Dimension and $\mathcal{Z}$-stability

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Classifying structures for operator algebras and dynamical systems

Dimension

Nuclear dimension generalizes covering dimension to C^*-algebras

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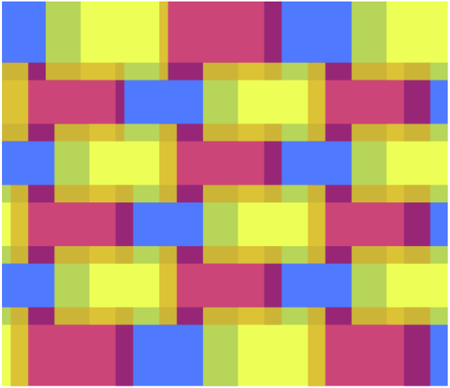

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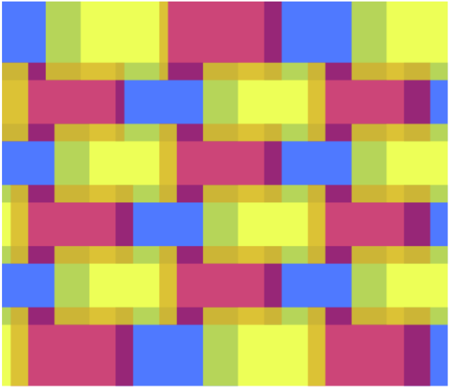
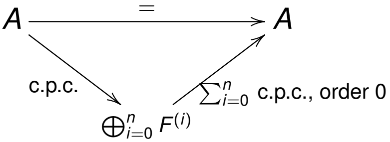
Nuclear dimension generalizes covering dimension to C^*-algebras



Comes naturally by treating approximations in the completely positive approximation property as **non-commutative partitions of unity.**

Dimension

Nuclear dimension $\leq n$:



Commuting pointwise-$\|\cdot\|$ approximately; $F^{(i)}$ is f.d.

Order 0 means orthogonality preserving,
$ab = 0 \Rightarrow \phi(a)\phi(b) = 0$.

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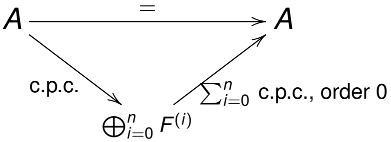

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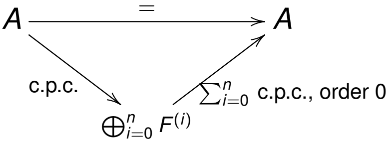

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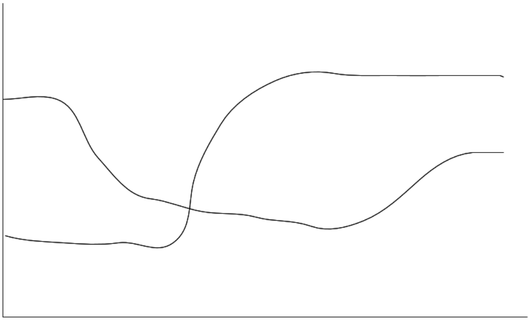
$$A \xrightarrow{=} A$$

$A \xrightarrow{\text{c.p.c.}} \bigoplus_{i=0}^{n} F^{(i)} \xrightarrow{\sum_{i=0}^{n} \text{ c.p.c., order } 0} A$



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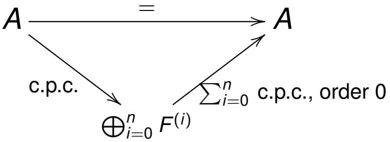

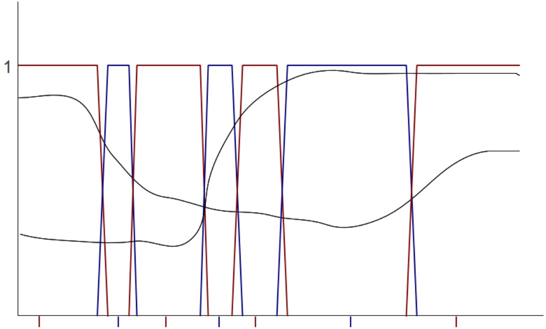

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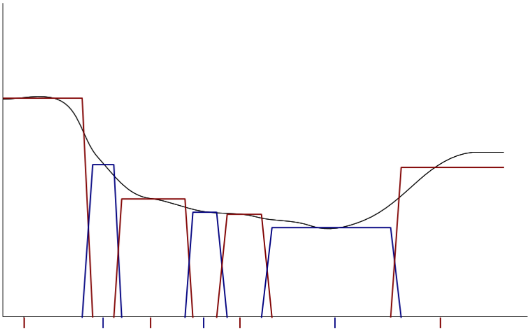

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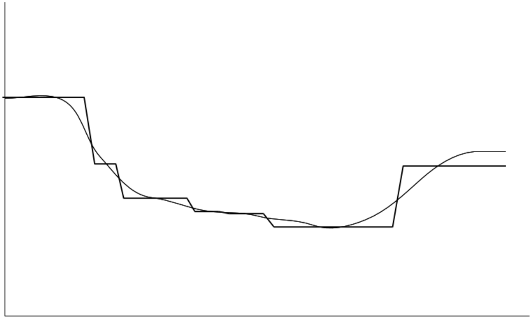

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Hyperfiniteness!

The Jiang-Su algebra

The Jiang-Su algebra $\mathcal{Z}$ is a C^*-algebra which:

• is self-absorbing ($\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$);

• has a lot of uniformity: any unital *-homomorphism $\mathcal{Z} \to \mathcal{Z}$ is approximately inner;

• makes good things happen to C^*-algebras by $\otimes$.

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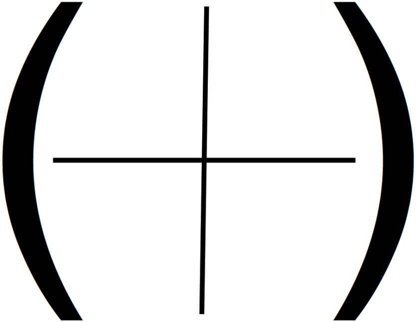
UHF algebras:



M_{2^∞}

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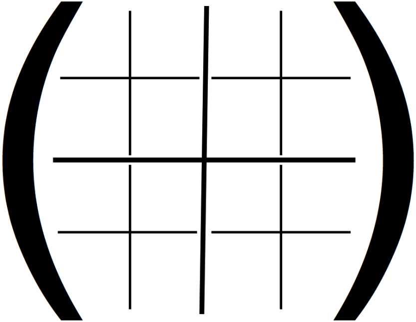

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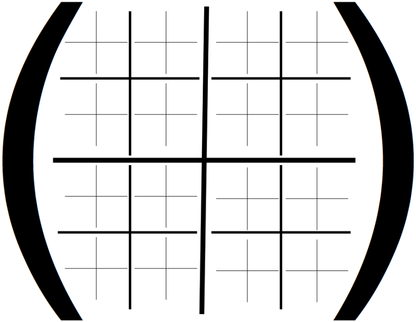

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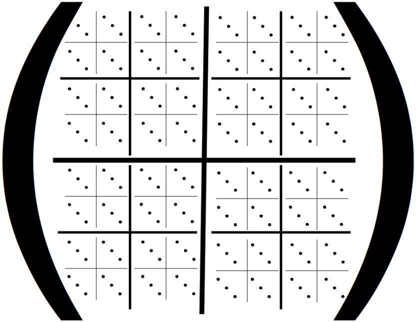

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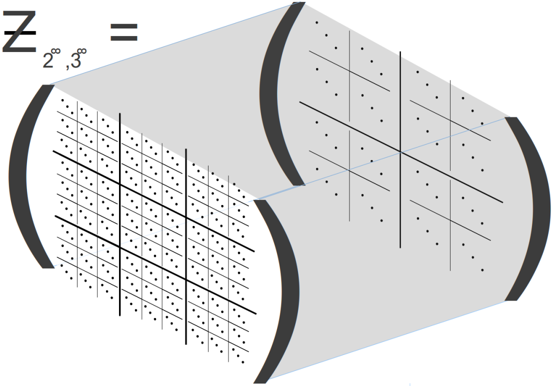

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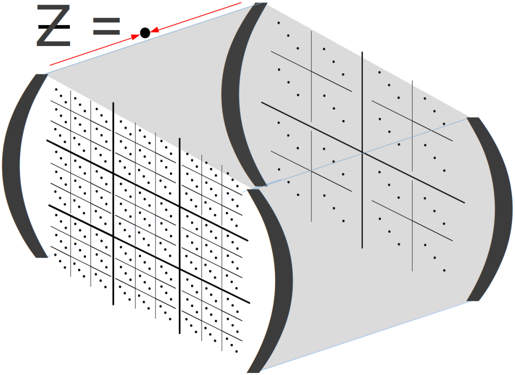

Nuclear dimension and $\mathcal{Z}$-stability

Conjecture (Toms-Winter)

Among simple, separable, nuclear, unital, non-type I C^*-algebras, the following are equivalent:

(i) A is $\mathcal{Z}$-stable;

(ii) A has finite nuclear dimension;

(iii) A has strict comparison of positive elements.

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Verifying the Toms-Winter conjecture

$\mathcal{Z}$-stability

$\dim_{nuc} < \infty$

strict comparison

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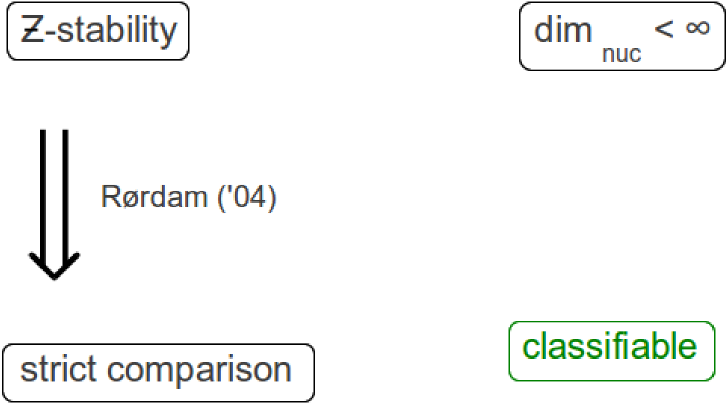

$\mathcal{Z}$-stability

$\dim_{nuc} < \infty$

$\Downarrow$ $\Uparrow$

with additional hypotheses:
Winter ('10), Matui-Sato ('11),
Toms-White-Winter ('12),
Kirchberg-Rordam ('12),
Sato ('12)

strict comparison

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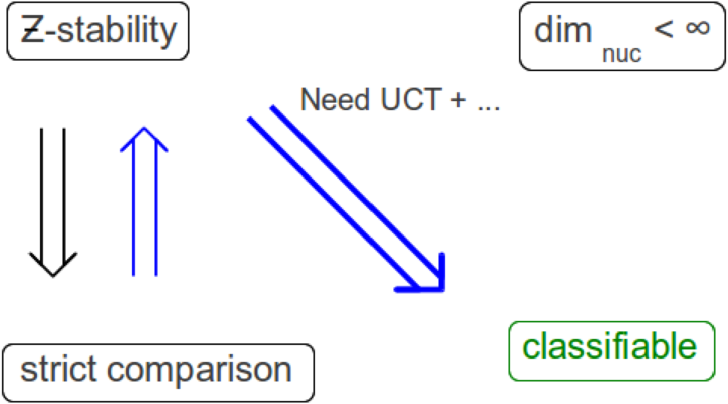

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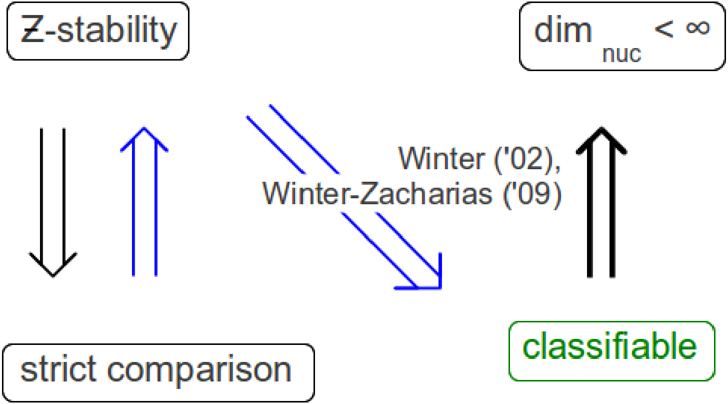

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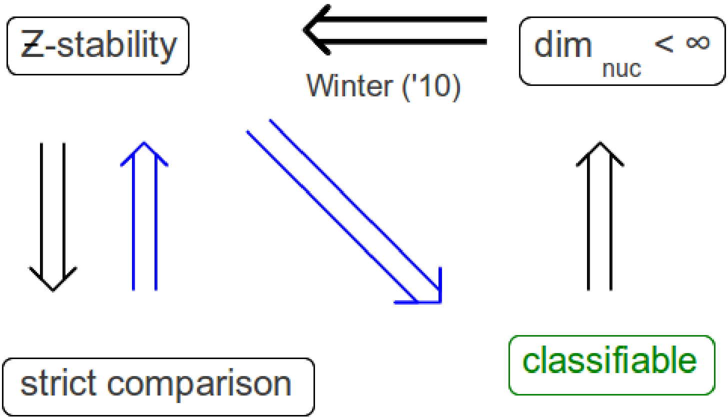

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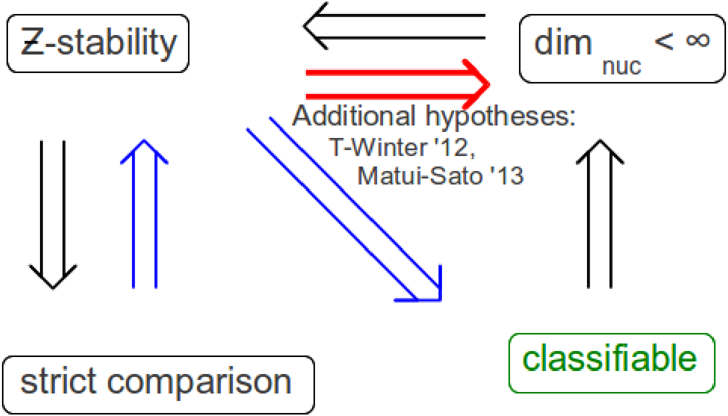

Dimension reduction

Theorem (Matui-Sato '13)

Let A be a simple, unital, nuclear, separable, quasidiagonal C^*-algebra with unique trace. Then $A \otimes \mathcal{Z}$ has decomposition rank at most 3.

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For any space X, $C_0(X) \otimes \cdot 1_{\mathcal{O}_2}) \subset C_0(X) \otimes \mathcal{O}_2$ factors (exactly!)

$$C_0(X) \to C_0(Y) \to C(X, \mathcal{O}_2),$$

where $\dim Y \leq 1$.

In particular, $C_0(X) \otimes \mathcal{O}_2$ has nuclear dimension at most 3.

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