Amenability of some groupoid extensions

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(joint work with Dana Williams)

Introduction

It is well known that for a locally compact groupoid G with Haar system,

$$G \quad \text{amenable} \quad \Rightarrow \quad C^*(G) \quad \text{nuclear}$$

and that this is a convenient way to prove the nuclearity of some C*-algebras.

A few years ago, I was recruited by an Australian team to help them to show that the C*-algebras of their topological higher rank graph C*-algebras were nuclear.

Deaconu-Renault groupoids

Just as graph C*-algebras, they can be described as C*-algebras of groupoids of the Deaconu-Renault type.

Here is the construction: let X be a topological space and T a local homeomorphism from an open subset dom(T) of X onto an open subset ran(T) of X. Then

$$G(X,\mathbb{N},T) = \{(x, m-n, y) : m, n \in \mathbb{N}, T^m x = T^n y\}$$

has a natural étale groupoid structure.

This is a semi-direct product construction and can be extended to arbitrary semi-groups $P \subset Q$ where Q is a not necessarily abelian group. Their essential feature is the canonical cocycle

$$c : G(X,P,T) \to Q$$

given by $c(x, mn^{-1}, y) = mn^{-1}$.

Two previous results

In our case $Q = \mathbb{Z}^d$ is abelian and the kernel $c^{-1}(0)$ is approximately proper. I quoted

Proposition (ADR 2000)

Let $c : G \to Q$ be a continuous cocycle. If $c(G^x) = Q$ for all $x \in G^{(0)}$, the amenability of Q and of $c^{-1}(0)$ imply the amenability of G.

to conclude that $G(X, P, T)$ is amenable. However, the above condition (strong surjectivity of c) is not always satisfied. Fortunately, we were saved by

Proposition (Spielberg 2011)

Let $c : G \to Q$ continuous, where G is étale and Q is a countable discrete abelian group. Then the amenability of $c^{-1}(0)$ implies the amenability of G.

A better result?

Clearly, there should be a result including these two cases. Moreover, the proof given by J. Spielberg goes against the groupoid philosophy: it makes a detour through C*-algebras, invoking a result of J. Quigg (and also C. K. Ng) on discrete coactions to obtain that $C^*(G)$ is nuclear.

The question is, given a locally compact groupoid with Haar system G, a locally compact group Q and a continuous cocycle $c : G \to Q$, find a sufficient condition on c such that the amenability of Q and of $c^{-1}(e)$ imply the amenability of G.

Our result

Theorem (R-Williams 2013)

Let G be a locally compact groupoid with Haar system G, Q a locally compact group and $c : G \to Q$ a continuous cocycle. Assume that Q and $c^{-1}(e)$ are amenable and that there exists a countable subset $D \subset Q$ such that

$$\forall x \in G^{(0)}, \quad c(G^x)D = Q,$$

then G is amenable.

This theorem covers the two previous results. More generally, it covers the case when the effective range of c, namely

$$\{(r(\gamma), c(\gamma)) : \gamma \in G\} \subset G^{(0)} \times Q$$

is an open subset of $G^{(0)} \times Q$ and $G^{(0)}$ and Q are second countable.

Borel amenability

Although it is likely that our theorem admits a direct proof, we obtain it by juggling with two notions of amenability, topological and Borel, which I recall below

Definition (Jackson-Kechris-Louveau, 2002)

A Borel groupoid G is said to be *Borel amenable* if there exists a *Borel approximate invariant mean*, i.e. a sequence $(m_n)_{n\in\mathbb{N}}$, where each m_n is a family $(m_n^x)_{x\in G^{(0)}}$ of probability measures m_n^x on $G^x = r^{-1}(x)$ such that:

1. for all $n \in \mathbb{N}$, m_n is Borel in the sense that for all bounded Borel functions f on G, $x \mapsto \int f dm_n^x$ is Borel;
2. $\|\gamma m_n^{s(\gamma)} - m_n^{r(\gamma)}\|_1 \to 0$ for all $\gamma \in G$.

topological amenability

Definition (R,1980; Anantharaman-R, 2000)

A locally compact groupoid G is said to be *topologically amenable* if there exists a *topological approximate invariant mean*, i.e. a sequence $(m_n)_{n\in\mathbb{N}}$, where each m_n is a family $(m_n^x)_{x\in G^{(0)}}$, m_n^x being a finite positive measure of mass not greater than one on $G^x = r^{-1}(x)$ such that

1. for all $n \in \mathbb{N}$, m_n is continuous in the sense that for all $f \in \mathcal{C}_c(G)$, $x \mapsto \int f dm_n^x$ is continuous;
2. $\|m_n^x\|_1 \to 1$ uniformly on the compact subsets of $G^{(0)}$;
3. $\|\gamma m_n^{s(\gamma)} - m_n^{r(\gamma)}\|_1 \to 0$ uniformly on the compact subsets of G.

Borel and topological amenability coincide

It turns out that, for groupoids we are interested in, both notions coincide:

Theorem (R, 2013)

Let (G, λ) be a σ-compact locally compact groupoid with Haar system. Then the following properties are equivalent:

- *G is topologically amenable;*
- *G is Borel amenable*

Therefore, we have the flexibility to use one or the other, whichever is the most convenient.

Groupoid equivalence

One can define Borel and topological groupoid equivalence:

Definition

Two Borel [resp. topological] groupoids G, H are said to be equivalent if there exists a Borel [resp. topological] space Z endowed with a left free and proper action of G and a right free and proper action of H such that the moment maps give identification maps

$$r : Z/H \to G^{(0)} \quad \text{and} \quad s : G\backslash Z \to H^{(0)}.$$

In the Borel case, proper means properly amenable, i.e. the existence of an invariant system of probability measures for the moment map.

Amenability is invariant under equivalence

Here is an important property of amenability

Theorem (ADR 2000, R-Williams 2013)

For Borel [resp. locally compact] groupoids, Borel [resp. topological] amenability is invariant under Borel [resp. topological] equivalence.

Both proofs follow the same pattern.

Skew-product

Given a cocycle $c : G \to Q$, where G is a groupoid and Q is a group, one can construct a groupoid much better behaved than the kernel $c^{-1}(e)$: it is the skew-product $G(c)$. It is simply the semi-direct product $G \ltimes Z$ where $Z = G^{(0)} \times Q$ and $\gamma(s(\gamma), a) = (r(\gamma), c(\gamma)a)$. As a set $G(c) = G \times Q$. Note that Q acts on $G(c)$ on the right by automorphisms: $(\gamma, a)b = (\gamma, ab)$.

Proposition (R, 1980)

Let G be a σ-compact locally compact groupoid with Haar system, Q a locally compact group and $c : G \to Q$ a continuous homomorphism. If Q and the skew-product $G(c)$ are topologically amenable, then G is topologically amenable.

The effective range of a cocycle

Definition

We define the effective range of a cocycle $c : G \to Q$ as

$$Y = \{(r(\gamma), c(\gamma)) : \gamma \in G\} \subset G^{(0)} \times Q$$

It is an invariant subset of the unit space of the skew-product $G(c)$ and a Borel subset since we assume that c is continuous and G is σ-compact.

Proposition

Under the above assumptions, the reduction $G(c)_{|Y}$ is Borel equivalent to $c^{-1}(e)$.

The equivalence is implemented by G.

Putting the pieces together

Thus, if $c^{-1}(e)$ is Borel amenable, $G(c)_{|Y}$ is also Borel amenable. Since right multiplication by $a \in Q$ is an automorphism, $G(c)_{|Ya}$ will also be Borel amenable for all $a \in Q$. The following lemma concludes the proof.

Lemma

Let G be a Borel groupoid. Assume that $G^{(0)}$ is the union of a countable family (Y_i) of invariant Borel subsets such that for all i, the reduction $G_{|Y_i}$ is Borel amenable. Then G is Borel amenable.

semi-group action

Definition

A right action of a semi-group P on a topological space X is a map

$$(x, n) \in X * P \quad \mapsto \quad xn \in X$$

where $X * P$ is an open subset of $X \times P$, such that

1. for all $x \in X$, $(x, e) \in X * P$ and $xe = x$;
2. if $(x, m) \in X * P$, then $(xm, n) \in X * P$ iff $(x, mn) \in X * P$; if this holds, we have $(xm)n = x(mn)$;
3. for all $n \in P$, the map defined by $T_n x = xn$ is a local homeomorphism with domain $U(n) = \{x \in X : (x, n) \in X * P\}$ and range $V(n) = \{xn : (x, n) \in X * P\}$.

semi-direct product

Definition

Let us say that a semi-direct group action (X, P, T) is directed if for all pair $(m, n) \in P \times P$ such that $U(m) \cap U(n)$ in non-empty, there exists $r = ma = nb$ such that $U(r) \supset U(m) \cap U(n)$.

Proposition

Let (X, P, T) be a directed semi-group action. Assume that P is a subsemi-group of a group Q. Then

$$G(X, P, T) = \{(x, mn^{-1}, y) \in X \times Q \times X : xm = yn\}$$

is a subgroupoid of $X \times Q \times X$ which carries an étale groupoid topology and a continuous cocycle $c : G(X, P, T) \to Q$.

amenability of a semi-direct product

Theorem

Let (X, P, T) be a directed semi-group action where X is a locally compact Hausdorff space. Assume that P is a quasi-lattice ordered subsemi-group of a countable amenable group Q. Then the semi-direct product groupoid $G(X, P, T)$ is topologically amenable.

To prove this result, we apply our theorem to the continuous cocycle $c : G(X, P, T) \to Q$. The only missing point is the amenability of $c^{-1}(e)$. This is an equivalence relation which can be written as an increasing union of proper equivalence relations

$$R_n = \{(x, y) \in X \times X : \exists m \leq n : xm = ym\}$$

higher rank graphs

Topological higher rank graphs provide semi-group actions, where the semi-group is $P = \mathbb{N}^d \subset Q = \mathbb{Z}^d$.

A k-graph is a topological category Λ together with a degree function $d : \Lambda \to P$ with a unique factorization property.

Each $\lambda \in \Lambda$ is viewed as a path and for each $m \leq n \leq d(\lambda)$ one defines the segment $\lambda[m, n]$ of this path.

There is an action on P on Λ, such that λp is defined if $p \leq d(\lambda)$ and

$$\lambda p[m, n] = \lambda[pm, pn]$$

whenever this makes sense.

One considers next the set X_Λ of all possible paths, finite or infinite. Under a technical condition called compact alignment, this set has a natural locally compact topology and the above action extends to an action T of P on X_Λ which is directed. The path groupoid is the semi-direct product $G(X_\Lambda, P, T)$.

Corollary (RSWY 2012)

Let Λ be a compactly aligned topological k-graph with path space X_Λ. Then its path groupoid $G(X_\Lambda, P, T)$ is amenable.

Since the graph C*-algebra A_Λ is isomorphic to $C^*(G(X, T))$, one obtains:

Corollary (RSWY 2012)

Let Λ be a compactly aligned topological k-graph. Then its C-algebra A_Λ is nuclear and satisfies UCT.*