Almost Calabi-Yau algebras associated to $SU(3)$ modular invariants

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Joint work with David E. Evans:

Ocneanu Cells for the $SU(3)$ $\mathcal{ADE}$ Graphs, Münster J. Math. **2** (2009), 95–142.

Realisation of $SU(3)$ modular invariants, Rev. Math. Phys. **21** (2009), 877–928.

A_2-planar algebras I, Quantum Topol., **1** (2010), 321–377.

A_2-planar algebras II: Planar modules, J. Funct. Anal. **261** (2011), 1923–1954.

Almost CY algebras, Comm. Math. Phys. **312** (2012), 179–222.

Homology of almost CY algebras, J. Algebra **368** (2012), 92–125.

CARDIFF UNIVERSITY PRIFYSGOL CAERDYDD

Aberystwyth, 16-20 September 2013

Coleg Cymraeg Cenedlaethol

Overview

- Braided subfactors and nimreps
- Almost Calabi-Yau algebras

Verlinde algebra

Braided system ${}_N\mathcal{X}_N$ of endomorphisms on type III_1 factor N

$$[\lambda] = [\lambda'] \qquad \Leftrightarrow \qquad \lambda = \mathrm{Ad}(u) \circ \lambda', \qquad \text{unitary } u \in N$$

$$[\lambda][\mu] = [\lambda\mu]$$

$$\lambda \in {}_N\mathcal{X}_N \Rightarrow \bar{\lambda} \in {}_N\mathcal{X}_N \quad ([\lambda\bar{\lambda}] = [1_N] \oplus \cdots\ )$$

$$\lambda\mu = \mathrm{Ad}u(\lambda,\mu)\,\mu\lambda, \qquad u(\lambda,\mu) =$$

Fusion rules: $\qquad \lambda\mu = \sum_\nu N_{\lambda\nu}^{\mu}\nu$

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$$N_\lambda N_\mu = \sum_\nu N_{\lambda\nu}^{\mu} N_\nu, \qquad N_\lambda = \{N_{\lambda\nu}^{\mu}\}_{\mu,\nu}$$

Verlinde algebra

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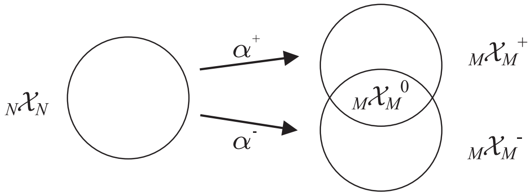
$$N_\lambda N_\mu = \sum_\nu N^\mu_{\lambda\nu} N_\nu, \qquad N_\lambda = \{N^\mu_{\lambda\nu}\}_{\mu,\nu}$$

Verlinde formula

$$N_\lambda = \sum_\sigma \frac{S_{\sigma\lambda}}{S_{\sigma 1}} S_\sigma S_\sigma^*, \qquad S_\sigma = \{S_{\sigma\mu}\}_\mu$$

α-induction

Braided subfactor $N \subset M$ $\qquad$ $(\theta \in \Sigma({}_N\mathcal{X}_N))$



$$\alpha^{\pm} : \quad \lambda \quad \longrightarrow \quad \alpha^{\pm}_{\lambda}$$

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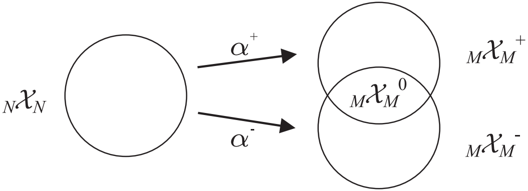


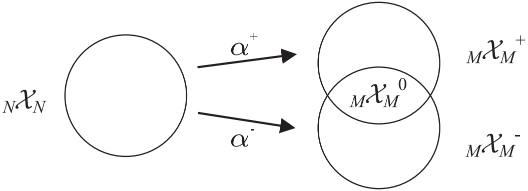
$$\alpha^{\pm} : \qquad \lambda \qquad \longrightarrow \qquad \alpha^{\pm}_{\lambda}$$

M-M sectors ${}_M\mathcal{X}_M^{\pm}$ from $\alpha^{\pm}_{\lambda}$

M-N sectors ${}_M\mathcal{X}_N$ from $\iota\lambda \quad (= \alpha^{\pm}_{\lambda}\iota\,) \qquad \iota : N \hookrightarrow M$

α-induction

Braided subfactor $N \subset M$ $\qquad$ $(\theta \in \Sigma({}_N\mathcal{X}_N))$



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M-N sectors ${}_M\mathcal{X}_N$ from $\iota\lambda \quad (= \alpha^{\pm}_{\lambda}\iota\,)$ $\qquad$ $\iota : N \hookrightarrow M$

$Z_{\lambda,\mu} = \langle \alpha^+_{\lambda}, \alpha^-_{\mu} \rangle$ is a modular invariant

Bockenhauer-Evans-Kawahigashi

Nimreps

Action of $\lambda \in {}_N\mathcal{X}_N$ on ${}_M\mathcal{X}_N$ gives M-N graph G_λ:

$$\xi\lambda = \sum_{\xi'} G^{\xi}_{\lambda,\xi'}\xi', \qquad G^{\xi}_{\lambda,\xi'} = \langle \xi\lambda, \xi' \rangle \in \mathbb{N}_0, \quad \xi, \xi' \in {}_M\mathcal{X}_N$$

nimrep: non-negative integer matrix representation of original Verlinde algebra

$$G_\lambda G_\mu = \sum_\nu N^{\mu}_{\lambda\nu} G_\nu, \qquad G_\lambda = (G^{\xi}_{\lambda,\xi'})_{\xi,\xi'}$$

Nimreps

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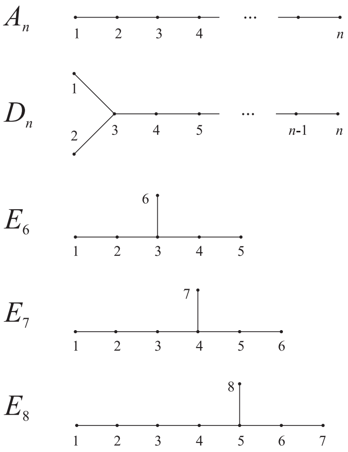
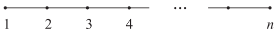
Spectrum of G_λ: Bockenhauer-Evans-Kawahigashi

$$\sigma(G_\lambda) = \{S_{\mu\lambda}/S_{\mu 1} \text{ with multiplicity } Z_{\mu,\mu}\}$$

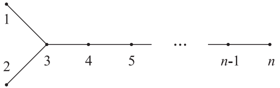
ADE Graphs



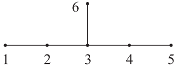
A_n



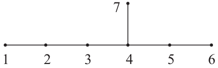
D_n



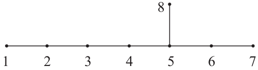
E_6



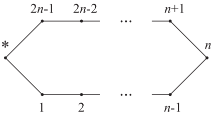
E_7



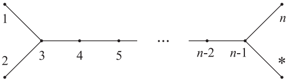
E_8



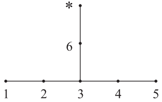
$A_{2n}^{(1)}$



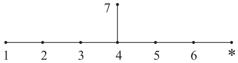
$D_n^{(1)}$



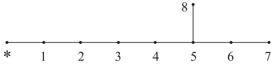
$E_6^{(1)}$



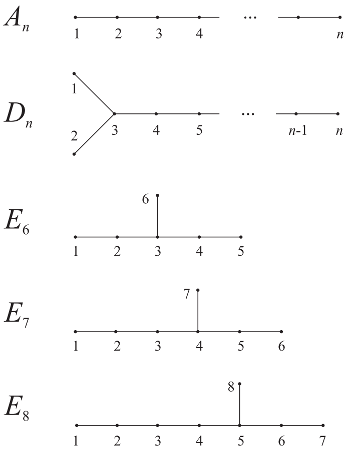
$E_7^{(1)}$



$E_8^{(1)}$



ADE Graphs



- $SU(2)$ modular invariants
- Realisation of $SU(2)$ modular invariants by braided subfactors
- Nimrep graphs:

$$G_\lambda G_\mu = \sum_\nu N_{\lambda\nu}^{\mu} G_\nu, \qquad G_\rho = \mathcal{G}$$

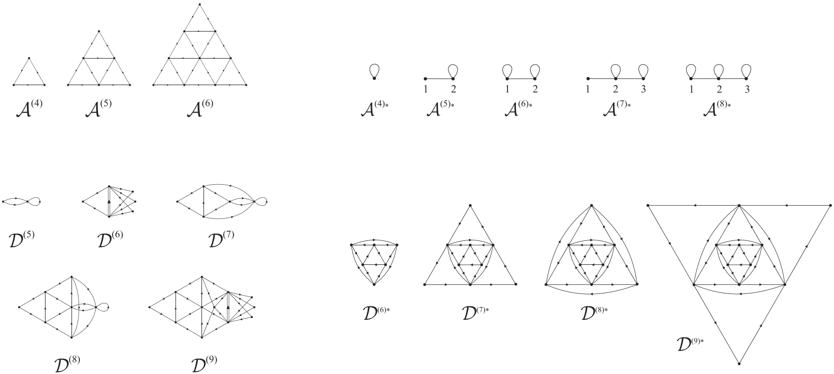
$$\sigma(G_\lambda) = \{S_{\mu\lambda}/S_{\mu 1}, \text{ mult. } Z_{\mu,\mu}\}$$

Example: E_6

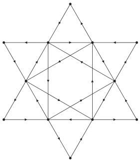
$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \sigma(E_6) &= \{\frac{S_{\rho,\mu}}{S_{1,\mu}} \text{ with multiplicity } Z_{\mu,\mu}\} \\ &= \{2\cos(\mu\pi/12) | \, \mu = 1, 4, 5, 7, 8, 11\} \end{aligned}$$

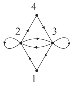
$SU(3)$ $\mathcal{ADE}$ Graphs



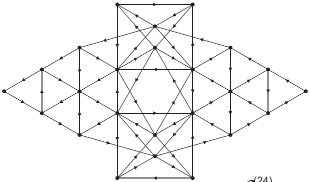
$SU(3)$ $\mathcal{ADE}$ Graphs



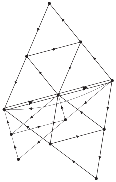
$\mathcal{E}^{(8)}$



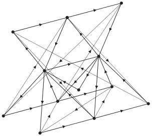
$\mathcal{E}^{(8)*}$



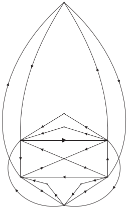
$\mathcal{E}^{(24)}$



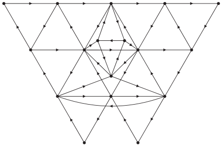
$\mathcal{E}_1^{(12)}$



$\mathcal{E}_2^{(12)}$



$\mathcal{E}_4^{(12)}$



$\mathcal{E}_5^{(12)}$

Subgroups of $SU(3)$

$\mathcal{ADE}$ graph	Subgroup $\Gamma \subset SU(3)$
(ADE)	B: finite subgroups of $SU(2) \subset SU(3)$
$\mathcal{A}^{(n)}$	A: $\mathbb{Z}_{n-2} \times \mathbb{Z}_{n-2}$
-	A: $\mathbb{Z}_m \times \mathbb{Z}_n \quad (m \neq n \neq 3)$
$\mathcal{D}^{(n)} \quad (n \equiv 0 \bmod 3)$	C: $\Delta(3(n-3)^2) = (\mathbb{Z}_{n-3} \times \mathbb{Z}_{n-3}) \rtimes \mathbb{Z}_3$
$\mathcal{D}^{(n)} \quad (n \not\equiv 0 \bmod 3)$	-
-	C: $\Delta(3n^2) = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$, $(n \not\equiv 0 \bmod 3)$
-	C: $(\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3 \quad (m \neq n)$
-	D: $\Delta(6n^2) = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3$
-	D: $(\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes S_3 \quad (m \neq n)$
$\mathcal{A}^{(n)*}$	-
$\mathcal{D}^{(n)*} \quad (n \geq 7)$	A: $\mathbb{Z}_{\lfloor (n+1)/2 \rfloor} \times \mathbb{Z}_3$
$\mathcal{E}^{(8)}$	$\mathrm{E} = \Sigma(36 \times 3) = \Delta(3.3^2) \rtimes \mathbb{Z}_4$
$\mathcal{E}^{(8)*}$	-
$\mathcal{E}_1^{(12)}$	$\mathrm{F} = \Sigma(72 \times 3)$
$\mathcal{E}_2^{(12)}$	$\mathrm{G} = \Sigma(216 \times 3)$
$\mathcal{E}_3^{(12)}$	B $\times \mathbb{Z}_3$: $BD_4 \times \mathbb{Z}_3$
$\mathcal{E}_4^{(12)}$	$\mathrm{L} = \Sigma(360 \times 3) \cong TA_6$
$\mathcal{E}_5^{(12)}$	$\mathrm{K} \cong TPSL(2,7)$
$\mathcal{E}^{(24)}$	-
-	$\mathrm{H} = \Sigma(60) \cong A_5$
-	$\mathrm{I} = \Sigma(168) \cong PSL(2,7)$
-	$\mathrm{J} \cong TA_5$

Subgroups of $SU(3)$

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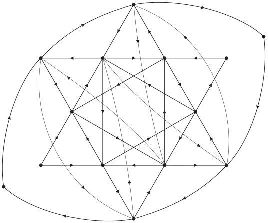

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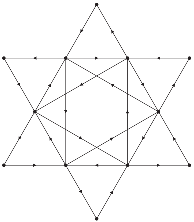
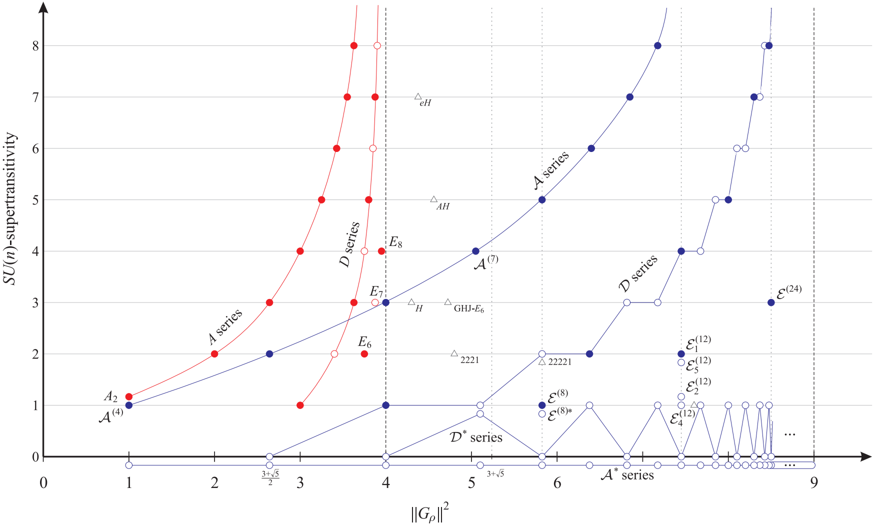


$\mathcal{E}^{(8)}$


SU(n)-supertransitivity
‖Gρ‖²
A series
D series
A series
D series
D* series
A* series
A2
A(4)
A(7)
E6
E7
E8
E(8)
E(8)*
E(12)1
E(12)5
E(12)2
E(12)4
E(24)
eH
AH
H
GHJ-E6
2221
22221
3+√5/2
3+√5
...

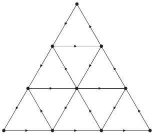
$SU(3)$ $\mathcal{ADE}$ Graphs

$$\triangle^{(\alpha\beta\gamma)} = i \xrightarrow{\alpha} j \xrightarrow{\beta} k \xrightarrow{\gamma} i$$

$$\rho \otimes \rho \otimes \rho \ni 1$$

Cell system, $W : \triangle^{(\alpha\beta\gamma)} \to \mathbb{C}$

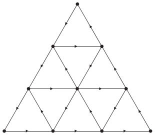
Ocneanu, Evans-P



$SU(3)$ $\mathcal{ADE}$ Graphs

$$\triangle^{(\alpha\beta\gamma)} = i \xrightarrow{\alpha} j \xrightarrow{\beta} k \xrightarrow{\gamma} i \qquad \rho \otimes \rho \otimes \rho \ni 1$$

Cell system, $W : \triangle^{(\alpha\beta\gamma)} \to \mathbb{C}$ Ocneanu, Evans-P



$SU(3)$ $\mathcal{ADE}$ Graphs:

- $SU(3)$ modular invariants
- Realisation of $SU(3)$ modular invariants by braided subfactors
  Ocneanu, Xu, Bockenhauer-Evans, Bockenhauer-Evans-Kawahigashi, Evans-P
- Nimrep graphs:

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$$\sigma(G_\lambda) = \{S_{\mu\lambda}/S_{\mu 1} \text{ with multiplicity } Z_{\mu,\mu}\}$$

Almost Calabi-Yau algebras

(graded) path algebra $\mathbb{C}\mathcal{G}$

$\mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}]$ cyclic paths up to cyclic permutation

Almost Calabi-Yau algebras

(graded) path algebra $\mathbb{C}\mathcal{G}$

$\mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}]$ cyclic paths up to cyclic permutation

Derivation $\partial_a : \mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}] \to \mathbb{C}\mathcal{G}$

$$\partial_a(a_1 \cdots a_n) = \sum_{j : a_j = a} a_{j+1} \cdots a_n a_1 \cdots a_{j-1}$$

(super)potential algebra: $\mathbb{C}\mathcal{G}/(\partial_a \Phi)$

Almost Calabi-Yau algebras

(graded) path algebra $\mathbb{C}\mathcal{G}$

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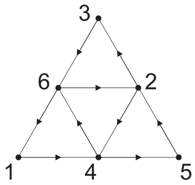
(super)potential algebra: $\mathbb{C}\mathcal{G}/(\partial_a \Phi)$

$\mathcal{G}$ an $SU(3)$ $\mathcal{ADE}$ graph, $\Phi = \sum_{i,j,k} W(\triangle^{(i,j,k)})\triangle^{(i,j,k)}$

Almost CY algebra:

$$A(\mathcal{G}, W) := \mathbb{C}\mathcal{G}/(\partial_a \Phi)$$

Example: $\mathcal{A}^{(5)}$



$$\Phi = \triangle^{(1,4,6)} + \triangle^{(2,3,6)} + \triangle^{(2,4,5)} + \widetilde{W}\triangle^{(2,4,6)}$$

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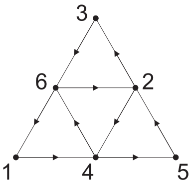

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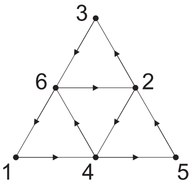

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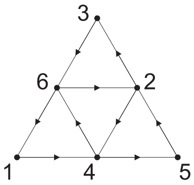

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Relations:

$$[146] = [461] = [245] = [524] = [362] = [623] = 0$$

$$[236] = -\widetilde{W}[246], \quad [452] = -\widetilde{W}[462], \quad [614] = -\widetilde{W}[624]$$

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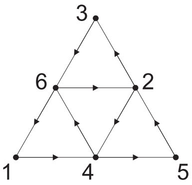

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Paths in $A(\mathcal{A}^{(5)}, W)$ (starting from vertices 1, 2):

$$[1], \quad [14], \quad [145], \qquad [2], \quad [23], \quad [24], \quad [236] = -\widetilde{W}[246]$$

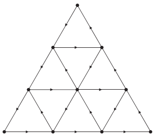
$$([1\underline{452}] = -\widetilde{W}[\underline{146}2] = 0, [\underline{236}1] = -\widetilde{W}[2\underline{461}] = 0, [2\underline{362}] = 0)$$

From ${}_N\mathcal{X}_N$ to almost Calabi-Yau algebras

Nimrep G and cell system W define monoidal functor
$F : {}_N\mathcal{X}_N \;\to\; {}_N\mathcal{X}_M$-bimod:

$$F(\lambda_{(m,l)}) = \bigoplus_{i,j \in \mathcal{G}_0} G_{\lambda_{(m,l)}}(i,j)\mathbb{C}_{i,j}$$

$\mathbb{C}_{i,j}$: 1-dimensional R-R bimodules, $\quad R = (\mathbb{C}\mathcal{G})_0$



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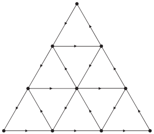

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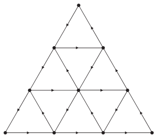

Define graded algebra:

$\bigoplus_{m=0}^{k} F(\lambda_{(m,0)}) \;\cong\; A(\mathcal{G}, W)$

Almost Calabi-Yau algebras

For $\mathcal{G}_\Gamma$, $\Gamma \subset SU(3)$, A is Calabi-Yau algebra of dimension 3:

$$0 \to A \otimes A \to A \otimes V^* \otimes A \to A \otimes V \otimes A \to A \otimes A \to A \to 0$$

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Almost Calabi-Yau algebra: Evans-P 2012

$$0 \to {}_1A_{\beta^{-1}} \to A \otimes A \to A \otimes V^* \otimes A \to A \otimes V \otimes A \to A \otimes A \to A \to 0$$

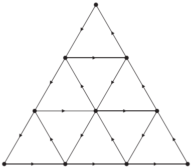
β Nakayama automorphism: $\qquad A^* = {}_1A_\beta$

$\beta =$ automorphism defined by $G_{\lambda_{(k,0)}}$ Evans-P 2012

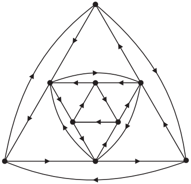
Nakayama automorphism for $A(\mathcal{G}, W)$

γ: automorphism of graph given by clockwise rotation by $2\pi/3$ ($\gamma^3 = \mathrm{id}$)

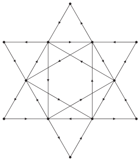
$$\beta = \begin{cases} \gamma^2 & \text{for} \quad \mathcal{A}^{(n)}, n \geq 4, \\ \gamma^{2n} & \text{for} \quad \mathcal{D}^{(n)*}, n \geq 5, \\ \gamma & \text{for} \quad \mathcal{E}^{(8)}, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$



$\mathcal{A}^{(n)}$



$\mathcal{D}^{(n)*}$



$\mathcal{E}^{(8)}$

Hochschild homology

$$0 \to {}_1A_{\beta^{-1}} \to A \otimes A \to A \otimes V^* \otimes A \to A \otimes V \otimes A \to A \otimes A \to A \to 0$$

$$\beta^3 = \mathrm{id} : \qquad \mathcal{N} := {}_1A_{\beta^{-1}}, \qquad \mathcal{N}^{(2)} := \mathcal{N} \otimes_A \mathcal{N} = {}_1A_{\beta^{-2}} = {}_1A_\beta$$

$$\mathcal{N} \otimes_A \mathcal{N} \otimes_A \mathcal{N} = A$$

Apply functor $- \otimes_A \mathcal{N}$ twice:

$$\cdots \to A \otimes A \to A \otimes \mathcal{N}^{(2)} \to A \otimes \widetilde{V} \otimes \mathcal{N}^{(2)} \to A \otimes V \otimes \mathcal{N}^{(2)} \to A \otimes \mathcal{N}^{(2)}$$
$$\to A \otimes \mathcal{N} \to A \otimes \widetilde{V} \otimes \mathcal{N} \to A \otimes V \otimes \mathcal{N} \to A \otimes \mathcal{N}$$
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Let $A^e := A^{op} \otimes_R A$, $\qquad$ A-A bimodule $\leftrightarrow$ left A^e-module

Hochschild homology: $\qquad HH_n(A) = \mathrm{Tor}_n^{A^e}(A, A)$

apply functor $- \otimes_{A^e} A$ $\quad \rightsquigarrow \quad$ Hochschild homology complex

Hochschild cohomology

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Hochschild cohomology: $HH^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$

apply functor $\mathrm{Hom}_{A^e}(\,-\,, A)$ $\rightsquigarrow$ Hochschild cohomology complex

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$$\begin{aligned} HH^i(A) &\cong HH_{3-i}(A)[-3], && i = 1, 2, \\ HH^i(A) &\cong HH_{15-i}(A)[-3h-3], && i = 3, \ldots, 12, \\ HH^{12+i}(A) &\cong HH^i(A)[-3h], && i = 1, 2, \ldots, \end{aligned}$$

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Graph $\mathcal{E}^{(8)*}$

$$\begin{aligned} &HH^0(A) \cong C^*[5] \oplus L, && HH^1(A) \cong C^*[5], \\ &HH^2(A) \cong C[-3], && HH^3(A) \cong C[-3] \oplus K^*[-3], \\ &HH^4(A) \cong C^*[-3] \oplus K[-3], && HH^{4+i}(A) \cong HH^i(A)[-8], \end{aligned}$$

$H_C(t) = 2t + t^2 + t^3 + t^5$, $H_L(t) = 4t^5$, $H_K(t) = 2$