Continuous orbit equivalence of one-sided shifts of finite type

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Main theorem

For an $N \times N$ matrix A with entries in $\{0,1\}$, we consider the associated one-sided shift of finite type (X_A, σ_A), where

$$X_A = \left\{ (x_n)_n \in \{1, \ldots, N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \quad \forall n \in \mathbb{N} \right\}.$$

Theorem (K. Matsumoto and M)

For irreducible matrices A and B, the following are equivalent.

1. (X_A, σ_A) *and* (X_B, σ_B) *are continuously orbit equivalent.*
2. *The étale groupoids* G_A *and* G_B *are isomorphic.*
3. *There exists an isomorphism* $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ *such that* $\Psi(C(X_A)) = C(X_B)$.
4. $(\mathrm{BF}(A^t), u_A) \cong (\mathrm{BF}(B^t), u_B)$ *and* $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$.

Continuous orbit equivalence

Two one-sided shifts of finite type (X_A, σ_A) and (X_B, σ_B) are said to be continuously orbit equivalent if there exists a homeomorphism $h : X_A \to X_B$, continuous maps $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \forall x \in X_A$$

and

$$\sigma_A^{k_2(x)}(h^{-1}(\sigma_B(x))) = \sigma_A^{l_2(x)}(h^{-1}(x)) \quad \forall x \in X_B.$$

For the two matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

(X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent, but not topologically conjugate.

Étale groupoids

The étale groupoid G_A for (X_A, σ_A) is given by

$$G_A = \left\{ (x, n, y) \in X_A \times \mathbb{Z} \times X_A \mid \exists k, l \in \mathbb{Z}_+,\ n = k{-}l,\ \sigma_A^k(x){=}\sigma_A^l(y) \right\}$$

with

$$(x, n, y) \cdot (x', n', y') = (x, n{+}n', y') \quad \text{if} \quad y = x',$$
$$(x, n, y)^{-1} = (y, -n, x).$$

The topology of G_A is generated by the sets

$$\left\{ (x, k{-}l, y) \in G_A \mid x \in V,\ y \in W,\ \sigma_A^k(x) = \sigma_A^l(y) \right\},$$

where $V, W \subset X_A$ are open and $k, l \in \mathbb{Z}_+$.

The unit space $(G_A)^{(0)} = \{(x, 0, x) \mid x \in X_A\}$ is identified with X_A.

$C_r^*(G_A)$ is equal to the Cuntz-Krieger algebra $\mathcal{O}_A$.

Bowen-Franks group

Let A be an $N \times N$ matrix with entries in $\{0, 1\}$ (or $\mathbb{Z}_+$).
The Bowen-Franks group is

$$\mathrm{BF}(A) = \mathbb{Z}^N / (\mathrm{id} - A)\mathbb{Z}^N.$$

The Bowen-Franks group is an invariant of flow equivalence for two-sided shifts of finite type (R. Bowen and J. Franks 1977).

The Bowen-Franks group is also related to the Cuntz-Krieger algebra $\mathcal{O}_A$. Namely, the Ext group of $\mathcal{O}_A$ is isomorphic to $\mathrm{BF}(A)$ (J. Cuntz and W. Krieger 1980), and $(K_0(\mathcal{O}_A), [1])$ is isomorphic to $(\mathrm{BF}(A^t), u_A)$, where $u_A \in \mathrm{BF}(A^t)$ is the equivalence class of $(1, 1, \ldots, 1) \in \mathbb{Z}^N$ (J. Cuntz 1981).

Main theorem

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For irreducible matrices A and B, the following are equivalent.

(1) (X_A, σ_A) *and* (X_B, σ_B) *are continuously orbit equivalent.*

(2) *The étale groupoids* G_A *and* G_B *are isomorphic.*

(3) *There exists an isomorphism* $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ *such that* $\Psi(C(X_A)) = C(X_B)$.

(4) $(\mathrm{BF}(A^t), u_A) \cong (\mathrm{BF}(B^t), u_B)$ *and* $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$.

(1)⇔(2) is easy.
(2)⇒(3) is clear, and (3)⇒(2) is due to J. Renault.

As mentioned before, $(K_0(\mathcal{O}_A), [1]) \cong (\mathrm{BF}(A^t), u_A)$.
So, $\mathcal{O}_A \cong \mathcal{O}_B$ implies $(\mathrm{BF}(A^t), u_A) \cong (\mathrm{BF}(B^t), u_B)$.

Flow equivalence

We denote the two-sided shift by $(\bar{X}_A, \bar{\sigma}_A)$.
The suspension space of $(\bar{X}_A, \bar{\sigma}_A)$ is the quotient of $\bar{X}_A \times \mathbb{R}$ by the relations

$$(x, t) \sim (\bar{\sigma}_A(x), t + 1) \quad x \in \bar{X}_A,\ t \in \mathbb{R}.$$

$(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are said to be flow equivalent if there exists an orientation preserving homeomorphism between their suspension spaces.

B. Parry and D. Sullivan in 1975 showed that $\det(\mathrm{id} - A)$ is an invariant of flow equivalence.

J. Franks in 1984 proved that the pair of $\mathrm{BF}(A)$ and $\det(\mathrm{id} - A)$ is a complete invariant of flow equivalence.

Clearly, $\# \mathrm{BF}(A) < \infty \iff \det(\mathrm{id} - A) \neq 0$,
and in this case $\# \mathrm{BF}(A) = |\det(\mathrm{id} - A)|$.

Proof of (4)⇒(3)

Theorem (K. Matsumoto)

If $(\mathrm{BF}(A^t), u_A) \cong (\mathrm{BF}(B^t), u_B)$ *and* $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$, *then* $(\mathcal{O}_A, C(X_A)) \cong (\mathcal{O}_B, C(X_B))$.

Proof.

By the theorem of Franks, $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent.
Then, by a result of Cuntz and Krieger, there exists an isomorphism $\Psi : \mathcal{O}_A \otimes \mathbb{K} \to \mathcal{O}_B \otimes \mathbb{K}$ such that $\Psi(C(X_A) \otimes \mathcal{C}) = C(X_B) \otimes \mathcal{C}$, where $\mathcal{C} \subset \mathbb{K}$ is the abelian subalgebra of diagonal operators.
Since $(\mathrm{BF}(A^t), u_A) \cong (\mathrm{BF}(B^t), u_B)$, thanks to a theorem of D. Huang, we may further assume $K_0(\Psi)(u_A) = u_B$.
Thus $K_0(\Psi)([1_{\mathcal{O}_A} \otimes e]) = [1_{\mathcal{O}_B} \otimes e]$, where $e \in \mathbb{K}$ is a minimal projection.
Hence we get $(\mathcal{O}_A, C(X_A)) \cong (\mathcal{O}_B, C(X_B))$. □

Main theorem

Theorem (K. Matsumoto and M)

For irreducible matrices A and B, the following are equivalent.

(1) (X_A, σ_A) *and* (X_B, σ_B) *are continuously orbit equivalent.*

(2) *The étale groupoids* G_A *and* G_B *are isomorphic.*

(3) *There exists an isomorphism* $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ *such that* $\Psi(C(X_A)) = C(X_B)$.

(4) $(\mathrm{BF}(A^t), u_A) \cong (\mathrm{BF}(B^t), u_B)$ *and* $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$.

It remains for us to show that
(1) or (2) or (3) implies $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$.

To this end, it suffices to show that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent, because $\det(\mathrm{id} - A)$ is an invariant for flow equivalence.

Boyle-Handelman's theorem

For a two-sided SFT $(\bar{X}_A, \bar{\sigma}_A)$, we let

$$\bar{H}^A = C(\bar{X}_A, \mathbb{Z}) / \{\xi - \xi \circ \bar{\sigma}_A \mid \xi \in C(\bar{X}_A, \mathbb{Z})\},$$

$$\bar{H}^A_+ = \{[\xi] \in \bar{H}^A \mid \xi(x) \geq 0 \quad \forall x \in \bar{X}_A\}.$$

$(\bar{H}^A, \bar{H}^A_+)$ is called the ordered cohomology group.

Theorem (M. Boyle and D. Handelman 1996)

For irreducible matrices A and B, the following are equivalent.

1. $(\bar{X}_A, \bar{\sigma}_A)$ *and* $(\bar{X}_B, \bar{\sigma}_B)$ *are flow equivalent.*
2. $(\bar{H}^A, \bar{H}^A_+) \cong (\bar{H}^B, \bar{H}^B_+)$.

2-sided versus 1-sided

For a one-sided SFT (X_A, σ_A), we let

$$H^A = C(X_A, \mathbb{Z})/\{\xi - \xi \circ \sigma_A \mid \xi \in C(X_A, \mathbb{Z})\},$$

$$H^A_+ = \{[\xi] \in H^A \mid \xi(x) \geq 0 \quad \forall x \in X_A\}.$$

Lemma

The canonical projection $\rho : \bar{X}_A \to X_A$ induces an isomorphism $\tilde{\rho}$ from (H^A, H^A_+) to $(\bar{H}^A, \bar{H}^A_+)$.

Let us see the surjectivity of $\tilde{\rho}$.
Take $\zeta \in C(\bar{X}_A, \mathbb{Z})$. There exists $n \in \mathbb{N}$ such that $\zeta(x)$ depends only on finitely many coordinates $x_{-n}, \ldots, x_0, \ldots, x_n$ of $x \in \bar{X}_A$.
Hence there exists $\xi \in C(X_A, \mathbb{Z})$ such that $\zeta \circ \bar{\sigma}_A^{n+1} = \xi \circ \rho$,
and so $\tilde{\rho}([\xi]) = [\xi \circ \rho] = [\zeta \circ \bar{\sigma}_A^{n+1}] = [\zeta]$ in $\bar{H}^A$.

Cohomology of groupoid (1/2)

For an étale groupoid G, we let $\mathrm{Hom}(G, \mathbb{Z})$ be the set of continuous homomorphisms $\omega : G \to \mathbb{Z}$. For $\xi \in C(G^{(0)}, \mathbb{Z})$, we can define $\partial(\xi) \in \mathrm{Hom}(G, \mathbb{Z})$ by $\partial(\xi) = \xi(r(g)) - \xi(s(g))$.
The cohomology group $H^1(G) = H^1(G, \mathbb{Z})$ is the quotient of $\mathrm{Hom}(G, \mathbb{Z})$ by $\{\partial(\xi) \mid \xi \in C(G^{(0)}, \mathbb{Z})\}$.

For the SFT groupoid G_A, it is known that $H^1(G_A)$ is isomorphic to H^A. The isomorphism is given as follows. Let $\omega \in \mathrm{Hom}(G, \mathbb{Z})$. For every $x \in X_A$, we consider an element $(x, 1, \sigma_A(x)) \in G_A$ and define $\xi \in C(X_A, \mathbb{Z})$ by

$$\xi(x) = \omega((x, 1, \sigma_A(x))).$$

Then $\omega \mapsto \xi$ gives rise to an isomorphism $\Phi : H^1(G_A) \to H^A$.

We have to characterize the positive cone of $H^1(G_A) \cong H^A$ in terms of groupoids.

Cohomology of groupoid (2/2)

Let $g \in G$ be such that $r(g) = s(g)$ and let $U \subset G$ be a compact open G-set containing g. Then $\pi_U = (r|U) \circ (s|U)^{-1}$ is a homeomorphism from $s(U)$ to $r(U)$.
We say that g is attracting if there exists U such that $r(U) \subset s(U)$ and

$$\lim_{n \to +\infty} (\pi_U)^n(y) = r(g) \quad \forall y \in s(U).$$

Lemma

There exists an isomorphism $\Phi : H^1(G_A) \to H^A$ *such that*

$$\Phi([\omega]) \in H^A_+ \iff \omega(g) \geq 0 \quad \forall \textit{attracting } g \in G_A.$$

Proof of $G_A \cong G_B \Rightarrow \det(\mathrm{id} - A) = \det(\mathrm{id} - B)$

Theorem (K. Matsumoto and M)

If irreducible one-sided SFT (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent, then the two-sided SFT $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent. In particular, $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$.

Proof.

There exists an isomorphism $\varphi : G_A \to G_B$. For any $g \in G_A$, g is attracting iff $\varphi(g)$ is attracting. Hence we get $(H^A, H^A_+) \cong (H^B, H^B_+)$.
This implies $(\bar{H}^A, \bar{H}^A_+) \cong (\bar{H}^B, \bar{H}^B_+)$.
By the Boyle-Handelman's theorem, we can conclude that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent.
By the Parry-Sullivan's theorem, we have $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$. □

This completes the proof of the main theorem.

Example

For the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

we have

$$\mathrm{BF}(A^t) = \mathrm{BF}(B^t) = \mathrm{BF}(C^t) = 0,$$

$$\det(\mathrm{id} - A^t) = -1, \quad \det(\mathrm{id} - B^t) = -1, \quad \det(\mathrm{id} - C^t) = 1.$$

So we have $\mathcal{O}_2 \cong \mathcal{O}_A \cong \mathcal{O}_B \cong \mathcal{O}_C$,

$$(\mathcal{O}_A, C(X_A)) \cong (\mathcal{O}_B, C(X_B)), \quad (\mathcal{O}_A, C(X_A)) \not\cong (\mathcal{O}_C, C(X_C)).$$

Stable isomorphism (1/2)

Let $\mathcal{C} \cong c_0(\mathbb{Z})$ be the maximal abelian subalgebra of $\mathbb{K} = \mathbb{K}(\ell^2(\mathbb{Z}))$ consisting of diagonal operators.

Corollary

Let $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ be irreducible two-sided SFT. The following are equivalent.

(1) $(\bar{X}_A, \bar{\sigma}_A)$ *and* $(\bar{X}_B, \bar{\sigma}_B)$ *are flow equivalent.*

(2) *There exists an isomorphism* $\Psi : \mathcal{O}_A \otimes \mathbb{K} \to \mathcal{O}_B \otimes \mathbb{K}$ *such that* $\Psi(C(X_A) \otimes \mathcal{C}) = C(X_B) \otimes \mathcal{C}$.

(1)⇒(2) is due to Cuntz and Krieger. We prove (2)⇒(1).

Stable isomorphism (2/2)

Let $\Psi : \mathcal{O}_A \otimes \mathbb{K} \to \mathcal{O}_B \otimes \mathbb{K}$ be an isomorphism such that $\Psi(C(X_A) \otimes \mathcal{C}) = C(X_B) \otimes \mathcal{C}$.

$K_0(\Psi)$ gives an isomorphism $\mathrm{BF}(A^t) \to \mathrm{BF}(B^t)$. Let $v = K_0(\Psi)(u_A)$. There exists an irreducible matrix C such that $(\mathrm{BF}(B^t), v) \cong (\mathrm{BF}(C^t), u_C)$ and $\det(\mathrm{id} - B) = \det(\mathrm{id} - C)$. Then $(\bar{X}_B, \bar{\sigma}_B)$ and $(\bar{X}_C, \bar{\sigma}_C)$ are flow equivalent. Moreover, by Huang's theorem, there exists an isomorphism $\Phi : \mathcal{O}_B \otimes \mathbb{K} \to \mathcal{O}_C \otimes \mathbb{K}$ such that $\Phi(C(X_B) \otimes \mathcal{C}) = C(X_C) \otimes \mathcal{C}$ and $K_0(\Phi)(v) = u_C$.

It follows that $\Phi \circ \Psi$ is an isomorphism $\mathcal{O}_A \otimes \mathbb{K} \to \mathcal{O}_C \otimes \mathbb{K}$ such that $(\Phi \circ \Psi)(C(X_A) \otimes \mathcal{C}) = C(X_C) \otimes \mathcal{C}$ and $K_0(\Phi \circ \Psi)(u_A) = u_C$. Hence we get $(\mathcal{O}_A, C(X_A)) \cong (\mathcal{O}_C, C(X_C))$. By the main theorem, we have $\det(\mathrm{id} - A) = \det(\mathrm{id} - C)$, which equals $\det(\mathrm{id} - B)$. Therefore $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent.

Topological full groups (1/2)

Let G be an étale groupoid whose unit space $G^{(0)}$ is a Cantor set.
We call

$$[[G]] = \{(r|U) \circ (s|U)^{-1} \in \mathrm{Homeo}(G^{(0)}) \mid U \subset G \text{ is compact and open}\}$$

the topological full group of G.

Corollary

For irreducible matrices A and B, the following are equivalent.

1. *The étale groupoids G_A and G_B are isomorphic.*
2. $(\mathrm{BF}(A^t), u_A) \cong (\mathrm{BF}(B^t), u_B)$ *and* $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$.
3. $[[G_A]] \cong [[G_B]]$.
4. $D([[G_A]]) \cong D([[G_B]])$.

Topological full groups (2/2)

Theorem

Let G_A be the étale groupoid arising from an irreducible one-sided SFT.

1. $D([[G_A]])$ *is simple.*
2. $[[G_A]]/D([[G_A]])$ *is isomorphic to* $\mathrm{Ker}(\mathrm{id} - A^t) \oplus (\mathrm{BF}(A^t) \otimes \mathbb{Z}_2)$.
3. $[[G_A]]$ *is finitely presented.*
4. $[[G_A]]$ *has the Haagerup property.*

When (X_A, σ_A) is the full shift over n symbols, $[[G_A]]$ is isomorphic to the Higman-Thompson group V_n.

So, $[[G_A]]$ for general SFT may be thought of as a generalization of the Higman-Thompson group V_n.