Local asymptotic normality for quantum Markov chains

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Stochastic Processes at the Quantum Level
Aberystwyth, October 2009

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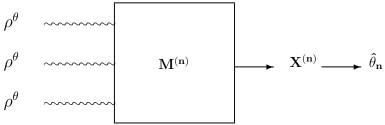
Quantum Statistics in the 70's: the classics

- Helstrom, Holevo, Belavkin, Yuen, Kennedy...
- Formulated and solved first quantum statistical decision problems
  - quantum statistical model $\mathcal{Q} = \{\rho_\theta \ : \ \theta \in \Theta\}$
  - decision problem (estimation, testing)
  - find optimal measurement (and estimator)
- Quantum Gaussian states, covariant families, state discrimination...
- Elements of a (purely) quantum statistical theory
  - Quantum Fisher Information
  - Quantum Cramér-Rao bound(s)
  - Holevo bound
  - ...

More recent trends in Quantum Statistics

- Asymptotics

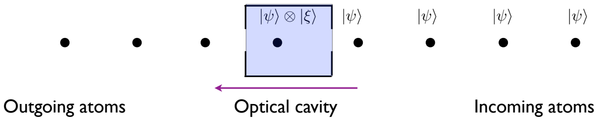


- New quantum statistical problems
  - Quantum (homodyne) tomography
  - Channel estimation
  - quantum cloning / broadcasting
  - quantum benchmarks for teleportation
  - Statistical inference for dynamical systems (system identification)

Plan

- Quantum Markov chains: the Schrödinger picture
- Local asymptotic normality for i.i.d. states
- Quantum Markov chains: the Heisenberg picture
- Theorem: L.A.N. for mixing quantum Markov chains
- Forgetful quantum Markov chains
- C. L.T. for forgetful quantum Markov chains

Quantum Markov chains



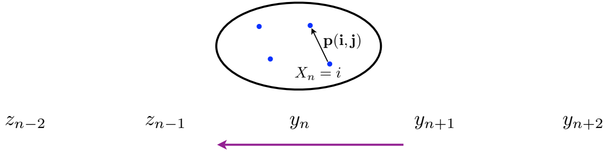
- Examples: quantum optical networks, atom maser, solid state cavity QED...

- Dynamics: unitary 'scattering' of atoms by cavity

$$U : M(\mathbb{C}^d \otimes \mathbb{C}^k) \to M(\mathbb{C}^d \otimes \mathbb{C}^k)$$

- System identification: estimate U by measuring outgoing atoms

Classical analogue



- Bernoulli shift Y_n
- Markov chain X_n driven by Y_n

$$X_{n+1} = F(X_n, Y_n)$$

- Observed (scattered) process Z_n

$$Z_n = S(X_n, Y_n)$$

A few examples

- Creation-annihilation coupling $U : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$

$$U = \exp\left[\alpha(\sigma_- \otimes \sigma_+ - \sigma_+ \otimes \sigma_-)\right]$$

- Jaynes-Cummings coupling $U : \mathbb{C}^2 \otimes \ell^2(\mathbb{N}) \to \mathbb{C}^2 \otimes \ell^2(\mathbb{N})$

$$U = \exp\left[\alpha(\sigma_- \otimes a^* - \sigma_+ \otimes a) + i\beta\sigma_z + i\gamma a^* a\right]$$

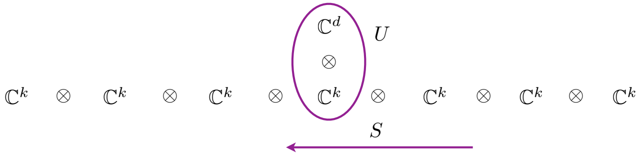
- Continuous-time quantum Markov process

$$U_t : \mathbb{C}^d \otimes \mathcal{F}(L^2(\mathbb{R}_+)) \to \mathbb{C}^d \otimes \mathcal{F}(L^2(\mathbb{R}_+))$$

$$dU_t = \left\{ L \otimes dA_t^* - L^* \otimes dA_t - \frac{1}{2} L^* L \, dt - iH \, dt \right\} U_t \qquad \text{(QSDE)}$$

Hilbert space evolution

- ‘system’ $\mathbb{C}^d$, ‘noise unit’ $\mathbb{C}^k$, interaction unitary U



- One step joint evolution: $W = S \circ U$

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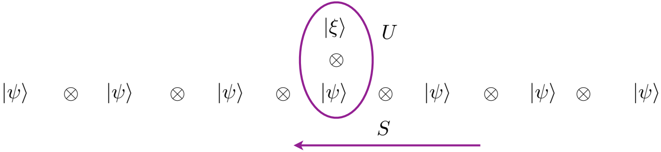

Hilbert space evolution

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$$U_{-1} \qquad |\xi\rangle$$

$$\otimes$$

$$|\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle$$

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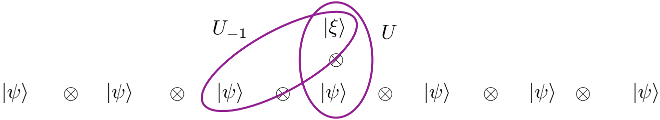

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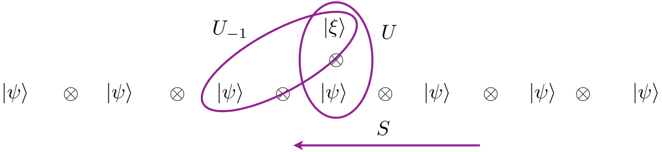

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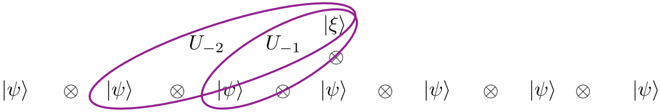

- Output state after n steps

$$|\psi_n\rangle := U_{-1} \circ \cdots \circ U_{-n} \, |\xi\rangle \otimes |\psi\rangle^{\otimes n} \in \mathbb{C}^d \otimes \mathbb{C}^k$$

INTERMEZZO

- Convergence of quantum statistical models
- Local asymptotic normality for i.i.d. states

Convergence of quantum statistical models

- Sequence of quantum statistical models $\mathcal{Q}_n := \{\rho_{\theta,n} \ : \ \theta \in \Theta\}$
- Statistical decision problem for $\mathcal{Q}_n$ (estimation, testing...)

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Guiding Principle

If $\mathcal{Q}_n$ 'converges' to $\mathcal{Q} := \{\rho_\theta \ : \ \theta \in \Theta\}$

then the optimal measurements M_n (and risks) 'converge' as well

$$\begin{tikzcd}
\mathcal{Q}_n \arrow[r] \arrow[d, "M_n"] & \mathcal{Q} \arrow[d, "M"] \\
\mathcal{P}_n \arrow[r] & \mathcal{P}
\end{tikzcd}$$

Weak and strong convergence for pure states models

Let $\mathcal{Q}_n := \{|\psi_{\theta,n}\rangle \ : \ \theta \in \Theta\}$ and $\mathcal{Q} := \{|\psi_\theta\rangle \ : \ \theta \in \Theta\}$

- $\mathcal{Q}_n$ converges weakly to $\mathcal{Q}$ if

$$\lim_{n\to\infty} \langle \psi_{\theta_1,n} \,|\, \psi_{\theta_2,n}\rangle = \langle \psi_{\theta_1} \,|\, \psi_{\theta_2}\rangle, \qquad \text{(for some choice of phases!)}$$

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- $\mathcal{Q}_n$ converges strongly to $\mathcal{Q}$ if there exist channels T_n, S_n such that

$$\lim_{n\to\infty} \sup_{\theta\in\Theta} \| T_n\left(|\psi_{\theta,n}\rangle\langle\psi_{\theta,n}|\right) - |\psi_\theta\rangle\langle\psi_\theta| \|_1 = 0$$

$$\lim_{n\to\infty} \sup_{\theta\in\Theta} \| |\psi_{\theta,n}\rangle\langle\psi_{\theta,n}| - S_n\left(|\psi_\theta\rangle\langle\psi_\theta|\right) \|_1 = 0$$

The quantum Gaussian shift model

- Quantum harmonic oscillator with canonical observables Q, P

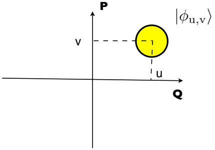
$$QP - PQ = i\mathbf{1} \qquad \text{(Heisenberg's commutation relations)}$$

- Vaccum (Gaussian) state $|\phi_0\rangle$

$$\langle \phi_0 \,|\, \exp(-ivQ - iuP) \,|\, \phi_0 \rangle = \exp(-(u^2 + v^2)/4)$$

- Coherent states

$$|\phi_{u,v}\rangle := \exp(-ivQ - iuP) \,|\, \phi_0\rangle$$



- Optimal measurements
  - one-parameter: $\hat{u} \sim N(u, 1/2)$ by measuring Q
  - two-parameter: $(\hat{u}, \hat{v}) \sim N((u, v), \mathbf{1})$ by 'joint' (Q, P) measurement

Local asymptotic normality for I.I.D. pure states

- $|0\rangle$ and $|1\rangle$ ON basis in $\mathbb{C}^2$
- Rotated spin $\quad |\psi_\theta\rangle := \exp(i\theta\sigma_x)|1\rangle = \cos\theta\,|1\rangle + \sin\theta\,|0\rangle$

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- n I.I.D. spins with local parametrisation $\theta = \theta_0 + u/\sqrt{n}$

$$|\psi_{u,n}\rangle := |\psi_{\theta_0+u/\sqrt{n}}\rangle^{\otimes n}$$

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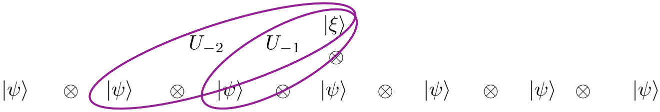
Local asymptotic normality

$\mathcal{Q}_n := \{|\psi_{u,n}\rangle : u \in \mathbb{R}\}$ converges to the Gaussian model $\{|\phi_{\sqrt{2}u}\rangle : u \in \mathbb{R}\}$

$$\langle\psi_{u,n}|\psi_{v,n}\rangle = \cos((u-v)/\sqrt{n})^n \longrightarrow e^{-\frac{1}{2}(u-v)^2} = \langle\phi_{\sqrt{2}u}|\phi_{\sqrt{2}v}\rangle$$

Back to quantum Markov chains

- ‘system’ $\mathbb{C}^d$, ‘noise unit’ $\mathbb{C}^k$, interaction unitary U



- One step joint evolution: $W = S \circ U$
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Markov (transition) semigroup and ergodicity

- $T : M(\mathbb{C}^d) \to M(\mathbb{C}^d)$ describes the 'reduced' evolution of the system

$$X \mapsto T(X) := \left\langle \psi \right| U^{-1} \left(X \otimes \mathbf{1} \right) U \left| \psi \right\rangle$$

$$\begin{array}{ccccccccccccc} & & & & & & X & & & & & & \\ & & & & & & \otimes & & & & & & \\ \mathbf{1} & \otimes & \mathbf{1} & \otimes & \mathbf{1} & \otimes & \mathbf{1} & \otimes & \mathbf{1} & \otimes & \mathbf{1} & \otimes & \mathbf{1} \end{array}$$

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$$\xrightarrow{\quad S^{-1} \,\cdot\, S \quad}$$

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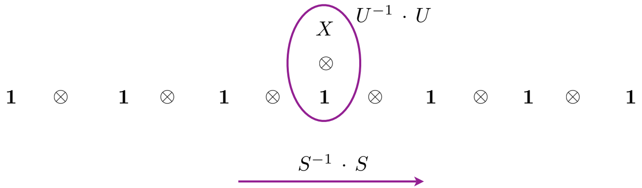

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$U_1^{-1} \cdot U_1$

X

$\otimes$

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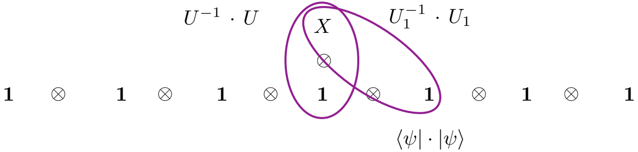

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- after n steps

$$X \mapsto T_n(X) := \left\langle \psi^{\otimes n} \,\middle|\, \hat{U}^{-n} \, (X \otimes \mathbf{1}) \, \hat{U}^{n} \,\middle|\, \psi^{\otimes n} \right\rangle = T^n(X)$$

Mixing quantum Markov chain

- The Markov chain (transition operator T) is called mixing if
  - $T(X) = X$ if and only if $X = \alpha\mathbf{1}$
  - All other eigenvalues λ satisfy $|\lambda| < 1$.

- Convergence to equilibrium

  If T is mixing then there exists a unique invariant state ρ_∞ on $M(\mathbb{C}^d)$ and

$$\lim_{n\to\infty} T_*^n(\sigma) = \rho_\infty, \qquad \text{for all initial states } \sigma$$

- Classical analogue

  Finite state irreducible aperiodic chain (Perron-Frobenius Therem)

Theorem: L.A.N. for (one parameter) coupling constant

- $U_\theta = \exp(i\theta H) \in \mathcal{U}(\mathbb{C}^d \otimes \mathbb{C}^k)$ with unknown coupling θ.
- Mixing transition operator $T_\theta(X) := \left\langle \psi \,\middle|\, U_\theta^{-1}\,(X \otimes \mathbf{1})\,U_\theta \,\middle|\, \psi \right\rangle.$

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Then the output state (statistical model)

$$|\,\psi_{u,n}\rangle \;:=\; \left(S \circ U_{\theta_0 + u/\sqrt{n}}\right)^n \left|\xi \otimes \psi^{\otimes n}\right\rangle$$

is asymptotically normal, i.e

$$\lim_{n\to\infty} \langle \psi_{u,n} \,|\, \psi_{v,n} \rangle = \left\langle \phi_{\sqrt{2V}u} \,|\, \phi_{\sqrt{2V}v} \right\rangle = \exp\left(-V(u-v)^2/2\right),$$

where $\{\,|\,\phi_{\sqrt{2V}u}\rangle : u \in \mathbb{R}\}$ is the quantum Gaussian shift with Fisher info $4V$.

Fisher information = variance of generator

- The 'variance' V is given by

$$\begin{aligned} V = V(H,H) \quad &:= \quad \mathbb{E}\left(H^2\right) + 2\sum_{k=1}^{\infty} \mathbb{E}\left(H \circ \left(W_{\theta_0}^{-k} H W_{\theta_0}^{k}\right)\right) \\ &= \quad \mathbb{E}\left(H^2\right) + 2\mathbb{E}\left(U_{\theta_0}^{-1}\left(H \circ (\mathrm{Id} - T_{\theta_0})^{-1}(K)\right) U_{\theta_0}\right) \end{aligned}$$

where

- $\mathbb{E} := \rho_\infty \otimes |\psi\rangle\langle\psi|^{\otimes\infty}$ is the stationary state at θ_0
- $K := \langle\psi|H|\psi\rangle$ is the conditional expectation of H onto the system,
- $A \circ B := (AB + BA)/2$

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$$= \quad \mathbb{E}\left(H^2\right) + 2\mathbb{E}\left(U_{\theta_0}^{-1}\left(H \circ (\mathrm{Id} - T_{\theta_0})^{-1}(K)\right) U_{\theta_0}\right)$$

  where

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  - $K := \langle\psi|H|\psi\rangle$ is the conditional expectation of H onto the system,
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- Interpretation:
  - limit model is family of coherent states $\tilde{\phi}_{\sqrt{2V}u} = \exp(iu\,\mathbb{G}(H))$
  - for optimal estimation of u measure conjugate variable of $\mathbb{G}(H)$

More insight into the limit model

- Forgetful quantum Markov chains
- Central Limit Theorem

Forgetful Markov chains [Kretschmann and Werner 2005]

- A quantum Markov chain is called forgetful if there exist linear maps

$$R_n : M(\mathbb{C}^d) \rightarrow M(\mathbb{C}^k)^{\otimes n}$$

such that

$$\lim_{n\rightarrow\infty} \|W^{-n}\ (X \otimes \mathbf{1})\ W^n - \mathbf{1} \otimes R_n(X)\| = 0, \qquad X \in M(\mathbb{C}^d)$$

$$U^{-1} \cdot U \qquad X \qquad U_1^{-1} \cdot U_1$$

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- Example: the creation-annihilation interaction on $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$U_\alpha := \exp(-\alpha(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+))$$

is forgetful in a neighbourhood of $\alpha = \pi/2$, and

- Conjecture: U_α is forgetful for all $\alpha \in (0, \pi)$

Properties of forgetful Markov chains

- Forgetfulness is equivalent to asymptotic abelianess

$$\lim_{n\to\infty} \left\| \left[W^{-n} \left(X \otimes \mathbf{1} \right) W^{n} ,\ Y \otimes \mathbf{1} \right] \right\| = 0, \qquad X, Y \in M\mathbb{C}^{d})$$

- Forgetfulness implies mixing property

- Controllability

  The system can be driven to any state asymptotically

- Observability

  Any measurement on the system can be performed indirectly

CLT for forgetful Markov chains

- Forgetful Markov chain with unitary $U \in M(\mathbb{C}^d \otimes \mathbb{C}^k)$
- 'Local observable' $A \in M(\mathbb{C}^d \otimes \mathbb{C}^k)$ such that $\mathbb{E}(A) = 0$
- Fuctuation operator associated to A

$$\mathbb{F}_n(A) := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} A(k), \qquad\qquad A(k) := W^{-k} A W^k$$

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$$\mathbb{F}_n(A) := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} A(k), \qquad A(k) := W^{-k} A W^k$$

Then $\mathbb{F}_n(A)$ converges in distribution to $N(0, V(A, A))$ where

$$\begin{aligned} V(A, A) &:= \mathbb{E}\left(A^2\right) + 2 \sum_{k=1}^{\infty} \mathbb{E}\left(A \circ \left(W^{-k} A W^k\right)\right) \\ &= \mathbb{E}\left(A^2\right) + 2\mathbb{E}\left(A \circ \left(U^{-1} \left(\mathrm{Id} - T\right)^{-1} (B) U\right)\right), \qquad B := \langle \psi | A | \psi \rangle \end{aligned}$$

Outlook

- Optimal measurement

  Solution of Gaussian optimisation problem

  Can it be implemented in the lab ?

- Strong convergence

  Extend LAN to strong convergence of statistical models and mixed states

- Continuous time

  LAN can be obtained by using a perturbation theorem of B. Davies

- Forgetfulness

  Is forgetfulness a 'generic' property ?

  Can we prove CLT with weaker assumptions ?

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