Equilibrium states on C^*-algebras of higher-rank graphs

Classifying structures for operator algebras and dynamical systems

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Higher-rank Cuntz-Krieger algebras

- Robertson and Steger studied C^*-algbras arising from $\mathbb{Z}^k$ actions on $\tilde{A}_k$-buildings.
- Data consists of k commuting binary matrices such that $A_i A_j A_l$ is binary valued for distinct i, j, l.
- Resulting C^*-algebra generated by copies of the Cuntz-Krieger algebras $\mathcal{O}_{A_i}$ subject to commutation relations encoded by the products $A_i A_j$.

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- Resulting C^*-algebra generated by copies of the Cuntz-Krieger algebras $\mathcal{O}_{A_i}$ subject to commutation relations encoded by the products $A_i A_j$.
- Kumjian and Pask recognised that such a family of matrices encodes a sort of higher-rank graph:

Definition (KP). A *k-graph* is a countable category Λ with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the factorisation property: whenever $d(\lambda) = m + n$ there are unique $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$.

Notation

- Λ^n denotes $d^{-1}(n)$.
- Factorisation property gives $\Lambda^0 = \{\mathrm{id}_o : o \in \mathrm{Obj}(\Lambda)\}$.
- The domain and codomain maps determine maps $s, r : \Lambda \to \Lambda^0$; and then $r(\lambda)\lambda = \lambda = \lambda s(\lambda)$ for all λ.
- Write, for example, $v\Lambda^n$ for $r^{-1}(v) \cap \Lambda^n$.
- $\mathrm{MCE}(\mu, \nu) = \{\lambda : d(\lambda) = d(\mu) \vee d(\nu) \text{ and } \lambda = \mu\mu' = \nu\nu'\}$.

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For today:

- Λ is "finite" in the sense that each Λ^n is finite; and
- Λ is strongly connected: each $v\Lambda w \neq \emptyset$.

Skeletons

- Suppose that E is a finite directed graph, endowed with a colouring map $c : E^1 \to \{1, \ldots, k\}$.
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- Suppose that for $fgh \in c^{-1}(ijl)$,

$$fgh \sim g_1 f_1 h \sim g_1 h_1 f_2 \sim h_2 g_2 f_2 \quad \text{and}$$
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yields $f_2 = f^2$, $g_2 = g^2$ and $h_2 = h^2$.

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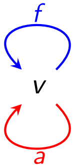
Proposition (PQR). $E^*/\sim$ is a k-graph and every finite k-graph arises this way.

Examples

In examples, we draw the skeleton with $c^{-1}(1)$ in blue and $c^{-1}(2)$ in red, and we list the relations $ef \sim f'e'$.



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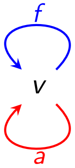

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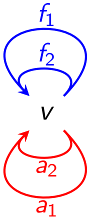

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Toeplitz-Cuntz-Krieger families

Definition (KP). Let Λ be a row-finite k-graph with no sources. Then $\mathcal{T}C^*(\Lambda)$ is universal for $\{t_\lambda : \lambda \in \Lambda\}$ such that:

(TCK1) $\{t_v : v \in E^0\}$ is a set of mutually orthogonal projections;

(TCK2) $t_\mu t_\nu = t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$.

(TCK3) $t_\mu^* t_\mu = t_{s(\mu)}$ for all μ, and

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$\mu \neq \nu \in \Lambda^n \implies \mathrm{MCE}(\mu, \nu) = \emptyset$. So $t_v \geq \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$.
$C^*(\Lambda)$ is universal for (TCK1)–(TCK4) plus

(CK) $t_v = \sum_{\mu \in v\Lambda^n} t_\mu t_\mu^*$ for all v, n.

Spanning elements

- For $\mu, \nu \in \Lambda$, have $t_\mu^* t_\nu = \sum_{\mu\alpha = \nu\beta \in \mathrm{MCE}(\mu,\nu)} t_\alpha t_\beta^*$, so

$$\mathcal{T}C^*(\Lambda) = \overline{\mathrm{span}}\{t_\mu t_\nu^* : s(\mu) = s(\nu)\}.$$

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- Universal property gives $\gamma : \mathbb{T}^k \to \mathrm{Aut}\,\mathcal{T}C^*(\Lambda)$ s.t. $\gamma_z(t_\lambda) = z^{d(\lambda)} t_\lambda$,
- so each $r \in [0,\infty)^k$ yields an action $\alpha^r : \mathbb{R} \to \mathrm{Aut}\,\mathcal{T}C^*(\Lambda)$ via the formula $\alpha_t^r = \gamma_{e^{itr}}$.

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- $\alpha_t^r(t_\mu t_\nu^*) = e^{itr\cdot(d(\mu)-d(\nu))} t_\mu t_\nu^*$, so $t \mapsto \alpha_t^r(t_\mu t_\nu^*)$ extends analytically to $\mathbb{C}$.

KMS states

- Given $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ and $\beta \in \mathbb{R}$, a state ϕ of A is KMS$_\beta$ for (A, α) if
$$\phi(ab) = \phi(b\alpha_{i\beta}(a))$$
whenever $t \mapsto \alpha_t(a), \alpha_t(b)$ have analytic extensions.
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KMS states

- Given $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ and $\beta \in \mathbb{R}$, a state ϕ of A is KMS_β for (A, α) if
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- It always suffices to check this KMS condition on your favourite set of analytic elements with dense linear span.
- Questions:
  - what are the KMS states for $(\mathcal{T}C^*(\Lambda), \alpha^r)$?
  - Which ones factor through $C^*(\Lambda)$?

First observation

Spse ϕ is KMS$_\beta$.
Put $m^\phi = (\phi(t_v))_{v\in\Lambda^0}$ and $A^n(v,w) = |v\Lambda^n w|$.
Following ideas of Enomoto-Fujii-Watatani, obtain

$$A^n m^\phi \leq e^{\beta r\cdot n} m^\phi.$$

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If ϕ factors through $C^*(\Lambda)$, we have equality.

Perron-Frobenius for commuting matrices

the A^n are not irreducible individually, so Perron-Frobenius doesn't immediately apply. Nevertheless, if $A_i := A^{e_i}$, then

Proposition (KP)

(1) If $y \in [0,\infty)^{\Lambda^0}$ and $\lambda_1, \ldots, \lambda_k$ satisfy $A_i y \leq \lambda_i y$ for all i, then $y_v > 0$ for all v and $\lambda_i \geq \rho(A_i)$ for all i; and

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(2) There is a unique $x^\Lambda \in [0,\infty)^{\Lambda^0}$ with $\|x^\Lambda\|_1 = 1$ which is a common eigenvector of the A_i; and then
$A^n x^\Lambda = \rho(A^n) x^\Lambda = \prod_{i=1}^k \rho(A_i)^{n_i} x^\Lambda$.

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Corollary

If ϕ is KMS_β for α^r, then $\beta r_i \ge \ln \rho(A_i)$ for all i. If ϕ factors through $C^*(\Lambda)$, the $\beta r_i = \ln \rho(A_i)$, and $m^\phi = x^\Lambda$.

Second observation

- If ϕ is KMS$_\beta$, then

$$\phi(t_\mu t_\nu^*) = e^{-\beta r\cdot d(\mu)}\phi(t_\nu^* t_\mu) = e^{-\beta r\cdot (d(\mu)-d(\nu))}\phi(t_\mu t_\nu^*).$$

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Proposition (aHLRS)

Suppose that $\beta r_i > \ln\rho(A_i)$ for all i. Then ϕ is KMS_β for $(\mathcal{T}C^*(\Lambda), \alpha^r)$ if and only if

$$\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta r\cdot d(\mu)} m^\phi_{s(\mu)} \text{ for all } \mu,\nu. \qquad (*)$$

Proof sketch.

"if" is a calculation. For "only if," need $\phi(t_\mu t_\nu^*) = 0$ if $d(\mu) \neq d(\nu)$ but $r \cdot d(\mu) \neq r \cdot d(\nu)$.

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$$\begin{aligned} |\phi(t_\mu t_\nu^*)| &\leq \sum_{\lambda \in s(\mu)\Lambda^{jn}} \phi(t_{\mu\lambda} t_{\mu\lambda}^*) \\ &= e^{-\beta r \cdot (jn + d(\mu))} \sum_w \sum_{\lambda \in s(\mu)\Lambda^{jn} w} \phi(t_w) \end{aligned}$$

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&= (e^{-\beta r\cdot n} A^n)^j_{s(\mu)} \phi(t_\mu t_\mu^*) \to 0.
\end{aligned}$$

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KMS states on $\mathcal{T}C^*(\Lambda)$

Theorem (aHLRS)

Suppose that $\beta r_i > \ln \rho(A_i)$ for all i. Then

1. *For $v \in \Lambda^0$, $\sum_{\mu \in \Lambda v} e^{-\beta r \cdot d(\mu)}$ converges to some $y_v > 1$. For $\epsilon \in [0, \infty)^{\Lambda^0}$, $m^\epsilon := \prod_{i=1}^{k} (1 - e^{-\beta r_i} A_i)^{-1} \epsilon$ satisfies $A_i m^\epsilon \leq e^{\beta r_i} m$ for all i, and $\|m^\epsilon\|_1 = 1$ iff $\epsilon \cdot y = 1$.*

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2. *If* $\epsilon \cdot y = 1$, *there is a* KMS_β *state* ϕ_ϵ *such that* $\phi_\epsilon(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta r \cdot d(\mu)} m^\epsilon_{s(\mu)}$.

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2. *If $\epsilon \cdot y = 1$, there is a KMS$_\beta$ state ϕ_ϵ such that $\phi_\epsilon(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta r \cdot d(\mu)} m^\epsilon_{s(\mu)}$.*
3. *$\epsilon \mapsto \phi_\epsilon$ is an affine isomorphism of $\{\epsilon : \epsilon \cdot y = 1\}$ with the KMS$_\beta$ simplex of $(\mathcal{T}C^*(\Lambda), \alpha^r)$.*

Proof sketch

(1) The terms in $\sum_{\mu\in\Lambda v} e^{-\beta r\cdot d(\mu)}$ are terms in the series expansion of $\prod_{i=1}^{k}(1-e^{-\beta r_i}A_i)^{-1}$, so the sum converges.

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$$\begin{aligned} e^{-\beta r_i} A_i (1 - e^{-\beta r_i} A_i)^{-1} &= \sum_{n=0}^{\infty} (e^{-\beta r_i} A_i)^{n+1} \\ &< \sum_{n=0}^{\infty} (e^{-\beta r_i} A_i)^{n} = (1 - e^{-\beta r_i} A_i)^{-1}. \end{aligned}$$

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(2) Define $T_\lambda \in \mathcal{B}(\ell^2(\Lambda))$ by $T_\lambda \xi_\mu = \delta_{s(\lambda),r(\mu)}\xi_{\lambda\mu}$. This is a TCK-family, so induces $\pi_T : \mathcal{T}C^*(\Lambda) \to \mathcal{B}(\ell^2(\Lambda))$.

Proof sketch

(1) The terms in $\sum_{\mu\in\Lambda v} e^{-\beta r\cdot d(\mu)}$ are terms in the series expansion of $\prod_{i=1}^{k}(1-e^{-\beta r_i}A_i)^{-1}$, so the sum converges. We calculate

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Check that $\Delta_\mu := e^{-\beta r\cdot d(\mu)}\epsilon_{s(\mu)}$ satisfies $\sum_{\mu\in\Lambda}\Delta_\mu = 1$. So

$$\phi_\epsilon(a) := \sum_\mu \Delta_\mu(\pi_T(a)\xi_\mu \mid \xi_\mu)$$

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KMS states on the Cuntz-Krieger algebra

Our proof that $\phi(t_\mu t_\nu^*) = 0$ if $d(\mu) \neq d(\nu)$ but $r \cdot d(\mu) = r \cdot d(\nu)$ breaks down if $\beta r_i = \ln \rho(A_i)$.

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Theorem (aHLRS)

There is a KMS_β *state for* $(C^*(\Lambda), \alpha^r)$ *if and only if* $\beta r_i = \ln \rho(A_i)$ *for all* i*; the formula* $\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} \rho(A^{d(\mu)})^{-1} x^\Lambda_{s(\mu)}$ *always defines such a state. If the* $\ln \rho(A_i)$ *are rationally independent, then this is the only KMS state for* $(C^*(\Lambda), \alpha^r)$.

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Proof.

We saw earlier that $\beta r_i = \ln \rho(A_i)$ is necessary. A weak*-compactness argument proves existence. The uniqueness follows from our calculation

$$\phi(t_\mu t_\nu^*) = e^{-\beta r \cdot d(\mu)} \phi(t_\nu^* t_\mu) = e^{-\beta r \cdot (d(\mu) - d(\nu))} \phi(t_\mu t_\nu^*)$$

earlier. □