Unitary double product integrals as implementors of Bogolubov transformations

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A programme for Paul Jones.

Plan of talk:

1. Double products
2. Properties
3. Unitary double products as Bogolubov implementors
4. The rotational case
5. The general case
6. Applications and extensions.

1. Double products

Let $dr = g \otimes g$, $g = \mathbb{C}\langle dA^\dagger, dA, dT \rangle$

	$dA^\dagger$	dA	dT
$dA^\dagger$	0	$\beta^2 dT$	0
dA	$\alpha^2 dT$	0	0
dT	0	0	0

$$\sigma^2 = \alpha^2 + \beta^2 \geq 1$$
$$\alpha^2 - \beta^2 = 1$$

Let $a < b$, $s < t \in \mathbb{R}_+$

Definition ${}^b_a\prod^t_s (1 + dr)$

$$=: \ (1) \quad {}^b_a\prod \left\{ 1 + \hat{\prod}{}^t_s (1 + dr) \right\}$$

$$=: \ (2) \quad \prod{}^t_s \left\{ 1 + {}^b_a\hat{\prod} (1 + dr) \right\}$$

Here, for $dK \in g \otimes A$ where A is a unital algebra, ${}^b_a\prod(1 + dK)$ is the solution $X(b)$ at b of the qsde

$$dX = X\,dK, \quad X(a) = 1_{A \otimes B(K)}$$

and, for $dL \in \mathfrak{B} \otimes g$ where $\mathfrak{B}$ is a non-unital algebra (eg $\mathfrak{B} = g$), $\hat{\prod}{}^t_s (1 + dL)$ is the solution $Y(t)$ at t of

$$dY = (Y + 1_{\mathfrak{B}(K)})\,dL, \quad Y(s) = 0.$$

Theorem: 1 = 2

Proof Construct both as sums of double iterated stochastic integrals of form

$$I^{1,2}_{m,n} = \sum \int_{a<x_1<\cdots<x_m<b} dL_1(x_1) \cdots dL_m(x_m) \otimes \int_{s<y_1<\cdots<y_n<t} dL'_1(y_1) \cdots dL'_m(x_n).$$

Show that $I^{1}_{m,n} = I^{2}_{m,n}$ using the discrete analog:

$$\prod_{j=1}^{m}\left\{\prod_{k=1}^{n} x_{jk}\right\} = \prod_{k=1}^{n}\left\{\prod_{j=1}^{m} x_{jk}\right\}$$

provided that x_{jk} commutes with $x_{j'k'}$ whenever $j \neq j'$ and $k \neq k'$

eg $m = n = 2$

$$x_{11}x_{12}x_{21}x_{22} = x_{11}x_{21}x_{12}x_{22}. \quad \square$$

Gaps: 1. Convergence of iterated integral expansions.

2. Uniqueness of solutions. especially in Fock case $\sigma^2 = 1$.

2. Properties

1. Bi-evolution: ${}_a^b\prod_s^t (1+dr)$ is an evolution in (a,b) for fixed (s,t) and an evolution in (s,t) for fixed (a,b).

A consequence: Partition $]a,b]$ and $]s,t]$ as

$$a = x_0 < x_1 < \cdots < x_m = b, \quad s = y_0 < y_1 < \cdots < y_n = t$$

and define $\prod_{jk} =: {}_{x_{j-1}}^{x_j}\prod_{y_{k-1}}^{y_k}(1+dr)$. Then

$$
\begin{aligned}
{}_a^b\prod_s^t (1+dr) &= \prod_{j=1}^{m}\Big\{\prod_{k=1}^{n} \prod_{jk}\Big\} \\
&= \prod_{k=1}^{n}\Big\{\prod_{j=1}^{m} \prod_{jk}\Big\}.
\end{aligned}
$$

(But we know these are equal by the discrete version of Fubini).

Gap Find an invariant domain on which all double products act.

Challenge to sceptics. Try to construct a noncommutative tri-evolution. (You can't).

2. Shift-covariance: For the isometric shift in $L^2(\mathbb{R}_+)$

$$S_c f(x) = \begin{cases} 0, & x < c \\ f(x-c), & x \geq c \end{cases}$$

denote by Γ_c its isometric implementor. Then

$$\Gamma_c^* \otimes \Gamma_u^* \, {}_a^b\prod_s^t (1+dr) \, \Gamma_c \otimes \Gamma_u$$

$$= {}_{a+c}^{b+c}\prod_{s+u}^{t+u} (1+dr).$$

3. Time-reversal covariance: For the time reversal isometry in $]a,b]$

$$R_a^b f(x) = \begin{cases} f(x) & \text{if } x \notin]a,b] \\ f(a+b-x) & \text{if } x \in]a,b] \end{cases}$$

denote by Γ_a^b its (unitary) implementor. Then

$$(\Gamma_a^b \otimes \Gamma_s^t)^* \, {}_a^b\prod_s^t (1+dr) \, (\Gamma_a^b \otimes \Gamma_s^t)$$

$$= \left({}_a^b\prod_s^t (1+dr) \right)^*.$$

4 Adjunction:

$$\left({}_a^b\prod_s^t (1+dr) \right)^* = {}_a^b\overleftarrow{\prod}_s^t (1+dr^*)$$

$\leftarrow$: all right side's become left side's.

dr^*: product involution in $\mathfrak{g}\otimes\mathfrak{g}$

$$(dL_1 \otimes dL_2)^* = dL_1^\dagger \otimes dL_2^\dagger .$$

5. Inversion:

$$\left({}_a^b\prod_s^t (1+dr) \right)^{-1} = {}_a^b\overleftarrow{\prod}_s^t (1+dr')$$

where dr' is the quasi-inverse to dr,

$$dr + dr' + dr\,dr' = dr' + dr + dr'dr = 0.$$

(which exists because $\mathfrak{g}$ is nilpotent).

<u>Consequence</u> $\prod(1+dr)$ is unitary-valued if and only if

$$dr^\dagger + dr + dr^\dagger dr = dr + dr^\dagger + dr\,dr^\dagger = 0.$$

3. Unitary double products as Bogolubov implementors

Underlying the Itô algebra $\mathfrak{g} = \mathbb{C}\langle dA^{+}, dA, dT\rangle$ and its associated stochastic calculus is a representation of the Weyl relation over $\mathfrak{h} = L^2(\mathbb{R}_+)$:

$$W(f)\,W(g) = \exp(-i\,\mathrm{Im}\langle f, g\rangle)\,W(f+g).$$

In the Fock case $\sigma^2 = 1$ this is the Fock representation defined in $\mathcal{H} = \mathcal{F}(\mathfrak{h})$ by

$$W_1(f)\,e(g) = \exp\left(-\tfrac{1}{2}\langle f, g\rangle - \tfrac{1}{2}\|f\|^2\right)e(f+g)$$

When $\sigma^2 = \alpha^2 + \beta^2 > 1$ it is defined in $\mathcal{K} = \mathcal{H} \otimes \overline{\mathcal{H}}$ by

$$W_\sigma(f) = W_1(\alpha f) \otimes \overline{W_1(\beta f)}$$

A Bogolubov transformation is a real-linear invertible map $B : \mathfrak{h} \to \mathfrak{h}$ such that

$$\mathrm{Im}\langle Bf, Bg\rangle = \mathrm{Im}\langle f, g\rangle \quad (f, g \in \mathfrak{h}).$$

Then if W satisfies the Weyl relation, so too does W_B where

$$W_B(f) = W(B^{-1}f).$$

If there is a unitary U_B such that $W_B(\cdot) = U_B\,W(\cdot)\,U_B^{-1}$, then U_B implements B.

Bogolubov implementors are morally of form

$$U = \exp L$$

where L is quadratic in the creation and annihilation fields.

Now consider a unitary double product $\prod(1+dr)$ where

$$dr + dr^{+} + dr\,dr^{+} = dr^{+} + dr + dr^{+}dr = 0.$$

Thus: $dr = \lambda\, dA^{+} \otimes dA - \bar{\lambda}\, dA \otimes dA^{+}$
$+ \Lambda\, dA^{+} \otimes dA^{+} - \bar{\Lambda}\, dA \otimes dA$
$+$ terms in $dA^{\#} \otimes dT$, $dT \otimes dA^{\#}$, and $dT \otimes dT$, $\lambda, \Lambda \in \mathbb{C}$.

Note that only the $dT \otimes dT$ term is conditioned by the correction $dr\,dr^{+}$.

Morally $\prod(1+dr) = \exp\{L + \text{linear} + \text{scalar}\}$
$= \exp L \exp\{\text{linear}' + \text{scalar}'\}$.

$\Rightarrow$ Programme: Construct unitary double products explicitly as Bogolubov implementors modulo Weyl operators.

4. The rotational case

$$ {}_a^b\prod_s^t \left(1+\lambda(dA^\dagger\otimes dA - dA\otimes dA^\dagger) - \lambda^2\alpha^2\beta^2 dT\otimes dT\right), \quad \lambda\in\mathbb{R} $$

Consider first the Fock case $\beta=0$.

Approximate ${}_a^b\prod_s^t (1+\lambda(dA^\dagger\otimes dA - dA\otimes dA^\dagger))$ as follows.

Partion $\left\{\begin{matrix}]a,b] \\ ]s,t] \end{matrix}\right\}$ into $\left\{\begin{matrix} m \\ n \end{matrix}\right\}$ subintervals of equal lengths. Define

$$ a_j^\# = \sqrt{\frac{m}{b-a}}\left(A^\#(a+\tfrac{j}{m}(b-a)) - A^\#(a+\tfrac{j-1}{m}(b-a))\right)\otimes 1 $$

$$ b_k^\# = \sqrt{\frac{n}{t-s}}\, 1\otimes\left(A^\#(s+\tfrac{k}{n}(t-s)) - A^\#(s+\tfrac{k-1}{n}(t-s))\right) $$

Then $(a_j, a_j^\dagger)$, $j=1,\dots,m$ and $(b_k, b_k^\dagger)$ $k=1,\dots,n$ are mutually commuting standard creation-annihilation pairs satisfying

$$ [a_{j_1}, a_{j_2}^\dagger] = 1, \quad [b_{k_1}, b_{k_2}^\dagger] = 1. $$

We expect that, for large m, n

$$ {}_a^b\prod_s^t (1+\lambda(dA^\dagger\otimes dA - dA\otimes dA^\dagger)) \simeq \prod_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n} \exp\left(\lambda\sqrt{\frac{(b-a)(t-s)}{mn}}\left(a_j^\dagger b_k - a_j b_k^\dagger\right)\right). $$

Consider a single term in this product:

$$\exp\left(\theta\left(a^\dagger b - a b^\dagger\right)\right), \quad \theta = \lambda\sqrt{\frac{(b-a)(t-s)}{mn}}$$

as a unitary operator in the Fock space $\Gamma(\mathbb{C}^2)$. Using the CCR,

$$[a^\dagger b - a b^\dagger, a^\dagger] = -b^\dagger$$
$$[a^\dagger b - a b^\dagger, b^\dagger] = a^\dagger$$

and so

$$[a^\dagger b - a b^\dagger, [a^\dagger b - a b^\dagger, a^\dagger]] = -a^\dagger$$
$$[a^\dagger b - a b^\dagger, [a^\dagger b - a b^\dagger, b^\dagger]] = -b^\dagger$$

and more generally

$$\exp\left(\theta\left(a^\dagger b - a b^\dagger\right)\right) \begin{bmatrix} a^\dagger \\ b^\dagger \end{bmatrix} \exp\left(-\theta\left(a^\dagger b - a b^\dagger\right)\right)$$

$$= \exp\left(\mathrm{Ad}_{\theta(a^\dagger b - a b^\dagger)}\right) \begin{bmatrix} a^\dagger \\ b^\dagger \end{bmatrix}$$

ie, take the commutator with $a^\dagger b - a b^\dagger$.

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a^\dagger \\ b^\dagger \end{bmatrix}$$

Thus the unitary $\exp\left(\theta\left(a^\dagger b - a b^\dagger\right)\right)$ in $\Gamma(\mathbb{C}^2)$ implements the Bogolubov transformation in $\mathbb{C}^2$ given by the rotation with matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \theta = \theta_{mn} = \lambda\sqrt{\frac{(b-a)(t-s)}{mn}}$$

It follows that our approximation

$$\prod_{(j,k)\in \mathbb{N}_m \times \mathbb{N}_m} \exp\left(\theta\left(a_j^* b_k - a_j b_k^*\right)\right),$$

regarded as a unitary in $\Gamma(\mathbb{C}^{m+n})$, implements the Bogolubov transformation in $\mathbb{C}^{m+n}$ given by

$$R_{m,n}(\theta_{m,n}) = \prod_{(j,k)\in \mathbb{N}_m \times \mathbb{N}_n} R^{j,k}_{m,n}$$

where $R^{j,k}_{m,n}$ is the $(m+n)\times(m+n)$ matrix

$$\begin{array}{c} \\ \\ (j) \\ \\ (m+k) \\ \\ \\ \end{array}
\begin{bmatrix}
1 & & 0 & & 0 & 0 \\
 & \ddots & & & & \\
0 & & \cos\theta & & -\sin\theta & 0 \\
 & & & 1 \ddots & & \\
0 & & \sin\theta & & \cos\theta & 0 \\
 & & & & 0 & \ddots \; 1 \\
0 & & 0 & & &
\end{bmatrix}$$

got by embedding $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ into the intersections of the j^{th} and $(m+k)$-th rows and columns, and completing with 1's in vacant diagonal and 0's in off-diagonal positions.

<u>Note</u> $[R^{j,k}_{m,n}, R^{j',k'}_{m,n}] = 0$ if $j \neq j'$ and $k \neq k'$.

I know no explicit general formula for

$$R_{m,n}(\theta) = \prod_{(j,k)\in \mathbb{N}_m \times \mathbb{N}_n} R^{j,k}_{m,n}.$$

But one easily calculates:

$$R_{1,n}(\theta) = \left[\begin{array}{c|cccccc}
\alpha^n & \beta & \alpha\beta & \alpha^2\beta & \cdots & \cdots & \alpha^{n-1}\beta \\
\hline
\gamma\alpha^{n-1} & \delta & \gamma\beta & \gamma\alpha\beta & \cdots & \cdots & \gamma\alpha^{n-2}\beta \\
\gamma\alpha^{n-2} & 0 & \delta & \gamma\beta & \cdots & \cdots & \gamma\alpha^{n-3}\beta \\
\gamma\alpha^{n-3} & 0 & 0 & \delta & \ddots & & \gamma\alpha^{n-4}\beta \\
\vdots & & & & & \ddots & \vdots \\
\gamma & 0 & 0 & 0 & \cdots & \cdots & \delta
\end{array}\right]$$

and

$$R_{m,1}(\theta) = \left[\begin{array}{cccccc|c}
\alpha & \beta\gamma & \beta\delta\gamma & \cdots & \cdots & \beta\delta^{m-2}\gamma & \beta\delta^{m-1} \\
0 & \alpha & \beta\gamma & \cdots & \cdots & \beta\delta^{m-3}\gamma & \beta\delta^{m-2} \\
0 & 0 & \alpha & \cdots & \cdots & \beta\delta^{m-4}\gamma & \beta\delta^{m-3} \\
 & & & \ddots & & & \vdots \\
0 & 0 & 0 & & & \alpha & \beta \\
\hline
\gamma & \delta\gamma & \delta^2\gamma & - & - & \delta^{m-1}\gamma & \delta^m
\end{array}\right]$$

where $\alpha = \delta = \cos\theta$, $-\beta = \gamma = \sin\theta$.

Embedding $\mathbb{C}^n$ and $\mathbb{C}^m$ into $L^2(\mathbb{R}_+)$ by mapping their canonical orthonormal bases to normalised indicator functions of the subintervals of $]s,t]$ and $]a,b]$, and remembering that $\theta = \theta_{m,n} = \lambda\sqrt{\frac{(b-a)(t-s)}{mn}}$ one can then evaluate

$$\lim_{n\to\infty} R_{1,n}(\theta), \quad \lim_{m\to\infty} R_{m,1}(\theta).$$

They are operators on $\mathbb{C} \oplus L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_+) \oplus \mathbb{C}$, of form

$$\begin{bmatrix} \alpha_m & \beta_m \\ \gamma_m & \delta_m \end{bmatrix}, \quad \begin{bmatrix} \alpha^{(n)} & \beta^{(n)} \\ \gamma^{(n)} & \delta^{(n)} \end{bmatrix}$$

where $\alpha_m, \delta^{(n)}$ are scalars (operators on $\mathbb{C}$)
$\gamma_m, \beta^{(n)}$ are vectors in $L^2(\mathbb{R}_+)$
$\beta_m, \gamma^{(n)}$ are co-vectors in $L^2(\mathbb{R}_+)$
$\delta_m, \alpha^{(n)}$ are operators on $L^2(\mathbb{R}_+)$.

For example

$$\alpha_m = \lim_{n\to\infty} \left(\cos \Theta_{m,n}\right)^n = \lim_{n\to\infty} \left(1 - \lambda^2 \frac{(b-a)(t-s)}{2mn}\right)^n = e^{-\frac{1}{2m}\lambda^2 (b-a)(t-s)}.$$

One can now replace $\alpha, \beta, \gamma, \delta$ in the previous calculations by these scalars, vectors, covectors and operators and evaluate

$$\lim_{m\to\infty}\left\{\lim_{n\to\infty} R_{mn}\right\} \text{ and } \lim_{n\to\infty}\left\{\lim_{m\to\infty} R_{mn}\right\}$$

as explicit operators on $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$.

One finds:

$$\lim_{m\to\infty}\left\{\lim_{n\to\infty} R_{mn}\right\} = \lim_{n\to\infty}\left\{\lim_{m\to\infty} R_{mn}\right\} =: {}^b_aW^t_s$$

where

$${}^b_aW^t_s = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where for example

$A : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ maps $L^2]a,b]$ non-trivially to $L^2]a,b]$, and is given by

$$A = \exp\left(-\frac{\lambda^2}{2}(t-s)\Delta^b_a\right)$$

where Δ^b_a is the integral operator with kernel $<^b_a(x,y) = \begin{cases} 1 \text{ if } a<x\leq y\leq b \\ 0 \text{ otherwise.}\end{cases}$

<u>Gap</u>

Explicit formulas for B, C, D are known certainly only when the double product is replaced by one of the form $\overrightarrow{\prod}\overleftarrow{\prod}$ in which right side's are replaced by left in the second time interval $]s,t]$. see:

RLH Markov Processes Relat. Fields <u>13</u> (2007), 169–190, JTLewis memorial volume, Likewise, the following properties of ${}_a^b W_s^t$ are only certainly verified in this modification. (gap)

1. Unitarity (hence Bogolubovicity)
2. Bi-evolution
3. Shift-covariance
4. Time-reversal covariance
5. The second quantisation

$$\Gamma({}_a^b W_s^t) = {}_a^b\prod_s^t\left(1+\lambda(dA^\dagger\otimes dA - dA\otimes dA)\right)$$

in so far as it does indeed solve the requisite sde's.

See RLH, QPIDA XX pp241-250 (2007).

So much for the Fock case $\sigma^2=1$.

What about non-Fock $\sigma^2>1$?

In $\mathcal{K}=\mathcal{H}\otimes\overline{\mathcal{H}}$ we can implement the Bogolubov transformation ${}_a^b W_s^t$ for the corresponding Weyl system using the unitary implementors $\Gamma({}_a^b W_s^t)\otimes\overline{\Gamma({}_a^b W_s^t)}$

But this implementor is NOT

$${}_a^b\prod_s^t\left(1+\lambda(dA^\dagger\otimes dA - dA\otimes dA^\dagger(-\lambda^2\kappa^2\beta^2\, dT\otimes dT)\right)$$

It does not belong to the von Neumann algebra $\mathfrak{N} \otimes \mathfrak{N}$, $\mathfrak{N} = \{W_0(f) : f \in h\}$.
However all is not lost.
Because ${}^b_a W^t_s = 1 +$ Hilbert-Schmidt, a variant of Shale's theorem ensures that $\Gamma({}^b_a W^t_s) \otimes \Gamma({}^b_a W^t_s)$ can be factorised in the form ${}^b_a\Lambda^t_s \, {}^b_a\Lambda'^t_s$, where ${}^b_a\Lambda^t_s \in \mathfrak{N} \otimes \mathfrak{N}$, ${}^b_a\Lambda'^t_s \in \mathfrak{N}' \otimes \mathfrak{N}' = (\mathfrak{N} \otimes \mathfrak{N})'$
This factorisation is non-unique.
But the non-uniqueness is limited to a scaler ${}^b_a\omega^t_s$ because $\mathfrak{N}$ is a factor.

Theorem (see RLH, Operator Theory, Advances and Applications 195, 173-184 (2009), but only for Π case)
There is a unique choice of implementors ${}^b_a\Lambda^t_s \in \mathfrak{N} \otimes \mathfrak{N}$ such that
① $({}^b_a\Lambda^t_s)_{a \le b,\, s \le t}$ is a bievolution
② It is covariant under shifts and time-reversal.

It can then be shown that for this choice
$${}^b_a\Lambda^t_s = {}^b_a\prod^t_s \left(1 + \lambda(dA^\dagger \otimes dA - dA \otimes dA^\dagger) - \lambda^2\kappa^2\beta^2 dt \otimes dt\right)$$

5. The general case

The generator is

$$dr = \lambda\, dA^\dagger \otimes dA - \bar\lambda\, dA \otimes dA^\dagger + \Lambda\, dA^\dagger \otimes dA^\dagger - \bar\Lambda\, dA \otimes dA + \text{terms involving } dT.$$

$\lambda, \Lambda \in \mathbb{C}$

Ignoring the latter consider, in $\Gamma(\mathbb{C}^2)$,

$$L = \theta\left(e^{i\omega} a_1^\dagger a_2 - e^{-i\omega} a_1 a_2^\dagger\right) + \Theta\left(e^{i\Omega} a_1^\dagger a_2^\dagger - e^{-i\Omega} a_1 a_2\right)$$

One finds that

$$\mathrm{ad}_L\, a_1^\dagger = -\theta e^{-i\omega} a_2^\dagger - \Theta e^{-i\Omega} a_2$$

$$\mathrm{ad}_L\, a_2^\dagger = \theta e^{i\omega} a_1^\dagger - \Theta e^{-i\Omega} a_1,$$

$$\mathrm{ad}_L^2\, a_1^\dagger = (\Theta^2 - \theta^2) a_1^\dagger$$

$$\mathrm{ad}_L^2\, a_2^\dagger = (\Theta^2 - \theta^2) a_2^\dagger$$

and so

$$\exp(\mathrm{ad}_L)\, a_1^\dagger = \cosh\sqrt{\Theta^2-\theta^2}\; a_1^\dagger + \frac{\sinh\sqrt{\Theta^2-\theta^2}}{\sqrt{\Theta^2-\theta^2}}\left(-\theta e^{-i\omega} a_2^\dagger - \Theta e^{-i\Omega} a_2\right)$$

The corresponding Bogolubov transformation in $\mathbb{C}^2$ is thus of the form

$$B\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{\Theta^2-\theta^2}\, z_1 + S\sqrt{\Theta^2-\theta^2}\left(-\theta e^{i\omega} z_2 - \Theta e^{-i\Omega} \bar z_2\right) \\ S\sqrt{\Theta^2-\theta^2}\left(\theta e^{-i\omega} z_1 - \Theta e^{-i\Omega} \bar z_1\right) + \cosh\sqrt{\Theta^2-\theta^2}\, z_2 \end{pmatrix}$$

17. Applications and extensions

1. Causal products

Replace

$$ {}_a^b\prod_s^t(1+dr) = \prod_{a<x\le b,\ s<y\le t}(1+dr(x,y)) $$

in $K \otimes K$ by

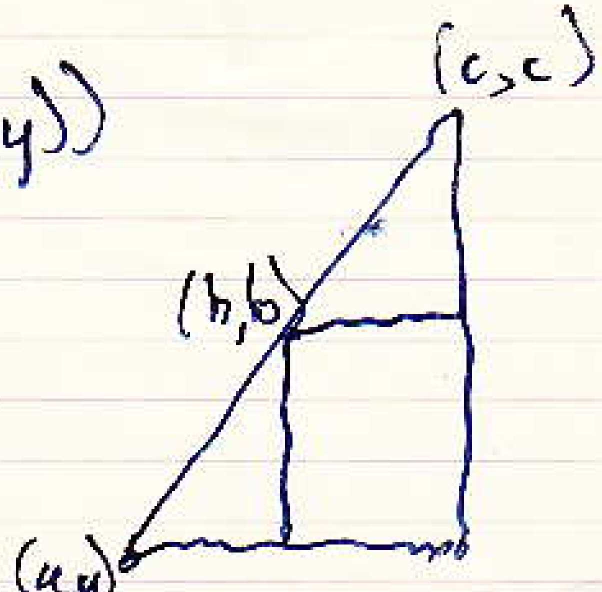
$$ \prod_{a<x\le y\le t}(1+dr(x,y)) $$

living in K.

There is a causal multiplicative Fubini theorem

The bi evolution property becomes

$$ \prod_{a<x\le y\le c}(1+dr(x,y)) $$



$$ = \prod_{a<x\le b}(1+dr(x,y)) $$

$$ {}_b^a\prod_b^c(1+dr) $$

$$ {}^a\prod^b_{b<x\le y\le c}(1+dr(x,y)) $$

insofar as $K_a^c = K_a \otimes K_b^c$

In particular, when $\Theta = \omega = 0$ this reduces to the (unitary) rotational case considered before. When $\Theta = 0$, $\omega \neq 0$ it is a straight forward modification (still unitary) in which $e^{\mp i\omega}$ appears in the off-diagonal terms B, C of ${}^{b}_{a}W^{t}_{s}$.

When $\Theta = \Omega = 0$ we obtain the pseudo rotation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cosh\Omega & -\sinh\Omega\, \kappa \\ -\kappa \sinh\Omega & \cosh\Omega \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

where κ is complex conjugation.

In all these cases, after verifying that the two forms of the limit ${}^{b}_{a}W^{t}_{s}$ are equal, one must check that

1. ${}^{b}_{a}W^{t}_{s}$ is Bogolubov
2. Bi-evolution
3. Covariance
4. Shale criterion for unitary implementability.

The programme will work as before as in the <u>non-Fock</u> case (even in Fock)