1 MONOTONE COMPLETE C*-ALGEBRAS AND GENERIC DYNAMICS

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This talk is on joint work with Kazuyuki SAITÔ.

I shall begin by talking about Monotone Complete C*-algebras. Then I will give a brief introduction to Generic Dynamics and its close connection to MCAs.

Introduction: Monotone complete C^*-algebras

Let A be a C^*-algebra. Its self-adjoint part, A_{sa}, is a partially ordered, real Banach space whose positive cone is $\{zz^* : z \in A\}$.If each upward directed, norm-bounded subset of A_{sa}, has a least upper bound then A is said to be *monotone complete.* Each monotone complete C^*-algebra has a unit element (this follows by considering approximate units). Unless we specify otherwise, all C^*-algebras considered will possess a unit element. Every von Neumann algebra is monotone complete but the converse is false.

Monotone complete C^*-algebras arise in several different areas. For example, each injective operator system can be given the structure of a monotone complete C^*-algebra, in a canonical way. Injective operator spaces can be embedded as "corners" of monotone complete C^*-algebras.

When a monotone complete C^*-algebra is commutative, its lattice of projections is a complete Boolean algebra. Up to isomorphism, every complete Boolean algebra arises in this way.

We recall that each commutative (unital) C^*-algebra can be identified with $C(X)$,the algebra of complex valued continuous functions on some compact Hausdorff space X. Then $C(X)$ is monotone complete precisely when X is *extremally disconnected*, that is, the closure of each open subset of X is also open.

Monotone complete C^*-algebras are a generalisation of von Neumann algebras. The theory of the latter is now very well advanced. In the seventies the pioneering work of Connes, Takesaki and other giants of the subject transformed our knowledge of von Neumann algebras. By contrast, the theory of monotone complete C^*-algebras is very incomplete with many fundamental questions unanswered. But considerable progress has been made in recent years.

In 2007 Saitô and I introduced a classification semi-group for small monotone complete C^*-algebras which divides them into 2^c distinct equivalence classes.

A monotone complete C^*-algebra is said to be a *factor* if its centre is one dimensional; we may regard factors as being as far removed as possible from being commutative. Just as for von Neumann algebras, each monotone complete factor is of Type I or Type II_1or Type II_∞,or Type III. Old results of Kaplansky imply that each Type I factor is a von Neumann algebra. This made it natural for him to ask if this is true for every factor. The answer is "no", in general. We call a factor which is not a von Neumann algebra *wild.*

A C^*-algebra is *separably representable* when it has an isometric $*$-representation on a separable Hilbert space. As a consequence of more general results, I showed that if a monotone complete factor is separably representable (as a C^*-algebra) then it is a von Neumann algebra. So, in these circumstances, Kaplansky's question has a positive answer.

Throughout this talk, a topological space is said to be *separable* if it has a countable dense subset; this is a weaker property than having a countable base of open sets. (But if the topology is metrisable, they coincide.) Akemann showed that a von Neumann algebra has a faithful representation on a separable Hilbert space if, and only if its state space is separable.

We call a C^*-algebra with a separable state space *almost separably representable.* Answering a question posed by Akemann, I gave examples of monotone complete C^*-algebras which have separable state spaces but which are NOT separably representable.

If a monotone complete factor M possess a strictly positive functional and is not a von Neumann algebra then, as an application of more general results, M must be of Type III. Whenever an algebra is almost separably representable then it possesses a strictly positive functional. So if a wild factor is almost separably representable then it must be of Type III.

A (unital) C^*-algebra A is said to be *small* if there exists a unital complete isometry of A into $L(H)$, where H is a separable Hilbert space.

Saitô showed that A is small if, and only if, $A \otimes M_n(\mathbb{C})$ has a separable state space for $n = 1, 2...$so clearly every small C^*-algebra is almost separably representable. We do not know if the converse is true, but it is true for monotone complete factors.

Examples of (small) wild factors were hard to find. The first examples were due, independently, to Dyer and Takenouchi. As a consequence of a strong uniqueness theorem of Sullivan-Weiss-Wright, it turned out that the Dyer factor and the Takenouchi factor were isomorphic.

Another method of finding wild factors was given by me. I showed that each C^*-algebra A could be embedded in its "regular $\sigma-$completion", $\widehat{A}$. When A is separably representable then $\widehat{A}$ is monotone complete and almost separably representable. Furthermore, when A is infinite dimensional, unital and simple, then $\widehat{A}$ is a wild factor. But it was very hard to distinguish between these factors. Indeed one of the main results of S-W-W showed that an apparently large class of wild factors were, in fact, a unique (hyperfinite) factor. Some algebras were shown to be different by Saitò. In 2001 a major breakthrough by Hamana showed that there were 2^c non-isomorphic (small) wild factors, where $c = 2^{\aleph_o}$. This pioneering paper has not yet received as much attention as it deserves.

In 2007, Saitô and I introduced a quasi-ordering between monotone complete C^*-algebras. From this quasi-ordering we defined an equivalence relation and used this to construct a classification semi-group $\mathcal{W}$ for large classes of monotone complete C^*-algebras. (In particular, all almost separably representable MCAs).This semi-group is abelian, partially ordered, and has the Riesz decomposition property. For each monotone complete, small C^*-algebra

A we assign a "normality weight", $w(A) \in \mathcal{W}$. If A and B are algebras then $w(A) = w(B)$, precisely when these algebras are equivalent. It turns out that algebras which are very different can be equivalent. In particular, the von Neumann algebras correspond to the zero element of the semi-group. It might have turned out that $\mathcal{W}$ is very small and fails to distinguish between more than a few algebras. This is not so; when applied to the class of small MCAs, the cardinality of $\mathcal{W}$ is 2^c, where $c = 2^{\aleph_0}$.

One of the useful properties of $\mathcal{W}$ is that it can sometimes be used to replace problems about factors by problems about commutative algebras. For example, let G_j be a countable group acting freely and ergodically on a commutative monotone complete algebra A_j $(j = 1, 2)$.By a cross-product construction using these group actions, we can obtain monotone complete C^*-factors B_j $(j = 1, 2)$. Then it is easy to show that $wA_j = wB_j$.So if the commutative algebras A_1 and A_2 are not equivalent, then $wB_1 \neq wB_2$. In particular, $B_1$and B_2 are not isomorphic.

Influenced by $K-$theory, it is natural to wish to form the Grothendieck group of the semi-group $\mathcal{W}$. This turns out to be futile, since this Grothendieck group is trivial, because every element of $\mathcal{W}$ is idempotent. By a known general theory this implies that $\mathcal{W}$ can be identified with a join semi-lattice. The Riesz Decomposition Property for the semigroup turns out to be equivalent to the semi-lattice being distributive. So the known theory of distributive join semi-lattices can be applied to $\mathcal{W}$.

To each monotone complete C^*-algebra A we associated a *spectroid* invariant ∂A. Just as a spectrum is a set which encodes information about an operator, a spectroid encodes information about a monotone complete C^*-algebra. It turns out that equivalent algebras have the same spectroid. So the spectroid may be used as a tool for classifying elements of $\mathcal{W}$. Both spectroids and the classifying "weight" semigroup can be applied to much more general objects than C^*- algebras.

One of the many triumphs of Connes in the theory of von Neumann algebras, was to show that the injective von Neumann factors are precisely those which are hyperfinite. It is natural to conjecture an analogous result for wild factors. But this is not true. For, by applying deep results of Hroth and Kechris, it is possible to exhibit a small, wild, hyperfinite factor which is not injective. Saitô and I will give details of this, and other more general results in a paper we are preparing.

(#)Let Λ be a set of cardinality 2^c, where $c = 2^{\aleph_0}$. Then we showed that there exists a family of monotone complete C^*-algebras $\{B_\lambda : \lambda \in \Lambda\}$ with the following properties. Each B_λ is a monotone complete wild factor of Type III, and also a small C^*-algebra. For $\lambda \neq \mu$, B_λ and B_μ have different spectroids and so $wB_\lambda \neq wB_\mu$ and, in particular, B_λ is not isomorphic to B_μ. We can now show that we may choose each B_λ such that it is generated by an increasing sequence of full matrix algebras.

2 Introduction: Generic dynamics

An elegant account of Generic Dynamics has been given by Weiss; the term first occurred in S-W-W, in these articles, the underlying framework is a countable group of homeomorphisms acting on a complete separable metric space with no isolated points (a perfect Polish space).This corresponds to dynamics on a canonical compact, Hausdorff, extremally disconnected space (the Stone space of the complete Boolean algebra of regular open subsets of $\mathbb{R}$).

Let us recall that a subset V of a topological space X is *nowhere dense* if its closure has empty interior; a subset W of X is *meagre* if it is the union of countably many nowhere dense sets. A subset Y of X is said to be *generic,* if $X \backslash Y$ is meagre.

Let G be a countable group. Unless we specify otherwise, G will always be assumed to be infinite and equipped with the discrete topology. Let X be a Hausdorff topological space with no isolated points. Further suppose that X is a Baire space i.e. such that the only meagre open set is the empty set. (This holds if X is compact or a G-delta subset of a compact Hausdorff space or is homeomorphic to a complete separable metric space.)

Let ε be an action of G on X as homeomorphisms of X.

In classical dynamics we would require the existence of a Borel measure on X which was G-invariant or quasi-invariant, and discard null sets. In topological dynamics, no measure is required and no sets are discarded. In generic dynamics, we discard meagre Borel sets.

2.1 Ergodic discrete group actions on topological spaces

When G is a group of bijections of X, and $y \in X$, we denote the orbit $\{g(y) : g \in G\}$ by $G[y]$.

Lemma 1 *Let G be a countable group of homeomorphisms of X.*

(i) If there exists $x_0 \in X$ such that the orbit $G[x_0]$ is dense in X then every $G-$invariant open subset of X is either empty or dense.

(ii) If every non-empty open $G-$invariant subset of X is dense then, for each x in X, the orbit $G[x]$ is either dense or nowhere dense.

Proof. (i) Let U be a $G-$invariant open set which is not empty. Since $G[x_0]$ is dense, for some $g \in G$,we have $g(x_0) \in U$. But U is $G-$invariant. So $x_0 \in U$. Hence $G[x_0] \subset U$. So U is dense in Y.

(ii) Suppose y is an element of X such that $G[y]$ is not dense in X. Then $X \backslash clG[y]$ is a non-empty $G-$invariant open set. So it is dense in X.So $clG[y]$ has empty interior. ■

When G is a group of homeomorphisms of X its action is said to be *ergodic* if each $G-$invariant open subset of Y is either empty or dense in Y.

Lemma 2 *Let Y be an extremally disconnected space. Let G be a group of homeomorphisms of Y. Then the action of G is ergodic, if, and only if, the only $G-$invariant clopen subsets are Y and $\varnothing$.*

We concentrate on the situation where, for some $x_0 \in X$, the orbit $\{\varepsilon_g(x_0) : g \in G\}$ is dense in X. Of course this cannot happen unless X is separable. Let S be the Stone space of the (complete) Boolean algebra of regular open sets of X. Then, it can be shown that the action ε of G on X induces an action $\widehat{\varepsilon}$ of G as homeomorphisms of S; which will also have a dense orbit.

When, as in S-W-W, X is a perfect Polish space, then, as mentioned above, S is unique; it can be identified with the Stone space of the regular open sets of $\mathbb{R}$.But if we let X range over all separable compact subspaces of the separable space,$2^{\mathbb{R}}$, then we obtain 2^c essentially different S; where S is compact, separable and extremally disconnected. For each such S, $C(S)$ is a subalgebra of ℓ^∞.

Let E be the relation of orbit equivalence on S.That is, sEt, if, for some group element g, $\widehat{\varepsilon}_g(s) = t$.Then we can construct a monotone complete C^*-algebra M_E from the orbit equivalence relation. When there is a free dense orbit, the algebra will be a factor with a maximal abelian subalgebra, A, which is isomorphic to $C(S)$. There is always a faithful, normal, conditional expectation from M_E onto A.

For $f \in C(S)$, let $\gamma^g(f) = f \circ \widehat{\varepsilon}_{g^{-1}}$. Then $g \to \gamma^g$ is an action of G as automorphisms of $C(S)$.Then we can associate a monotone complete C^*-algebra $M(C(S), G)$, the *monotone cross-product* with this action. When the action $\widehat{\varepsilon}$ is free, then $M(C(S), G)$ is naturally isomorphic to M_E. **In other words, the monotone cross-product does not depend on the group, only on the orbit equivalence relation.**

This was a key point in Sullivan-Weiss-Wright where a strong uniqueness theorem was proved.

In the recent work by Saitô and me, we consider 2^c algebras $C(S)$, each taking different values in the weight semi-group $\mathcal{W}$. (Here $c = 2^{\aleph_0}$, the cardinality of $\mathbb{R}$.) We can also suppose that each S is separable and a subspace of $2^{\mathbb{R}}$.

There is no uniqueness theorem but, to our surprise, there is a strong generalisation of S-W-W.

Saito and I show the following:

Let G be any countably infinite group. Let α be an action of G (as homeomorphisms) on S and suppose this action has at least one orbit which is dense and free. Then, modulo meagre sets, the orbit equivalence relation obtained can also be obtained by an action of $\bigoplus \mathbb{Z}_2$ as homeomorphisms of S.So $M(C(S), G)$ can be identified with $M(C(S), \bigoplus \mathbb{Z}_2)$.

(An orbit $\{\varepsilon_g(x_0) : g \in G\}$ is said to be **free** if, for $g \neq \imath$, $\varepsilon_g(x_0) \neq x_0$.Equivalently, for $g \neq \imath$, ε_g leaves no point of the orbit fixed)

This should be compared with the situation in classical dynamics. e.g. In classical dynamics, any action by an amenable group is orbit equivalent to an action of $\mathbb{Z}$.But, in general, non-amenable groups give rise to orbit equivalence relations which do not come from actions of $\mathbb{Z}$.

The projections in $C(S)$ form a complete Boolean algebra, $ProjC(S)$. When $ProjC(S)$ is countably generated then $M(C(S), \bigoplus \mathbb{Z}_2)$ is generated by an increasing sequence of finite dimensional matrix algebras. Hence, $M(C(S), G)$ is Approximately Finite Dimensional. (In contrast to the von Neumann situation, we do not know if we can always choose the approximating finite dimensional algebras to be full matrix algebras.)

We construct 2^c,essentially different, compact extremally disconnected spaces, S_η, where $ProjC(S_\eta)$ is countably generated. Simultaneiously, we construct a natural action of $\bigoplus \mathbb{Z}_2$ with a free, dense orbit on each S_η This gives rise to a family of monotone complete C^*-algebras, $(B_\lambda, \lambda \in \Lambda)$ with the properties (#) described above.That is:

(#)Let Λ be a set of cardinality 2^c, where $c = 2^{\aleph_0}$. Then exists a family of monotone complete C^*-algebras $\{B_\lambda : \lambda \in \Lambda\}$ with the following properties. Each B_λ is a monotone complete wild factor of Type III, and also a small C^*-algebra. For $\lambda \neq \mu$, B_λ and B_μ have different spectroids and so $wB_\lambda \neq wB_\mu$ and, in particular, B_λ is not isomorphic to B_μ. We also show each B_λ is generated by an increasing sequence of full matrix algebras.

3 CLASSIFICATION SEMIGROUPS

I shall give a brief sketch of classification semigroups.

Our focus is on small monotone complete C*-algebras but much of the work here can be done in much greater generality. In one direction, weight semigroups can be defined, with no extra difficulty, for monotone complete C*-algebras of arbitrary size. To avoid some set theoretic difficulties we fix a large Hilbert space $H^\#$ and, for the rest of this section, only consider algebras which are isomorphic to subalgebras of $L(H^\#)$. For (unital) small C*-algebras, their Pedersen Borel envelopes, or more generally any (unital) C*-algebra of cardinality $c = 2^{\aleph_0}$, it suffices if $H^\#$ has an orthonormal basis of cardinality $c = 2^{\aleph_0}$. We call the corresponding classification semigroup, W, the normality weight semigroup.

Let A and B be monotone complete C*-algebras and let $\phi : A \to B$ be a positive linear map.

Then ϕ is *faithful* if $x \geq 0$ and $\phi(x) = 0$ implies $x = 0$.

Then ϕ is *normal* if, whenever D is a downward directed set of positive elements of ϕ maps the infimum of D to the infimum of $\{\phi(d) : d \in D\}$.

When defining the classification semigroup we shall use positive linear maps which are faithful and normal. By varying the conditions on ϕ we get a slightly different theory. For example, we could strengthen the conditions on ϕ by also requiring it to be completely positive. Essentially all the construction in this section can also be carried out with completely positive maps. We then get a semigroup which has a natural quotient onto the semigroup W. On the other hand, we could weaken the conditions by (i) only requiring ϕ to be σ-normal and (ii) only requiring the algebras to be monotone σ-complete; we still obtain a classification semigroup but can no longer show that it has the Riesz decomposition property. When we restrict our attention to small C*-algebras, we find that monotone σ-complete implies monotone complete. So in this setting, normal and σ-normal maps coincide.

Let Ω be the class of all monotone complete C*-algebras which are isomorphic to norm closed *-subalgebras of $L(H^{\#})$; and let $\Omega^{\#}$ be the set of all C*-subalgebras of $L(H^{\#})$ which are monotone complete (in themselves, they cannot be monotone closed subalgebras of $L(H^{\#})$ unless they are von Neumann algebras). So every $A \in \Omega$ is isomorphic to an algebra in $\Omega^{\#}$.

We define a relation on Ω by $A \precsim B$ if there exists a positive linear map $\phi : A \to B$ which is faithful and normal.

It can be proved that $\precsim$ is a quasi-ordering of Ω. We can now define an equivalence relation $\sim$ on Ω by $A \sim B$ if $A \precsim B$ and $B \precsim A$.

Let π be an isomorphism of A onto B. Then π and π^{-1} are both normal so $A \precsim B$ and $B \precsim A$.Now suppose π is an isomorphism of A onto a subalgebra of B. Then π need not be normal. It will only be normal if its range is a monotone closed subalgebra of B. In particular, if A is a monotone closed subalgebra of B, then by taking the natural injection as π, we see that $A \precsim B$.

For each $A \in \Omega^{\#}$ let $[A]$ be the corresponding equivalence class. Let $\mathcal{W}$ be the set of equivalence classes.

We can try to define $[A] + [B]$ to be $[A \oplus B]$.

It is not obvious this makes sense but it turns out to work OK. With this definition of addition, $\mathcal{W}$ is an abelian semi-group. It has a zero element $[\mathbb{C}]$

It is natural to try to form its Grothendieck group but this only gives the trivial group, because each element is idempotent.

We can partially order the semi-group by:

$[A] \leq [B]$ if $A_1 \sim A$ and $B_1 \sim B$ with $A_1 \precsim B_1$.

We find, appropriately, that 0 is its smallest element.

It then turns out that the partially ordered semi-group has the Riesz Decomposition Property:

If $0 \leq x \leq a + b$ then $x = a_1 + b_1$where $0 \leq a_1 \leq a$ and $0 \leq b_1 \leq b$.

For each $A \in \Omega$, A is isomorphic to $A^{\#} \in \Omega^{\#}$.

We define $w(A) = [A^{\#}]$.

In the partially ordered semigroup W, $w(A) + w(B)$ is the least upper bound of $w(A)$ and $w(B)$.

Also $w(A) \leq w(B)$ *if and only if* $w(A) + w(B) = w(B)$.

Then $\mathcal{W}$ is a join semi-lattice with the semi-group operation as join operation i.e. $a \vee b = a + b$

Translating the Riesz Decomposition Property in terms of the join operation is equivalent to the join semi-lattice being distributive.

PROPOSITION *Let A be an MCA in Ω. Then $w(A) = 0$ if, and only if, A is a von Neumann algebra with a faithful normal state.*

PROOF: The map $\lambda \to \lambda 1$ shows that $\mathbb{C} \precsim A$.So $A \sim \mathbb{C}$ if, and only if, $A \precsim \mathbb{C}$. But this is equivalent to the existence of a faithful normal functional $\phi : A \to \mathbb{C}$. By a well known theorem of Kadison, the existence of a faithful normal state on a monotone complete C*-algebra implies that A is a von Neumann algebra. Conversely, if A is a von Neumann algebra with a faithful normal state, then $A \precsim \mathbb{C}$.

The classification given here maps each small von Neumann algebra to the zero of the semigroup. It could turn out that W is very small and fails to distinguish between many algebras. We shall see later that this is far from true. Even when w is restricted to special subclasses of algebras, we can show that its range in W is huge, 2^c, where $c = 2^{\aleph_0}$. In the next section we shall introduce the spectroid of an algebra and show that it is, in fact, an invariant for elements of W.

THE SPECTROID AND REPRESENTING FUNCTIONS

Although our main interest is focused on monotone complete C*- algebras, in this section we shall also use the larger class of monotone σ-complete C*-algebras.

For any non-empty set J we let $F(J)$ be the collection of all finite subsets of J, including the empty set. In particular we note that $F(\mathbb{N})$, where $\mathbb{N}$ is the set of natural numbers, is countable.

3.1 DEFINITION A *representing function* for a monotone $\sigma-$complete C*-algebra, A, is a function $f : F(\mathbb{N}) \to A^+$ such that

(i) $f(k) \geq 0$ and $f(k) \neq 0$ for all k.

(ii) f is downward directed, that is, when k, l are finite subsets of $\mathbb{N}$, then $f(k \cup l) \leq f(k)$ and $f(k \cup l) \leq f(l)$.

(iii) $\bigwedge_{k \in F(\mathbb{N})} f(k) = 0$.

Let T be a set of cardinality $2^{\aleph_0} = c$. Let $\mathbf{N} : T \to \mathcal{P}(\mathbb{N})$ be an injection and let $\mathbf{N}(t)$ be infinite for each t. We do not require that $\{\mathbf{N}(t) : t \in T\}$ contains every infinite subset of $\mathbb{N}$. We shall regard T and the function $\mathbf{N}$ as fixed until further notice.

3.2 DEFINITION Let A be a monotone σ-complete C*-algebra and let $f : F(\mathbb{N}) \to A$ be a representing function. Then let $R_{(T,\mathbf{N})}(f)$ be the subset of T defined by

$$\{t \in T : \bigwedge_{k \in F(\mathbf{N}(t))} f(k) = 0\}.$$

The set $R_{(T,\mathbf{N})}(f)$ is said to be *represented by f in A, modulo* $(T, \mathbf{N})$.

Any subset of T which can be represented in A is said to be a representing set of A (modulo $(T, \mathbf{N})$).

3.3 DEFINITION Let A be a monotone σ-complete C*-algebra. Then the *spectroid* of A (modulo $(T, \mathbf{N})$), written $\partial_{(T,\mathbf{N})}A$, is the collection of all sets which can be represented in A,modulo $(T, \mathbf{N})$, by some representing function $f : F(\mathbb{N}) \to A^+$, that is,

$$\partial_{(T,\mathbf{N})}A = \{R_{(T,\mathbf{N})}(f) : \ f \text{ is a representing function for } A\}.$$

When it is clear from the context which $(T, \mathbf{N})$ is being used, we shall sometimes write ∂A.

Let us recall that $\#S$ denotes the cardinality of a set S.

3.4 PROPOSITION *Let* $(T, \mathbf{N})$ *be fixed and let* A *be any monotone* σ-complete C^**-algebra of cardinality* c*. Then* $\partial_{(T,\mathbf{N})}(A)$ *is of cardinality not exceeding* c*.*

PROOF: Each element of $\partial_{(T,\mathbf{N})}(A)$ arises from a representing function for A. But the cardinality of all functions from $F(\mathbb{N})$ into A is $\#A^{F(\mathbb{N})} = c^{\aleph_0} = c$. So $\#\partial_{(T,\mathbf{N})}(A) \leq c$.

3.5 COROLLARY *Let* $(T, \mathbf{N})$ *be fixed and let* A *be a small monotone* complete C^**-algebra. Then* $\partial_{(T,\mathbf{N})}(A)$ *is of cardinality not exceeding* c*.*

PROOF: By Proposition 1.3 $\#A = c$.

3.6 LEMMA *Let* $(T, \mathbf{N})$ *be fixed and let* S *be the set of all spectroids, modulo* $(T, \mathbf{N})$*, of monotone* σ*-complete* C^**-algebras of cardinality* c*. Then* $\#\mathcal{S} \leq 2^c$*.*

PROOF: Each subset of cardinality $\leq c$ is the range of a function from $\mathbb{R}$ into $\mathcal{P}(T)$.So $\#S \leq \#\mathcal{P}(\mathbb{R})^{\mathbb{R}} \leq \#\mathcal{P}(\mathbb{R} \times \mathbb{R}) = 2^c$.

Suppose that A and B are monotone σ-complete C*- algebras and $\phi : A \to B$ is a faithful positive linear map. Let us recall that ϕ is *σ-normal* if, whenever $(a_n)(n = 1, 2, ...)$ is a monotone decreasing sequence in A with $\bigwedge_{n=1}^{\infty} a_n = 0$ then $\bigwedge_{n=1}^{\infty} \phi(a_n) = 0$.

3.7 LEMMA *Let* A *and* B *be monotone* σ*- complete* C^**-algebras. Let* $\phi : A \to B$ *be a positive, faithful* σ*-normal linear map.*

Let D *be a downward directed subset of* A^+ *which is countable. Then* $\bigwedge\{d : d \in D\} = 0$ *if, and only if* $\bigwedge\{\phi(d) : d \in D\} = 0$*.*

PROOF: By Remark 1.7 there exists a monotone decreasing sequence $(c_n)(n = 1, 2, ...)$ in D such that $d \in D$ implies $d \geq c_n$ for some n. So $\bigwedge\{\phi(d) : d \in D\} = \bigwedge_{n=1}^{\infty} \phi(c_n) = \phi(\bigwedge_{n=1}^{\infty} c_n) = \phi(\bigwedge\{d : d \in D\})$.

Since ϕ is faithful, $\phi(\bigwedge\{d : d \in D\}) = 0$ if, and only if, $\bigwedge\{d : d \in D\} = 0$. The lemma follows.

3.8 DEFINITION Let A and B be monotone σ-complete C*- algebras. If there exists a positive, faithful σ-normal linear map $\phi : A \to B$ *w*e write $A \precsim_\sigma B$. Then the relation $\precsim_\sigma$ is a quasi ordering of the class of monotone σ-complete C*-algebras.

When $A \precsim_\sigma B$ and $B \precsim_\sigma A$ we say that A and B are *σ-normal equivalent* and write $A \sim_\sigma B$.This is an equivalence relation on the class of monotone σ-complete C*-algebras. Clearly, if A and B are monotone complete C*-algebras and $A \precsim B$ then $A \precsim_\sigma B$. So if $A \sim B$ it follows that $A \sim_\sigma B$.

3.9 PROPOSITION *Let* $(T, \mathbf{N})$ *be fixed and let* A *and* B *be monotone* σ*-complete* C^**-algebras. Let* $A \precsim_\sigma B$*. Then* $\partial_{(T,\mathbf{N})}(A) \subset \partial_{(T,\mathbf{N})}(B)$*.*

PROOF: Each element of ∂A is of the form $R_{(T,\mathbf{N})}(f)$ where f is a representing function for A. It is straightforward to verify that ϕf is a representing function for B. Since ϕ is faithful it follows from Lemma 2.5 that $R_{(T,\mathbf{N})}(f) = R_{(T,\mathbf{N})}(\phi f)$. Thus $\partial A \subset \partial B$.

It is clear that the spectroid is an isomorphism invariant but from Proposition 3.9 it is also invariant under σ-normal equivalence.

3.10 COROLLARY *Let* $(T, \mathbf{N})$ *be fixed and let* A *and* B *be monotone* σ*-complete* C^**-algebras. Let* $A \sim_\sigma B$. *Then* $\partial_{(T,\mathbf{N})}(A) = \partial_{(T,\mathbf{N})}(B)$.

3.11 COROLLARY *Let* A *and* B *be monotone complete* C^**-algebras with* $w(A) = w(B)$. *Then* $\partial_{(T,\mathbf{N})}(A) = \partial_{(T,\mathbf{N})}(B)$ *for any given* $(T, \mathbf{N})$.

So the spectroid is an invariant for the semigroup W and we may talk about the spectroid of an element of the semigroup.

For the rest of this section we shall consider only monotone complete C*-algebras, although some of the results (and proofs) are still valid for monotone σ-complete C*-algebras. Let $\mathcal{M}$ be the class of all small monotone complete C^*-algebras. We shall use W to denote the semigroup constructed in Section 2; but we shall assume that w has been restricted to the class of all small monotone complete C*-algebras and, from now on, use W to denote the semigroup $\{w(A) : A \in \mathcal{M}\}$. (So, in effect we are taking a sub semigroup of the one constructed in Section 2, and abusing our notation by giving it the same name.)

3.12 THEOREM *Let* $(T, \mathbf{N})$ *be fixed and consider only spectroids modulo* $(T, \mathbf{N})$. *Let* $\{A_\lambda : \lambda \in \Lambda\}$ *be a collection of small monotone complete* C^**-algebras such that the union of their spectroids has cardinality* 2^c. *Then there is a subcollection* $\{A_\lambda : \lambda \in \Lambda_0\}$ *where* Λ_0 *has cardinality* 2^c *and* $\partial(A_\lambda) \neq \partial(A_\mu)$ *whenever* λ *and* μ *are distinct elements of* Λ_0.

PROOF: Let us define an equivalence relation on Λ by $\lambda \approx \mu$ if, and only if, $\partial(A_\lambda) = \partial(A_\mu)$. By using the Axiom of Choice we can pick one element from each equivalence class to form Λ_0. Clearly $\partial(A_\lambda) \neq \partial(A_\mu)$ whenever λ and μ are distinct elements of Λ_0. Also $\bigcup\{\partial(A_\lambda) : \lambda \in \Lambda_0\}$ is equal to the union of all the spectroids of the original collection. So (a) $2^c = \# \bigcup\{\partial(A_\lambda) : \lambda \in \Lambda_0\}$.

By Corollary 3.5, $\#\partial(A_\lambda) \leq c$ for each $\lambda \in \Lambda_0$. Hence, from (a), $2^c \leq c \times \#\Lambda_0$. It follows that we cannot have $\#\Lambda_0 \leq c$. So $c \times \#\Lambda_0 = \#\Lambda_0$. So $2^c \leq \#\Lambda_0$.From Lemma 3.6 we get $\#\Lambda_0 \leq 2^c$.So $\#\Lambda_0 = 2^c$.

3.13 COROLLARY *Given the hypotheses of the theorem, whenever* λ *and* μ *are distinct elements of* Λ_0 *then* $wA_\lambda \neq wA_\mu$. *So* A_λ *is not equivalent to* A_μ. *In particular, they cannot be isomorphic.*

PROOF:Apply Corollary 3.11.

We have seen that the small monotone complete C^*-algebras can be classified by elements of W and also by their spectroids. Since w maps every small von Neumann algebra to the zero of the semigroup, this classification might be very coarse, possibly W might be too small to distinguish between more than a few classes of algebras. But we shall see in Section 7 that this is far from the truth. By applying Theorem 3.12 for appropriate $(T, \mathbf{N})$ we shall see that $\#W = 2^c$.

Representing sets for Boolean algebras appear in [20] and the generalisation of representing functions from the context of Boolean algebras to that of monotone complete algebras is given in [14]; the notion of spectroid appears to be new.

REMARK It is straightforward to generalise the classification semigroup to more general classes of partially ordered set; but the Riesz decomposition property may fail.

Let M be a partially ordered set with a smallest element 0. Then M is said to be *feasible* if each monotone decreasing sequence has a greatest lower bound. Then we can easily extend the notion of spectroid to feasible sets. Since the positive cone of a monotone $\sigma-$complete C*-algebra is a feasible set, we can re-define the spectroid of the algebra as the spectroid of its positive cone.