

Exchangeability, Braidability and Quantum Independence

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Stochastic Processes at the Quantum Level

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Motivation

*Though many probabilistic symmetries are conceivable [...], four of them - **stationarity**, **contractability**, **exchangeability** and **rotatability** - stand out as especially interesting and important in several ways: Their study leads to some deep structural theorems of great beauty and significance [...].*

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Can one transfer the related concepts to **noncommutative probability** and do they turn out to be fruitful in the study of the structure of **operator algebras** and **quantum dynamics**?

Remark

Noncommutative probability = classical & quantum probability

References

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Hierarchy of distributional symmetries

invariant objects	transformations
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Topic of this talk:

- invariant objects are generated by an **infinite sequence** of random variables
- only the **first three symmetries** are considered
- **contractable = spreadable** (= subsymmetric)

Motivating example for de Finetti's theorem

"Any exchangeable process is an average of i.i.d. processes."

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$$P(X_1 = e_1, \dots, X_n = e_n) = P(X_{\pi(1)} = e_1, \dots, X_{\pi(n)} = e_n)$$

holds for all $n \in \mathbb{N}$, permutations π and every $e_1, \dots, e_n \in \{0, 1\}$.

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Then there exists a unique probability measure ν on $[0, 1]$ such that

$$P(X_1 = e_1, \dots, X_n = e_n) = \int_0^1 p^s (1-p)^{n-s} \nu(dp),$$

where $s = e_1 + e_2 + \dots + e_n$.

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 - a von Neumann algebra \mathcal{A} acting on some Hilbert space \mathcal{H}
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- The **automorphisms of a probability space** $\text{Aut}(\mathcal{A}, \varphi)$ are *-automorphisms α of \mathcal{A} with $\varphi \circ \alpha = \varphi$.

Noncommutative distributions

Given the probability space (\mathcal{A}, φ) , two sequences $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ in \mathcal{A} have the same **distribution** if

$$\varphi(x_{i(1)} x_{i(2)} \cdots x_{i(n)}) = \varphi(y_{i(1)} y_{i(2)} \cdots y_{i(n)})$$

for all n -tuples $\mathbf{i}: \{1, 2, \dots, n\} \rightarrow \mathbb{N}_0$ and $n \in \mathbb{N}$.

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Notation

$$(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (y_0, y_1, y_2, \dots)$$

Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence $(x_n)_{n \geq 0}$ is

- **exchangeable** if $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_{\pi(0)}, x_{\pi(1)}, x_{\pi(2)}, \dots)$ for any finite permutation $\pi \in \mathbb{S}_\infty$ of \mathbb{N}_0 .

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Lemma (Hierarchy of distributional symmetries)

Exchangeability \Rightarrow Spreadability \Rightarrow Stationarity \Rightarrow Identical distr.

Noncommutative conditional independence

Given the probability space (\mathcal{A}, φ) , let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ be three von Neumann subalgebras of \mathcal{A} with φ -preserving conditional expectations $E_i: \mathcal{A} \rightarrow \mathcal{A}_i$ ($i = 0, 1, 2$).

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- If $\mathcal{A} = L^\infty(\mathbb{A})$ we obtain conditional independence with respect to a sub- σ -algebra.
- There are many different forms of noncommutative independence!
- \mathbb{C} -independence & Speicher's universality rules
 \rightsquigarrow tensor independence or free independence

Conditional independence of sequences

A sequence of random variables $(x_n)_{n \in \mathbb{N}_0}$ in \mathcal{A} is **(full) \mathcal{B} -independent** if

$$\bigvee \{x_i \mid i \in I\} \vee \mathcal{B} \quad \text{and} \quad \bigvee \{x_j \mid j \in J\} \vee \mathcal{B}$$

are \mathcal{B} -independent whenever $I \cap J = \emptyset$ with $I, J \subset \mathbb{N}_0$.

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- \mathcal{B} may **not** be contained in $\bigvee \{x_i \mid i \in I\}$
- Example of mixed coin tosses: $\dim \bigvee \{x_i\} = 2$, but $\mathcal{B} \simeq L^\infty([0, 1], \nu)$ may be infinite dimensional!

Classical dual version of extended De Finetti theorem

Let $(x_n)_{n \in \mathbb{N}_0}$ be random variables in (\mathcal{A}, φ) with tail algebra

$$\mathcal{A}^{\text{tail}} := \bigcap_{n \geq 0} \bigvee_{k \geq n} \{x_k\},$$

and consider:

- (a) $(x_n)_{n \in \mathbb{N}_0}$ is exchangeable
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Here tail algebra is commutative \rightsquigarrow ergodic decompositions

Noncommutative De Finetti theorems with commutativity conditions

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Theorem (Accardi & Lu '93, ..., K. '08)

$\mathcal{A}^{\text{tail}}$ is abelian: (a) \Rightarrow (e)

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Theorem (K. '07-'08, Gohm & K. '08)

$(a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$, but $(a) \not\Leftarrow (c) \not\Leftarrow (d) \not\Leftarrow (e)$

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- (a) $(x_n)_{n \in \mathbb{N}_0}$ is exchangeable
- (c) $(x_n)_{n \in \mathbb{N}_0}$ is spreadable
- (d) $(x_n)_{n \in \mathbb{N}_0}$ is stationary and $\mathcal{A}^{\text{tail}}$ -independent
- (e) $(x_n)_{n \in \mathbb{N}_0}$ is identically distributed and $\mathcal{A}^{\text{tail}}$ -independent

Theorem (K. '07-'08, Gohm & K. '08)

$(a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$, but $(a) \not\Leftarrow (c) \not\Leftarrow (d) \not\Leftarrow (e)$

Remark

'Spreadability' implies '(full) conditional independence' is the hard part!

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- All reverse implications in the noncommutative extended de Finetti theorem fail due to deep structural reasons!
- This will become clear from braidability...

Artin braid groups \mathbb{B}_n

Algebraic Definition (Artin 1925)

\mathbb{B}_n is presented by $n - 1$ generators $\sigma_1, \dots, \sigma_{n-1}$ satisfying

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i - j| = 1 \quad (\text{B1})$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| > 1 \quad (\text{B2})$$



Figure: Artin generators σ_i (left) and σ_i^{-1} (right)

$\mathbb{B}_1 \subset \mathbb{B}_2 \subset \mathbb{B}_3 \subset \dots \subset \mathbb{B}_\infty$ (inductive limit)

Braidability

Definition (Gohm & K. '08)

A sequence $(x_n)_{n \geq 0}$ in (\mathcal{A}, φ) is **braidable** if there exists a representation $\rho: \mathbb{B}_\infty \rightarrow \text{Aut}(\mathcal{A}, \varphi)$ satisfying:

$$\begin{aligned}x_n &= \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1) x_0 && \text{for all } n \geq 1; \\x_0 &= \rho(\sigma_n) x_0 && \text{if } n \geq 2.\end{aligned}$$

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Braidability extends exchangeability

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- $(x_n)_{n \geq 0}$ is exchangeable $\Leftrightarrow \begin{cases} (x_n)_{n \geq 0} \text{ is braidable and} \\ \rho(\sigma_n^2) = \text{id for all } n. \end{cases}$

Braidability implies spreadability

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Open Problem

Construct spreadable sequence which fails to be braidable.

Examples for Braidability

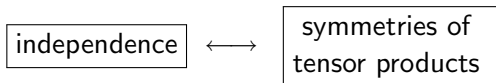
There are many!

- subfactor inclusion with small Jones index
- left regular representation of \mathbb{B}_∞
- ...

For further details see publication in Comm. Math. Phys.

Is there a free analogue of de Finetti's theorem?

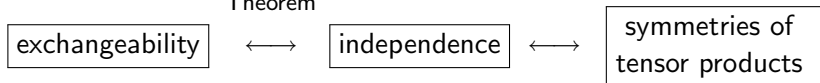
Classical Probability



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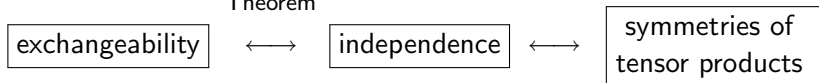
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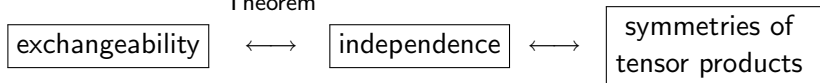


↔ Subject of distributional symmetries and invariance principles

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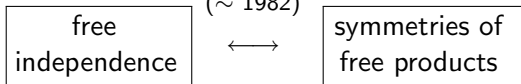
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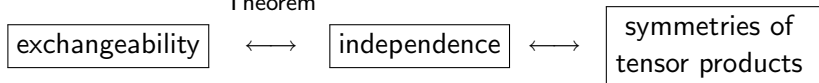
Voiculescu
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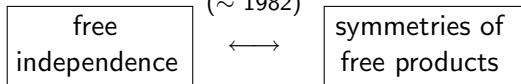
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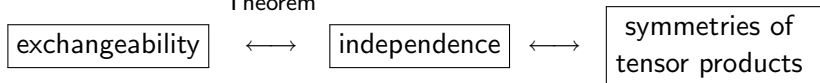


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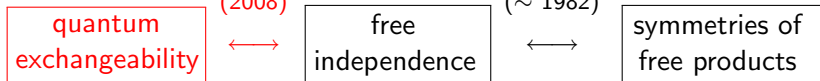


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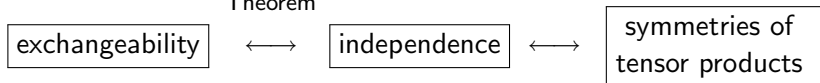


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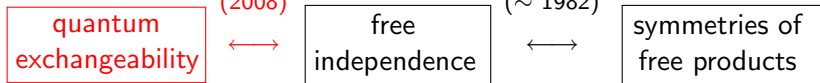


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↪ Foundation of free probability theory

↪ New direction of research in free probability

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 - ◇ Replace X_k 's by **operators** x_1, x_2, \dots
 - ◇ Replace $C(\mathbb{S}_k)$ by **quantum permutation group** $A_s(k)$

Quantum Permutation Groups

Definition and Theorem (Wang 1998)

The **quantum permutation group** $A_s(k)$ is the universal unital C^* -algebra generated by e_{ij} ($i, j = 1, \dots, k$) subject to the relations

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The **abelianization** of $A_s(k)$ is $C(\mathbb{S}_k)$, the continuous functions on the symmetric group \mathbb{S}_k

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for all $k \times k$ -matrices $(e_{ij})_{ij}$ satisfying defining relations for $A_s(k)$.

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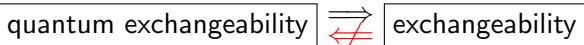
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Remark



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The following are equivalent for an infinite sequence of random variables x_1, x_2, \dots in (\mathcal{A}, φ) :

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- (b) the sequence is identically distributed and **freely independent with amalgamation over \mathcal{T}**

Here \mathcal{T} denotes the **tail von Neumann algebra**

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- (c) the sequence canonically embeds into $\star_{\mathcal{T}}^{\mathbb{N}} \text{vN}(x_1, \mathcal{T})$, a von Neumann algebraic **amalgamated free product over \mathcal{T}**

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Theorem (Thoma '64, Kerov & Vershik '81, Okounkov '97, Gohm & K. '09)

An extremal character of the group \mathbb{S}_∞ is of the form

$$\chi(\sigma) = \prod_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} a_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} b_j^k \right)^{m_k(\sigma)}.$$

Here $m_k(\sigma)$ is the number of k -cycles in the permutation σ and the two sequences $(a_i)_{i=1}^{\infty}, (b_j)_{j=1}^{\infty}$ satisfy

$$a_1 \geq a_2 \geq \dots \geq 0, \quad b_1 \geq b_2 \geq \dots \geq 0, \quad \sum_{i=1}^{\infty} a_i + \sum_{j=1}^{\infty} b_j \leq 1.$$

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- A new approach is opened to the theory of asymptotic representations of symmetric groups.
- ...

Thank you for your attention!