\mathcal{S}(x, t) = \tilde{S}(\Phi_t^{-1}(x), t)$$

i.e.

$$\mathcal{S}(x, t) = \tilde{S}(x_0, t), \quad \text{where } x_0 = \Phi_t^{-1}(x) \Leftrightarrow \Phi_t(x_0) = x$$

Stochastic H-J

Theorem

Assume Φ_t *satisfies no-caustic condition for* $0 \leq t \leq T(\omega)$.

1. $$\dot{\Phi}_t(x_0) = \nabla \mathcal{S}(\Phi_t(x_0), t) - a(\Phi_t(x_0), t) \quad a.e.\ \omega \in \Omega.$$
2. $\mathcal{S}$ *satisfies H.J. equation,*
$$\mathrm{d}\mathcal{S}(x,t) + \frac{1}{2}|\nabla \mathcal{S}(x,t) - a(x,t)|^2 \,\mathrm{d}t + V(x)\,\mathrm{d}t + k(x)\,\mathrm{d}W_t = 0.$$
3. *Define* $\rho(x,t) = \left|\det\left(D_x \Phi_t^{-1}(x)\right)\right|$ *then,*
$$\mathrm{d}\rho(x,t) + \nabla \cdot \{\rho(x,t)(\nabla \mathcal{S}(x,t) - a(x,t))\}\,\mathrm{d}t = 0.$$

Asymptotic expansions

Asymptotic series solution for viscous HJ equation,

$$\frac{\mu^2}{2}\Delta\mathcal{S}^\mu(x,t)\,\mathrm{d}t$$
$$= \mathrm{d}\mathcal{S}^\mu(x,t) + \frac{1}{2}|\nabla\mathcal{S}^\mu(x,t) - a(x,t)|^2\,\mathrm{d}t + V(x)\,\mathrm{d}t + k(x,t)\,\mathrm{d}W_t.$$

Suppose,

$$\mathcal{S}^\mu(x,t) \sim \sum_{j=0}^{\infty}\mu^{2j}\mathcal{S}_j(x,t).$$

Formally equating coefficients of μ^2,

$$\mathrm{d}\mathcal{S}_j + \frac{1}{2}\sum_{\substack{i_1,i_2\geq 0,\\ i_1+i_2=j}}\nabla\mathcal{S}_{i_1}\cdot\nabla\mathcal{S}_{i_2} - a\cdot\nabla\mathcal{S}_j = \frac{1}{2}\Delta\mathcal{S}_{j-1}.$$

Iterated continuity equations

$$T_0(y,t) = T_0(y),$$

$$T_j(y,t) = \int_0^t \frac{1}{\sqrt{\rho(\tilde{y},s)}} \Delta_{\tilde{y}} \left(T_{j-1}(\Phi_s^{-1}(\tilde{y}), s)\sqrt{\rho(\tilde{y},s)} \right)\Bigg|_{\tilde{y}=\Phi_s(y)} \mathrm{d}s,$$

and set,

$$\psi_j(x,t) = T_j(\Phi_t^{-1}(x), t)\sqrt{\rho(x,t)}.$$

Lemma (Iterated continuity equations)

For a.e. $\omega \in \Omega$, *and* $0 \le t \le T(\omega)$,

$$\mathrm{d}\psi_j + \nabla\psi_j \cdot (\nabla\mathcal{S} - a)\,\mathrm{d}t = -\frac{1}{2}\psi_j \Delta\mathcal{S}\,\mathrm{d}t + \Delta\psi_{j-1}\,\mathrm{d}t,$$

for $j = 0, 1, 2, \ldots$ *with the convention* $\psi_{-1} = 0$.

Iterated H-J equations

Theorem

For $0 \leq t \leq T(\omega)$ *then for a.e.* $\omega \in \Omega$*, the solutions of,*

$$\mathrm{d}\mathcal{S}_j + \frac{1}{2} \sum_{i_1, i_2 \geq 0,\, i_1 + i_2 = j} \nabla \mathcal{S}_{i_1} \cdot \nabla \mathcal{S}_{i_2} \, \mathrm{d}t - a \cdot \nabla \mathcal{S}_j \, \mathrm{d}t = \frac{1}{2} \Delta \mathcal{S}_{j-1} \, \mathrm{d}t,$$

are given by,

$$\mathcal{S}_0(x,t) = \mathcal{S}(x,t), \qquad \mathcal{S}_1(x,t) = -\ln \psi_0(x,t),$$

$$\mathcal{S}_j(x,t) = \frac{1}{2^{j-1}} \left(-\frac{\psi_{j-1}}{\psi_0} + \frac{1}{2\psi_0^2} \sum_{\substack{i_1, i_2 \geq 1, \\ i_1 + i_2 = j-1}} \psi_{i_1} \psi_{i_2} + \right.$$

$$\left. + \ldots + (-1)^{j-1} \frac{1}{(j-1)\psi_0^{j-1}} \psi_1^{j-1} \right).$$

Sketch of generalisation

For a general Hamiltonian Kernel $H(p, q, \circ\partial t)$ consider,

$$\mathrm{d}\mathcal{S}(x,t) + H(\nabla\mathcal{S}(x,t), x, \circ\partial t) = 0.$$

Underlying mechanical system,

$$\begin{aligned}
\partial q_t(x_0) &= \nabla_p H(p_t(x_0), q_t(x_0), \circ\partial t), \\
\partial\tilde{S}(x_0, t) &= p_t \circ \partial q_t(x_0) - H(p_t(x_0), q_t(x_0), \circ\partial t), \\
\partial p_t(x_0) &= -\nabla_q H(p_t(x_0), q_t(x_0), \circ\partial t),
\end{aligned}$$

with initial conditions,

$$q_0(x_0) = x_0, \qquad \tilde{S}(x_0, 0) = S_0(x_0), \qquad p_0(x_0) = \nabla S_0(x_0).$$

Then q is a diffeomorphism up to caustic time.

Sketch of Generalisation

Define

$$\mathcal{S}(x,t) = \tilde{S}(q_t^{-1}(x), t).$$

Theorem

Assume q_t satisfies no-caustic condition for $0 \le t \le T(\omega)$.

1. $$\nabla \mathcal{S}(q_t(x_0), t) = p_t(x_0),$$
2. $$\mathrm{d}\mathcal{S}(x,t) + H(\nabla \mathcal{S}(x,t), x, \circ\partial t) = 0.$$
3. *Define,*
$$\rho(x,t) = \left| \det\left(Dq_t^{-1}(x) \right) \right|,$$
then,
$$\partial\rho(x,t) + \nabla \cdot \left(\rho(x,t)\partial q_t(q_t^{-1}(x)) \right) = 0.$$

Sketch of Generalisation

Viscous H-J equation,

$$\partial \mathcal{S}^{\mu}(x,t) + H(\nabla \mathcal{S}^{\mu}(x,t), x, \circ\partial t) = \frac{\mu^2}{2} \sum_{l_1,l_2=1}^{n} \frac{\partial^2 H}{\partial p_{l_1} \partial p_{l_2}}(\nabla \mathcal{S}(x,t), x, \circ\partial t) \frac{\partial^2 \mathcal{S}^{\mu}}{\partial x_{l_1} \partial x_{l_2}}(x,t).$$

Explicitly find formal series,

$$\mathcal{S}^{\mu}(x,t) \sim \sum_{j=0}^{\infty} \mu^{2j} \mathcal{S}_j(x,t),$$

using iterated continuity equations as before.

Solutions of Heat equation

Before $T(\omega)$, change of measure with Feynman Kac,

$$\mathrm{d}Z_s^\mu = a(Z_s^\mu, t-s)\,\mathrm{d}s - \sum_{j=0}^{m} \mu^{2j}\nabla\mathcal{S}_j(Z_s^\mu, t-s)\,\mathrm{d}s + \mu\,\mathrm{d}B_s.$$

Theorem

$$u^\mu(x,t) = \exp\left(-\frac{1}{\mu^2}\sum_{j=0}^{m}\mu^{2j}\mathcal{S}_j(x,t)\right)$$
$$\times\,\mathbb{E}\Bigg\{\exp\Bigg(-\frac{\mu^{2m}}{2}\int_0^t \Delta\mathcal{S}_m(Z_s^\mu, t-s)\,\mathrm{d}s$$
$$+\frac{1}{2}\int_0^t \sum_{j=m+1}^{2m}\mu^{2(j-1)}\sum_{0\le i_1,i_2\le m,\, i_1+i_2=j}\nabla\mathcal{S}_{i_1}\cdot\nabla\mathcal{S}_{i_2}\Bigg)\Bigg\}$$

Solutions of Burgers equation

Applying the Hopf-Cole transformation,

$$v^{\mu}(x,t) = -\mu^2 \nabla \ln u^{\mu}(x,t) - a(x,t).$$

$$v_0(x,t) = \nabla \mathcal{S}_0(x,t) - a(x,t), \qquad v_j(x,t) = \nabla \mathcal{S}_j(x,t), \quad j \geq 1.$$

Theorem

$$v^{\mu}(x,t) = \sum_{j=0}^{m} \mu^{2j} v_j(x,t)$$

$$- \mu^2 \nabla \ln \mathbb{E}\Bigg\{ \exp\Bigg(-\frac{\mu^{2m}}{2} \int_0^t \nabla \cdot v_m(Z_s^{\mu}, t-s)\, \mathrm{d}s$$

$$+ \sum_{j=m+1}^{2m} \frac{\mu^{2(j-1)}}{2} \sum_{\substack{m \geq i_1, i_2 \geq 0, \\ i_1 + i_2 = j}} \int_0^t v_{i_1}(Z_s^{\mu}, t-s) \cdot v_{i_2}(Z_s^{\mu}, t-s)\, \mathrm{d}s \Bigg) \Bigg\}$$

After the caustic time

Before caustic time $T(\omega)$,

$$v^0(x,t) = \dot{\Phi}_t\left(\Phi_t^{-1}x\right), \quad \text{with probability 1.}$$

Caustic forms at $T(\omega)$

$$C_t = \left\{x : \quad \det\left(\frac{\partial \Phi_t(x_0)}{\partial x_0}\right) = 0\right\}$$

Assume $\Phi_t^{-1}\{x\} = \{x_0(i)(x,t) : i = 1, 2, \ldots n\}$

Schilder asymptotic expansions

$$u^\mu(x,t) \sim \sum_{i=1}^{n} \theta_i \exp\left(-\frac{S_0^i(x,t)}{\mu^2}\right),$$

$$S_0^i(x,t) := S_0\left(x_0(i)(x,t)\right) + A\left(x_0(i)(x,t), x, t\right).$$

After the caustic time

Hamilton-Jacobi level surface

$$H_t^c = \left\{ x : \quad S_0^i(x,t) = c \text{ for some } i \right\}.$$

Inviscid limit of the Burgers fluid velocity

$$v^0(x,t) = \dot{\Phi}_t(\tilde{x}_0(x,t)),$$

where $\tilde{x}_0(x,t)$ is the minimising $x_0(i)(x,t)$.

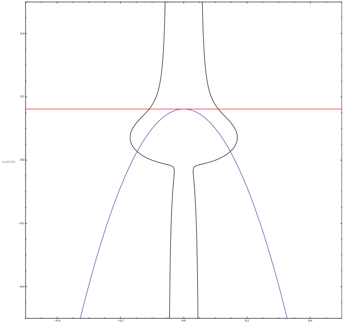
Maxwell set

$$M_t = \{x : \quad x = \Phi_t(x_0) = \Phi_t(\check{x}_0), \quad x_0 \neq \check{x}_0, \\ \mathcal{A}(x_0,x,t) = \mathcal{A}(\check{x}_0,x,t)\}.$$

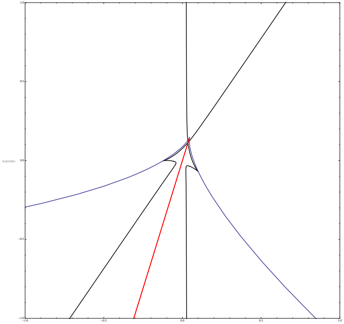
Cusp initial condition $S_0(x_0) = x_0^2 y_0$

$$a(x,t) = -\frac{1}{2}(x,y,z)\wedge(0,0,\Omega), \qquad V(x,t) = -\frac{1}{2}(x^2+y^2+z^2)\omega^2$$

Caustic, level surface & Maxwell set

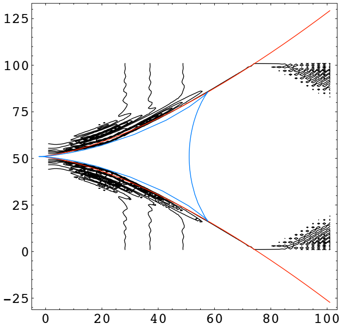


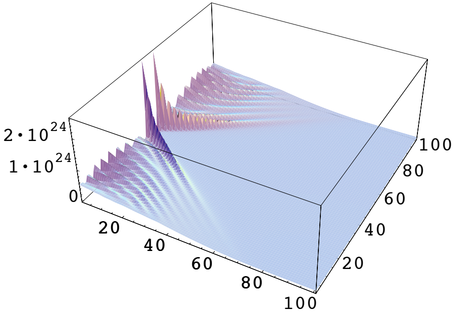
Φ_t
$\Rightarrow$



Cusp initial condition $S_0(x_0) = x_0^2 y_0$

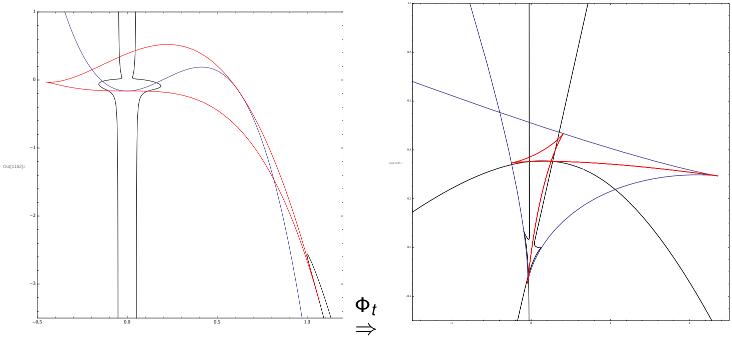
Numerical simulation of $v(x, t)$,





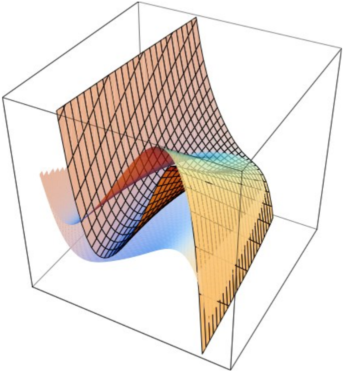
Swallowtail initial condition $S_0(x_0) = x_0^5 + x_0^2 y_0$

Caustic, level surface & Maxwell set

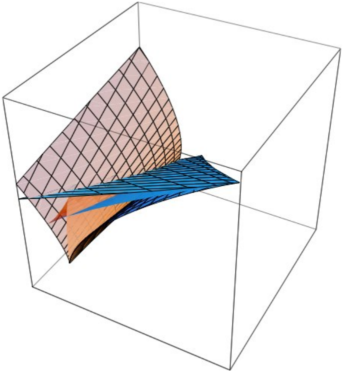


3D Swallowtail initial condition $S_0(x_0) = x_0^5 + x_0^2 y_0$

Caustic, & Maxwell set



Φ_t
$\Rightarrow$



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