Building C*-algebras and their K-theory

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Outline

Cayley graphs

From a group to a graph:
Let Γ be a group and $S = \{s_1, \ldots, s_n\}$ a set of generators.
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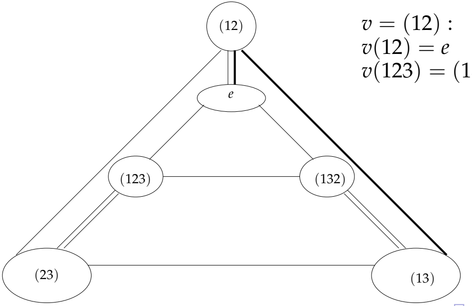
Example: $G = \mathcal{S}_3 = \{e, (12), (13), (23), (123), (132)\}, \quad S = \{(12), (123)\}.$



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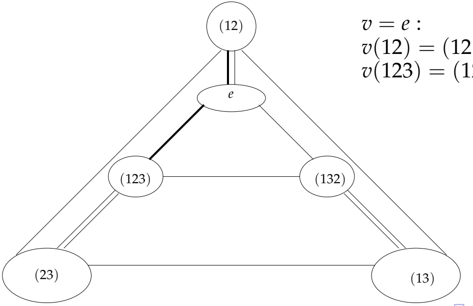

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- The Cayley graph of Γ with respect to the generating set $\{a, b\}$, $Cay(\Gamma, \{a, b\})$, is a homogeneous tree of degree 4.
- The vertices of the tree are elements of Γ *i.e.* reduced words in $S = \{a, b, a^{-1}, b^{-1}\}$.

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- if $x \in \Gamma$ then let $\Omega(x)$ be all semi-infinite words with the prefix x
- Then $\Omega(x)$ is open and closed in Ω and the sets $g\Omega(x)$ and $g(\Omega \setminus \Omega(x))$, where $g \in \Gamma$ and $x \in S$, form a base for the topology of Ω.

Cuntz-Krieger algebras

Left multiplication by $x \in \Gamma$ induces an action α of Γ by

$$\alpha(x)f(w) = f(x^{-1}w).$$

$C(\Omega) \rtimes \Gamma$ is generated by $C(\Omega)$ and the image of a unitary representation π of Γ such that $\alpha(g)f = \pi(g)f\pi^*(g)$ for $f \in C(\Omega)$ and $g \in \Gamma$ and every such C^*-algebra is a quotient of $C(\Omega) \rtimes \Gamma$.

For $x \in \Gamma$, let p_x denote the projection defined by the characteristic function $\mathbf{1}_{\Omega(x)} \in C(\Omega)$. For $g \in \Gamma$, we have

$$gp_xg^{-1} = \alpha(g)\mathbf{1}_{\Omega(x)} = \mathbf{1}_{g\Omega(x)}$$

and therefore for each $x \in S$,

$$p_x + xp_{x^{-1}}x^{-1} = \mathbf{1}.$$

$$p_a + p_{a^{-1}} + p_b + p_{b^{-1}} = \mathbf{1}$$

For $x \in S$ we define a *partial isometry* $s_x \in C(\Omega) \rtimes \Gamma$ by

$$s_x = x(\mathbf{1} - p_{x^{-1}}).$$

Then,

$$s_x s_x^* = x(\mathbf{1} - p_x)x^{-1} = p_x$$

and

$$s_x^* s_x = \mathbf{1} - p_{x^{-1}} = \sum_{y \neq x^{-1}} s_y s_y^*.$$

These relations show that the partial isometries s_x, for $x \in S$, generate the Cuntz-Krieger algebra $\mathcal{O}_A$

Where

$$A = \begin{pmatrix} \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} \end{pmatrix}$$

relative to $\{a, a^{-1}, b, b^{-1}\} \times \{a, a^{-1}, b, b^{-1}\}$.

We follow Cuntz-Krieger 1980 and construct a C^*-algebra from a matrix $A = (A(i,j))_{i,j\in\Sigma}$, Σ a finite set, $A(i,j) \in \{0,1\}$ and where every row and every column of A is non-zero. A C^*-algebra is then generated by partial isometries $S_i \neq 0$ $(i \in \Sigma)$ that act on a Hilbert space in such a way that their support projections $Q_i = S_i^* S_i$ and their range projections $P_i = S_i S_i^*$ satisfy the relations

$$P_i P_j = 0 \ (i \neq j), \ Q_i = \sum_{j\in\Sigma} A(i,j) P_j \ (i,j \in \Sigma). \tag{1}$$

The matrix A is used in symbolic dynamics as a transition matrix to construct one-sided and two-sided sub-shifts. The one-sided subshift σ_A acts on the compact space

$$X_A = \{(x_k)_{k \in \mathbb{N}} \in \Sigma^{\mathbb{N}} | A(x_k, x_{k+1}) = 1 \ (k \in \mathbb{N})\}$$

and is defined by

$$(\sigma(x))_k = x_{k+1} \ (k \in \mathbb{N}, x \in X_A).$$

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- X is a union of tessellated planes (apartments)
- for any two chambers there is an apartment containing both of them
- if two apartments F_1 and F_2 have non-trivial intersection, then there is an isomorphism from F_1 to F_2, fixing $F_1 \cap F_2$ pointwise.

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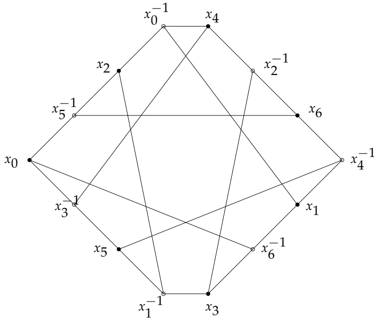
Take a sphere of a small radius at a point of the polyhedron. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point.

Example of a link



Interpreting x_i as POINTS and x_i^{-1} as LINES yields the incidence relations of a finite projective plane.

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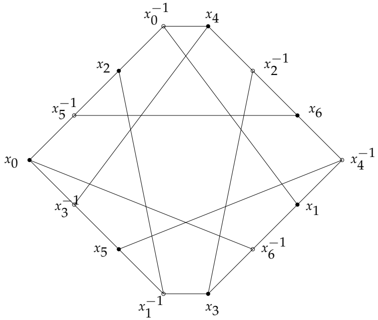

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Polyhedra and links

Theorem (Ballmann, Brin 1994)

Let X be a compact two-dimensional thick polyhedron with equilateral triangles as faces. If all links are incidence graphs of generalized m-gons, then the universal cover of the polyhedron is a two-dimensional building.

Natural question: how to construct such a polyhedron?

Polygonal presentation

Polygonal presentation is a set of words satisfying certain combinatorial properties (quite long and technical)

Theorem (AV,2002)

A polyhedron K which corresponds to a polygonal presentation $\mathcal{K}$ has graphs $G_1, G_2, \ldots, G_n$ as the links.

Groups acting on triangular buildings

Theorem (Kangaslampi, Vdovina)

There are 23 non-isomorphic groups acting on triangular buildings with the smallest generalized 4-gon as a link, for example

T_1 :	T_2 :
(x_1, x_2, x_7)	(x_1, x_{10}, x_1)
(x_1, x_8, x_{11})	(x_1, x_{15}, x_2)
(x_1, x_{14}, x_5)	(x_2, x_{11}, x_9)
(x_2, x_4, x_{13})	(x_2, x_{14}, x_3)
(x_{12}, x_4, x_2)	(x_3, x_7, x_4)
(x_4, x_9, x_3)	(x_3, x_{15}, x_{13})
(x_6, x_8, x_3)	(x_4, x_8, x_6)
(x_{14}, x_6, x_3)	(x_{12}, x_{11}, x_4)
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(x_{11}, x_{10}, x_7)	(x_7, x_{12}, x_7)
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Triangular buildings

We start with the infinite group

$$\Gamma = \langle x_0, x_1, \ldots, x_6 \mid x_0x_1x_3, x_1x_2x_4, \ldots, x_6x_0x_2 \rangle.$$

Let $S = \{x_0, x_1, \ldots x_6\}$. Then $Cay(\Gamma, S)$ is a one-skeleton of a thick Euclidean building with the following properties:

- lots of (equilateral) triangles (chambers)

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Γ belongs to a family considered by Edjvet,Howie 1989, and with relation to buildings by Cartwright/Steger/Mantero/Zappa 1993.

Building C*-algebras

- buildings (instead of trees)

if Γ is a group of type rotating automorphisms of Δ, then the C^*-algebra $C(\Lambda) \rtimes \Gamma$ is isomorphic to a higher rank Cuntz–Krieger algebra O_{A_1,A_2}. This is a particular (rank two) case of more general higher rank generalizations of Cuntz–Krieger algebras, associated to a finite collection of transition matrices A_j, $j = 1, \ldots, r$, with entries in $\{0, 1\}$, associated to shifts in r different directions, with the transition matrices satisfying compatibility conditions (see conditions (H0)–(H3) of Robertson-Steger).
The matrices give admissibility conditions for r-dimensional words in an assigned alphabet.

Building C*-algebras

- buildings (instead of trees)
- boundary Ω is defined by an equivalence relation on sectors (just as in the case of trees it is given by an equivalence relations on words).

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In the case of the Cuntz–Krieger algebra $O_A = C(\Lambda_\Gamma) \rtimes \Gamma$ one can choose as generators the partial isometries $S_{u,v} = T_{uv^{-1}} P_v$, for $u, v \in \Gamma$, with $t(u) = t(v)$ (same tail as edges in the Cayley graph).
Similarly, in the higher rank case, one has generators that are partial isometries $S_{u,v}$, where u and v are words in the given alphabet, with $t(u) = t(v)$. These satisfy the relations

$$\begin{aligned} S^*_{u,v} &= S_{v,u} & S_{u,v}S_{v,w} &= S_{u,w} \\ S_{u,v} &= \sum S_{uw,vw} & S_{u,u}S_{v,v} &= 0,\ \forall u \neq v \end{aligned} \qquad (2)$$

The sum here is over r-dimensional words w with source $s(w) = t(u) = t(v)$ and with shape $\sigma(w) = e_j$, for $j = 1, \dots, r$, where e_j is the j-th standard basis vector in $\mathbb{Z}^r$

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- Explicit computations of K-theory in infinitely many cases, but depending on the structure of the group (AV, 2006)
- (joint work with Johan Konter) Explicit computations of K-theory independent on the structure of the group
- More examples of multi-dimensional alphabets.

Results

Theorem

The order of the class $[\mathbf{1}]$ *of the identity element* $\mathbf{1}$ *of* $C(\Omega) \rtimes \Gamma$ *in* $K_0(C(\Omega) \rtimes \Gamma)$ *is* $q-1$*, where* Γ *is a Howie-CMSZ group for infinitely many* q*.*

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Theorem (J.Konter,AV)

*The order of the class [**1**] of the identity element **1** of $C(\Omega) \rtimes \Gamma$ in $K_0(C(\Omega) \rtimes \Gamma)$ is $q-1$, where Γ is a group acting on a triangular Euclidean building with three orbits and $q = 2^{2l-1}, l \in \mathbb{Z}$.*