Spectral Triples and Noncommutative Metric Spaces

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(joint with various)

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C^*-algebras

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Many more examples in between; however these two are key examples:

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The answer leads to spectral triples.

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Let A be a C^*-algebra. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ on A is given by a dense $*$-subalgebra $\mathcal{A}$ of A represented on the Hilbert space $\mathcal{H}$ via $\pi : A \to B(\mathcal{H})$ (faithful) along with a densely defined self-adjoint operator D on $\mathcal{H}$ (Dirac operator) such that

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$(\mathcal{A}, \mathcal{H}, D)$ is p-summable if $\sum_n \frac{1}{(1+\lambda_n^2)^{p/2}} < \infty$, where $D = \mathrm{Diag}(\lambda_n)$.

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In those examples summability is directly linked to dimension.

In the examples above we can express the distance on the manifold M by

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Basic idea of theory of noncommutative metric spaces

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ on a C^*-algebra A let ω_1, ω_2 be states on A and define

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Possible answer (Rieffel): If it induces the weak-$*$-topology on the state space. (The one given by pseudonorms $a \to |\phi(a)|$, $\phi \in A^*$)

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(Rieffel) This is the case iff the image of $\{a \in \mathcal{A} \mid \|[D, a]\| \leq 1\}$ in the quotient $A/\mathbb{C}1$ is relatively compact (in norm).

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Rieffel calls such structures (A, L) quantum metric spaces (if the seminorm gives a metric compatible with the weak-*-topology).

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2. $(\mathcal{A}, \mathcal{H}, D)$ is regular if the corresponding metric induces the weak-$*$-topology on the state space.

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$$\ell(s) = \inf\{k \mid \exists s_{i_1}, \ldots, s_{i_k} \in S : s = s_{i_1} \ldots s_{i_k}\}$$

for $s \neq e$ and $\ell(e) := 0$ defines a length function on Γ.

Consider the Hilbert space

$$\ell^2(\Gamma) = \ell^2\Gamma = \{(\alpha_t)_{t\in\Gamma} \mid \sum_{t\in\Gamma} |\alpha_t|^2 < \infty\}$$

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The norm closure of the $*$-algebra

$$\mathbb{C}[\Gamma] = \{\sum_{\text{finite}} \alpha_t\lambda(t) \mid \alpha_t \in \mathbb{C}\} \subseteq B(\ell^2\Gamma)$$

is the reduced group C^*-algebra $C^*_r(\Gamma)$.

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4) Ex. 3) also works if we replace ℓ by a translation bounded function l.

E.g. for $\Gamma = \mathbb{Z}$ we have the triple

$D_\ell e_n = |n| e_n$ $(\ell(n) = |n|)$

but can also define

$D_l e_n = n e_n$ $(l(n) = n)$.

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6) Quasidiagonal algebras.

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(In the spirit of permanence properties.)

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The crossed product

$$A \rtimes_\alpha \Gamma$$

is the C^*-algebra generated by all $a \in A$ and the unitaries u_g of a unitary representation $g \mapsto u_g$, where $\forall g$

$$u_g a u_g^* = \alpha_g(a)$$

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The C^*-algebra generated by $\widetilde{\pi}(A)$ and all the u_h is the reduced crossed product $A \rtimes_{\alpha,r} \Gamma$.

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Standard realisation: Regular representation.
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Let $\Theta \in [0, 1]$ be irrational; define $\alpha \in \mathrm{Aut}(C(\mathbb{T})$ by

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These algebras are simple and are often regarded as ‘deformations’ of the 2-torus.

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Consider Hilbert space

$$\ell^2(\Gamma) \otimes \mathcal{H} \oplus \ell^2(\Gamma) \otimes \mathcal{H}$$

with action of $A \rtimes_{\alpha,r} \Gamma$ and use the operator

$$D = \begin{bmatrix} 0 & D_l \otimes 1 - i \otimes D_A \\ D_l \otimes 1 + i \otimes D_A & 0 \end{bmatrix}$$

as Dirac operator. (Motivated by Kasparov theory.)

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$$\sup\{\|[D_A, \alpha_g(a)]\| \mid g \in \Gamma\} < \infty$$

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Can find spectral triples on $C(\Sigma)$ which are invariant.

Theorem

(Hawkins, Skalski, White, Z; several others independently) If $(\mathcal{A}, \mathcal{H}, D_A)$ is equicontinuous then

$$(\mathbb{C}[\Gamma, \alpha, \mathcal{A}], \ell^2(\Gamma, \mathcal{H}) \oplus \ell^2(\Gamma, \mathcal{H}), D)$$

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1. If D_l and D_A are p_1 and p_2-summable respectively, then D is $p_1 + p_2$-summable.

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Work in progress (same notations as before):

Theorem

If Γ is a finitely generated free group and l the word length function and $(\mathcal{A}, \mathcal{H}, D_A)$ is regular, then so is $(\mathbb{C}[\Gamma, \alpha, \mathcal{A}], \ell^2(\Gamma, \mathcal{H}) \oplus \ell^2(\Gamma, \mathcal{H}), D)$.

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It could generalise to hyperbolic groups.

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Assume that there is an infinite dim projection $P \in B(\mathcal{H}_A)$ such that

$$[P, \pi_A(A)] \subseteq \mathcal{K}(P\mathcal{H}_A),$$

$\|P\pi_A(a)P\| = \|a\|, \forall a$ and $P\pi_A(A)P$ does not meet the compacts (except in 0).

Then

$$E = \mathcal{K}(P\mathcal{H}_A) \otimes \pi_B(B) + P\pi_A(A)P \otimes \mathbb{C}I$$

forms a C^*-algebra E giving an extension

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0 \quad (*)$$

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If $[D_A, P] = 0$ then one can produce a Dirac operator D on

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \oplus \mathcal{H}_A \otimes \mathcal{H}_B.$$

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This results has several applications, e.g. giving regular spectral triples on $SU_q(2)$.

Thanks!