Exchangeability, Braidability and Quantum Independence

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Motivation

Though many probabilistic symmetries are conceivable [...], four of them - **stationarity**, **contractability**, **exchangeablity** *and* **rotatability** *- stand out as especially interesting and important in several ways: Their study leads to some deep structural theorems of great beauty and significance [...].*

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Can one transfer the related concepts to **noncommutative probability** and do they turn out to be fruitful in the study of the structure of **operator algebras** and **quantum dynamics**?

Remark

Noncommutative probability = classical & quantum probability

References

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invariant objects	transformations
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Topic of this talk:

- invariant objects are generated by an **infinite sequence** of random variables
- only the **first three symmetries** are considered
- **contractable = spreadable** (= subsymmetric)

Motivating example for de Finetti's theorem

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$$P(X_1 = e_1, \ldots, X_n = e_n) = P(X_{\pi(1)} = e_1, \ldots, X_{\pi(n)} = e_n)$$

holds for all $n \in \mathbb{N}$, permutations π and every $e_1, \ldots, e_n \in \{0,1\}$.

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Then there exists a unique probability measure ν on $[0,1]$ such that

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where $s = e_1 + e_2 + \ldots + e_n$.

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  THIS TALK: Random variables are selfadjoint operators in $\mathcal{A}$.
- The **automorphisms of a probability space** $\mathrm{Aut}(\mathcal{A}, \varphi)$ are *-automorphisms α of $\mathcal{A}$ with $\varphi \circ \alpha = \varphi$.

Noncommutative distributions

Given the probability space $(\mathcal{A}, \varphi)$, two sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ in $\mathcal{A}$ have the same **distribution** if

$$\varphi\big(x_{\mathbf{i}(1)} x_{\mathbf{i}(2)} \cdots x_{\mathbf{i}(n)}\big) = \varphi\big(y_{\mathbf{i}(1)}\, y_{\mathbf{i}(2)} \cdots y_{\mathbf{i}(n)}\big)$$

for all n-tuples $\mathbf{i}\colon \{1, 2, \ldots, n\} \to \mathbb{N}_0$ and $n \in \mathbb{N}$.

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Notation

$$(x_0, x_1, x_2, \ldots) \stackrel{\text{distr}}{=} (y_0, y_1, y_2, \ldots)$$

Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence $(x_n)_{n\geq 0}$ is

- **exchangeable** if $(x_0, x_1, x_2, \ldots) \stackrel{\text{distr}}{=} (x_{\pi(0)}, x_{\pi(1)}, x_{\pi(2)}, \ldots)$ for any finite permutation $\pi \in \mathbb{S}_\infty$ of $\mathbb{N}_0$.

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Lemma (Hierarchy of distributional symmetries)

Exchangeability $\Rightarrow$ Spreadability $\Rightarrow$ Stationarity $\Rightarrow$ Identical distr.

Noncommutative conditional independence

Given the probability space $(\mathcal{A}, \varphi)$, let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ be three von Neumann subalgebras of $\mathcal{A}$ with φ-preserving conditional expectations $E_i \colon \mathcal{A} \to \mathcal{A}_i$ $(i = 0, 1, 2)$.

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- If $\mathcal{A} = L^\infty(\mathbb{A})$ we obtain conditional independence with respect to a sub-σ-algebra.

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- There are many different forms of noncommutative independence!
- $\mathbb{C}$-independence & Speicher's universality rules $\rightsquigarrow$ tensor independence or free independence

Conditional independence of sequences

A sequence of random variables $(x_n)_{n\in\mathbb{N}_0}$ in $\mathcal{A}$ is **(full) $\mathcal{B}$-independent** if

$$\bigvee\{x_i \,|\, i \in I\} \vee \mathcal{B} \quad \text{and} \quad \bigvee\{x_j \,|\, j \in J\} \vee \mathcal{B}$$

are $\mathcal{B}$-independent whenever $I \cap J = \emptyset$ with $I, J \subset \mathbb{N}_0$.

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- $\mathcal{B}$ may **not** be contained in $\bigvee\{x_i \mid i\in I\}$
- Example of mixed coin tosses: $\dim\bigvee\{x_i\}=2$, but $\mathcal{B}\simeq L^\infty([0,1],\nu)$ may be infinite dimensional!

Classical dual version of extended De Finetti theorem

Let $(x_n)_{n\in\mathbb{N}_0}$ be random variables in $(\mathcal{A},\varphi)$ with tail algebra

$$\mathcal{A}^{\mathrm{tail}} := \bigcap_{n\geq 0} \bigvee_{k\geq n} \{x_k\},$$

and consider:

(a) $(x_n)_{n\in\mathbb{N}_0}$ is exchangeable

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Theorem (De Finetti '31, Ryll-Nardzewski '57,..., K. '08)

$\mathcal{A} \simeq L^\infty(\mathbb{A})$ *implies:* $\quad$ *(a)* $\Leftrightarrow$ *(c)* $\Leftrightarrow$ *(e)*

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Here tail algebra is commutative $\rightsquigarrow$ ergodic decompositions

Noncommutative De Finetti theorems with commutativity conditions

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Theorem (Accardi & Lu '93, ..., K. '08)

$\mathcal{A}^{\mathrm{tail}}$ *is abelian:* $\quad$ *(a)* $\Rightarrow$ *(e)*

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Theorem (K. '07-'08, Gohm & K. '08)

(a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e), but (a) $\not\Leftarrow$ (c) $\not\Leftarrow$ (d) $\not\Leftarrow$ (e)

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(a) $(x_n)_{n\in\mathbb{N}_0}$ is exchangeable

(c) $(x_n)_{n\in\mathbb{N}_0}$ is spreadable

(d) $(x_n)_{n\in\mathbb{N}_0}$ is stationary and $\mathcal{A}^{\text{tail}}$-independent

(e) $(x_n)_{n\in\mathbb{N}_0}$ is identically distributed and $\mathcal{A}^{\text{tail}}$-independent

Theorem (K. '07-'08, Gohm & K. '08)

(a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e), but (a) $\not\Leftarrow$ (c) $\not\Leftarrow$ (d) $\not\Leftarrow$ (e)

Remark

'Spreadability' implies '(**full**) conditional independence' is the hard part!

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- All reverse implications in the noncommutative extended de Finetti theorem fail due to deep structural reasons!
- This will become clear from braidability...

Artin braid groups $\mathbb{B}_n$

Algebraic Definition (Artin 1925)

$\mathbb{B}_n$ is presented by $n-1$ generators $\sigma_1, \dots, \sigma_{n-1}$ satisfying

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \qquad \text{if } \mid i-j \mid = 1 \tag{B1}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } \mid i-j \mid > 1 \tag{B2}$$

Figure: Artin generators σ_i (left) and σ_i^{-1} (right)

$\mathbb{B}_1 \subset \mathbb{B}_2 \subset \mathbb{B}_3 \subset \dots \subset \mathbb{B}_\infty$ (inductive limit)

Braidability

Definition (Gohm & K. '08)

A sequence $(x_n)_{n\geq 0}$ in $(\mathcal{A}, \varphi)$ is **braidable** if there exists a representation $\rho \colon \mathbb{B}_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$ satisfying:

$$\begin{aligned} x_n &= \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1) x_0 && \text{for all } n \geq 1; \\ x_0 &= \rho(\sigma_n) x_0 && \text{if } n \geq 2. \end{aligned}$$

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- If $\rho(\sigma_n^2) = \mathrm{id}$ for all n, one has a representation of $\mathbb{S}_\infty$.
- $(x_n)_{n\geq 0}$ is exchangeable $\Leftrightarrow \begin{cases} (x_n)_{n\geq 0} \text{ is braidable and} \\ \rho(\sigma_n^2) = \mathrm{id} \text{ for all } n. \end{cases}$

Braidability implies spreadability

It turns out that we can insert braidability between exchangeability and spreadability in the noncommutative de Finetti theorem and obtain a large and interesting class of spreadable sequences in this way.

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Open Problem

Construct spreadable sequence which fails to be braidable.

Examples for Braidability

There are many!

- subfactor inclusion with small Jones index
- left regular representation of $\mathbb{B}_\infty$
- ...

For further details see publication in Comm. Math. Phys.

Is there a free analogue of de Finetti's theorem?

Classical Probability

$$\boxed{\text{independence}} \quad \longleftrightarrow \quad \boxed{\begin{array}{c}\text{symmetries of}\\ \text{tensor products}\end{array}}$$

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$\rightsquigarrow$ Foundation of free probability theory

$\rightsquigarrow$ New direction of research in free probability

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- Rewrite exchangeability (for a classical sequence X) as a co-symmetry using the Hopf C*-algebra $C(\mathbb{S}_k)$

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- Rewrite exchangeability (for a classical sequence X) as a co-symmetry using the Hopf C*-algebra $C(\mathbb{S}_k)$
- Apply 'quantization':
  - ◇ Replace X_k's by **operators** $x_1, x_2, \ldots$
  - ◇ Replace $C(\mathbb{S}_k)$ by **quantum permutation group** $A_s(k)$

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The **quantum permutation group** $A_s(k)$ is the universal unital C*-algebra generated by e_{ij} $(i, j = 1, \ldots k)$ subject to the relations

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The **abelianization** of $A_s(k)$ is $C(\mathbb{S}_k)$, the continuous functions on the symmetric group $\mathbb{S}_k$

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$$\varphi(x_{i_1} \cdots x_{i_n}) \mathbb{1}_{A_s(k)} = \sum_{j_1, \ldots, j_n = 1}^{k} e_{i_1 j_1} \cdots e_{i_n j_n} \, \varphi(x_{j_1} \cdots x_{j_n})$$

for all $k \times k$-matrices $(e_{ij})_{ij}$ satisfying defining relations for $A_s(k)$.

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Remark

$$\boxed{\text{quantum exchangeability}} \;\substack{\Longrightarrow \\ \not\Longleftarrow}\; \boxed{\text{exchangeability}}$$

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Here $\mathcal{T}$ denotes the **tail von Neumann algebra**

$$\mathcal{T} = \bigcap_{n \in \mathbb{N}} \mathrm{vN}(x_k | k \geq n)$$

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(c) the sequence canonically embeds into $\bigstar_{\mathcal{T}}^{\mathbb{N}} \mathrm{vN}(x_1, \mathcal{T})$, a von Neumann algebraic **amalgamated free product over** $\mathcal{T}$

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A new application of exchangeability: Thoma's theorem

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Theorem (Thoma '64, Kerov & Vershik '81, Okounkov '97, Gohm & K. '09)

An extremal character of the group $\mathbb{S}_\infty$ is of the form

$$\chi(\sigma) = \prod_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} a_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} b_j^k \right)^{m_k(\sigma)} .$$

Here $m_k(\sigma)$ is the number of k-cycles in the permutation σ and the two sequences $(a_i)_{i=1}^{\infty}, (b_j)_{j=1}^{\infty}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq 0, \qquad b_1 \geq b_2 \geq \cdots \geq 0, \qquad \sum_{i=1}^{\infty} a_i + \sum_{j=1}^{\infty} b_j \leq 1.$$

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- Quantum exchangeability gives a beautiful characterization of freeness with amalgamation and opens a new research direction in free probability. The free version of the de Finetti theorem shows that a quantum symmetry leads to a quantum probabilistic invariant.

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- Quantum exchangeability gives a beautiful characterization of freeness with amalgamation and opens a new research direction in free probability. The free version of the de Finetti theorem shows that a quantum symmetry leads to a quantum probabilistic invariant.
- A new approach is opened to the theory of asymptotic representations of symmetric groups.
- ...

Thank you for your attention!