Derivations and local multipliers of $C^*$-algebras

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A $C^*$-algebra

$d : A \rightarrow A$ linear, derivation if $d(xy) = x(dy) + (dx)y$ $(x, y \in A)$

Sakai 1960: $d$ bounded

$d$ inner if $dx = xa - ax = [x, a] = d_a(x)$ for all $x$ and some $a$;

Sakai 1968: $d$ is inner if $A$ simple, unital

in general, $d$ not inner;
e.g., $A = K(H)$, $d = d_a|_{K(H)}$ where $a \notin K(H) + \mathbb{C}$;

Sakai 1971: $d$ is inner in $M(A)$ if $A$ simple,
where $M(A)$ multiplier algebra of $A$. 
A separable $C^*$-algebra


Every derivation $d : A \to A$ is inner in $M(A)$ if and only if $A$ is the direct sum of a continuous trace $C^*$-algebra and a $C^*$-algebra with discrete spectrum.

**Theorem (Pedersen 1978)**

Every derivation $d : A \to A$ extends uniquely to a derivation $\bar{d} : M_{loc}(A) \to M_{loc}(A)$ and there is $a \in M_{loc}(A)$ such that $\bar{d} = d_{a}$.

here, $M_{loc}(A)$ denotes the local multiplier algebra.
1978 Pedersen introduces $M_{\text{loc}}(A)$

as “algebra of essential multipliers”

Definition
For every $C^*$-algebra $A$,

$$M_{\text{loc}}(A) = \lim_{\longrightarrow} I \in \mathcal{I}_{\text{ce}}(A) M(I),$$

is its \textit{local multiplier algebra}, where

\[ M(I) = \{ y \in B(H) \mid yI + Iy \subseteq I \} \text{ multiplier algebra of } I. \]
joint work with Pere Ara (Barcelona)


- Automorphisms
- Derivations
- Elementary Operators
- Jordan Homomorphisms
- Lie Derivations, Lie Isomorphisms
- Centralising and Commuting Mappings
- Bi-derivations
- Commutativity Preserving Mapping
Our **algebraic approach** to $M_{\text{loc}}(A)$ enables us to solve complicated operator equations, e.g.,

\[
\left( ([x, z]y[z, q(x)] - [z, q(x)]y[x, z]) r ([x^2, z]y[x, z] - [x, z]y[x^2, z]) \\
- ([x^2, z]y[x, z] - [x, z]y[x^2, z]) r ([x, z]y[z, q(x)] - [z, q(x)]y[x, z]) \right) \times \\
\times u \left( [w^2, v] t[w, v] - [w, v] t[w^2, v] \right) = 0
\]

for fixed $x, y, z \in A$ and all $r, t, u, v, w \in A$. 
joint work with Pere Ara (Barcelona)


Theorem (Pedersen 1978)

Every derivation $d : A \to A$ extends uniquely to a derivation $\bar{d} : M_{\text{loc}}(A) \to M_{\text{loc}}(A)$.
If $A$ is separable, there is $a \in M_{\text{loc}}(A)$ such that $\bar{d} = da$.

Question 1

Is every derivation $d : M_{\text{loc}}(A) \to M_{\text{loc}}(A)$ inner?

Question 2

Is $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$ for every $C^*$-algebra $A$?

in general, $A \subseteq M_{\text{loc}}(A) \subseteq M_{\text{loc}}(M_{\text{loc}}(A)) \subseteq \ldots \subseteq I(A)$
### Theorem (Ara–Mathieu 2006)

There exist unital, separable, primitive AF-algebras $A$ such that $M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A)$.

### Theorem (Somerset 2000)

Let $A$ be a unital separable C*-algebra such that $\text{Prim}(A)$ contains a dense $G_\delta$ subset consisting of closed points.

Then $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$.

Moreover, every derivation on $M_{\text{loc}}(A)$ is inner.

### Theorem (Gogič 2013)

Let $A$ be a C*-algebra such that every irreducible representation of $A$ is finite dimensional. Then $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$.

Moreover, every derivation on $M_{\text{loc}}(A)$ is inner.
More on Q2

from Somerset,

A \textit{unital, separable, type I} \implies M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A).

Argerami, Farenick, Massey 2009:

\( M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A) \) if \( A = C[0, 1] \otimes K(H) \).

Ara–Mathieu 2008:

\( M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A) \) if \( A = C(X) \otimes B(H) \) where \( X \) Stonean with some additional properties.
A dichotomy answer to Pedersen’s question

\[ X \text{ perfect compact metric space} \]
\[ B \text{ separable simple (nuclear) } C^*\text{-algebra} \]
(Elliott’s programme)

\[ A = C(X) \otimes B \]

\[ B \text{ unital} \]
\[ M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A) \]

\[ B \text{ non-unital} \]
\[ M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A) \]
More on Q1

every derivation on $M_{\text{loc}}(A)$ is inner if

(i) $M_{\text{loc}}(A) = A$ and every derivation on $A$ is inner:
   - $A$ von Neumann algebra (Kadison–Sakai);
   - $A$ AW*-algebra (Olesen);
   - $A$ simple unital (Sakai).

(ii) $M_{\text{loc}}(A) = M(A)$ and every derivation on $A$ is inner in $M(A)$:
   - $A$ simple (Sakai).

(iii) $M_{\text{loc}}(A)$ simple (possible by Ara–Mathieu 1999!)

(iv) $M_{\text{loc}}(A)$ AW*-algebra:
   - $A$ commutative;
   - $A$ unital separable type I (Somerset);
   - $A$ with all irred. representations finite dimensional (Gogič);

in all these cases $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$
Summary

- we have no example in which $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$ and we do not know that every derivation of $M_{\text{loc}}(A)$ is inner;

- we have no example in which $M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A)$ and we know every derivation of $M_{\text{loc}}(A)$ is inner.
An if-and-only-if condition

**Theorem 1 (Ara–MM, 2011)**

Let $B$ and $C$ be separable $C^*$-algebras and suppose that at least one of them is nuclear. Suppose further that $B$ is simple and non-unital and that $\text{Prim}(C)$ contains a dense $G_δ$ subset consisting of closed points. Let $A = C \otimes B$. Then

$$M_{\text{loc}}(A) = M_{\text{loc}}(M_{\text{loc}}(A))$$

if and only if $\text{Prim}(A)$ contains a dense subset of isolated points.
A sufficient condition

**Theorem 2 (Ara–MM, 2011)**

Let $A$ be a quasi-central separable $C^*$-algebra such that $\text{Prim}(A)$ contains a dense $G_\delta$ subset consisting of closed points. Let $B$ be a $C^*$-subalgebra of $M_{\text{loc}}(A)$ containing $A$. Then $M_{\text{loc}}(B) \subseteq M_{\text{loc}}(A)$. In particular, $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$.

A quasi-central if no primitive ideal of $A$ contains $Z(A)$; e.g., $A$ unital or $A$ commutative

$B$ simple; $B$ quasi-central $\iff B$ unital.

**Corollary**

Let $A$ be a quasi-central separable $C^*$-algebra such that $\text{Prim}(A)$ contains a dense $G_\delta$ subset consisting of closed points. Then every derivation of $M_{\text{loc}}(A)$ is inner.
Outline of the argument

let $d : M_{\text{loc}}(A) \to M_{\text{loc}}(A)$, let $A \subseteq B \subseteq M_{\text{loc}}(A)$ separable $C^*$-subalgebra such that $dB \subseteq B$;
extend $d|_B$ uniquely to $d_{M_{\text{loc}}(B)} : M_{\text{loc}}(B) \to M_{\text{loc}}(B)$;

next extend both these derivations to the respective injective envelopes, but since $I(B) = I(M_{\text{loc}}(B))$ we have $d_{I(B)} = d_{I(M_{\text{loc}}(B))}$;
now extend $d$ to $I(M_{\text{loc}}(A))$; since $I(B) = I(A) = I(M_{\text{loc}}(A))$, 

$$d_{I(M_{\text{loc}}(A))] = d_{I(B)} = d_{I(M_{\text{loc}}(B))}. \tag{Theorem 2}$$

$\overset{\text{Pedersen}}{\Longrightarrow} d_{M_{\text{loc}}(B)} = dy$ some $y \in M_{\text{loc}}(B)$

consequently, $d = dy$ on $M_{\text{loc}}(A)$. $\square$
New Formulas for $M_{loc}(A)$ and $I(A)$

A $C^*$-algebra

\[ M_{loc}(A) = \text{alg lim}_{T \in T} \Gamma_b(T, A_{\mathcal{M}_A}) \]

\[ I(A) = \text{alg lim}_{T \in T} \Gamma_b(T, A_{\mathcal{J}_A}) \]

where $A_{\mathcal{M}_A}$ and $A_{\mathcal{J}_A}$ are the upper semicontinuous $C^*$-bundles associated to the multiplier sheaf $\mathcal{M}_A$ and the injective envelope sheaf $\mathcal{J}_A$ of $A$, respectively;

$T$ is the downwards directed family of dense $G_\delta$ subsets of $\text{Prim}(A)$;

$\Gamma_b(T, -)$ denotes the bounded continuous local sections on $T$. 

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Derivations and local multipliers of $C^*$-algebras
New Formulas for $M_{\text{loc}}(A)$ and $I(A)$

A $C^*$-algebra

$$M_{\text{loc}}(A) = \text{alg lim}_{T \in \mathcal{T}} \Gamma_b(T, A_m A)$$

$$I(A) = \text{alg lim}_{T \in \mathcal{T}} \Gamma_b(T, A_J A)$$

these descriptions are compatible: $A_m A \hookrightarrow A_J A$

Consequence:

$y \in M_{\text{loc}}(M_{\text{loc}}(A)) \subseteq I(A)$ is contained in some $C^*$-subalgebra $\Gamma_b(T, A_J A)$ and will belong to $M_{\text{loc}}(A)$ once we find $T' \subseteq T$, $T' \in \mathcal{T}$ such that $y \in \Gamma_b(T', A_m A)$.

Sheaves of $C^*$-algebras

$X$ a topological space;
$\mathcal{O}_X$ category of open subsets (with open subsets $U$ as objects and $V \to U$ if and only if $V \subseteq U$).

$C^*$ category of $C^*$-algebras.

**Definition**

A **presheaf of $C^*$-algebras** is a contravariant functor $\mathbb{A}: \mathcal{O}_X \to C^*$.
A **sheaf of $C^*$-algebras** is a presheaf $\mathbb{A}$ such that $\mathbb{A}(\emptyset) = 0$ and, for every open subset $U$ of $X$ and every open cover $U = \bigcup_i U_i$, the maps $\mathbb{A}(U) \to \mathbb{A}(U_i)$ are the limit of the diagrams $\mathbb{A}(U_i) \to \mathbb{A}(U_i \cap U_j)$ for all $i, j$. 
Sheaves of $C^*$-algebras

**Universal Property:**

$$
\mathcal{A}(U) \xrightarrow{\rho} \prod_i \mathcal{A}(U_i) \xrightarrow{\nu} \prod_{i,j} \mathcal{A}(U_i \cap U_j)
$$

$U_i \cap U_j \rightarrow U_i$ yields $\rho_{ji} : \mathcal{A}(U_i) \rightarrow \mathcal{A}(U_i \cap U_j)$; similarly, $\rho_i : \mathcal{A}(U) \rightarrow \mathcal{A}(U_i)$

requirement $\nu \circ \rho = \mu \circ \rho$;

if $(B, \sigma)$ has like properties as $(\mathcal{A}(U), \rho)$ then $\exists! B \rightarrow \mathcal{A}(U)$. 

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Derivations and local multipliers of $C^*$-algebras
Sheaves of $C^*$-algebras

**Notation and Terminology:**

the $C^*$-algebra $\mathcal{A}(U)$ is the *section algebra* over $U \in \mathcal{O}_X$;
by $s|_V$, $V \subseteq U$ open, we mean the “restriction” of $s \in \mathcal{A}(U)$ to $V$;
i.e., the image of $s$ in $\mathcal{A}(V)$ under $\mathcal{A}(U) \to \mathcal{A}(V)$;
the **unique gluing property** of a sheaf can be expressed as follows:

for each compatible family of sections $s_i \in \mathcal{A}(U_i)$, i.e.,
$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j$, there is a unique section $s \in \mathcal{A}(U)$
such that $s|_{U_i} = s_i$ for all $i$. 
Sheaves of $C^*$-algebras

**Example 1. Sheaves from bundles**

Let $(A, \pi, X)$ be an upper semicontinuous $C^*$-bundle. Then

$$\Gamma_b(-, A): \mathcal{O}_X \to C_1^*, \quad U \mapsto \Gamma_b(U, A)$$

defines the sheaf of bounded continuous local sections of $A$, where $C_1^*$ is the category of unital $C^*$-algebras.

$$\Gamma_b(U, A) \to \Gamma_b(V, A), \ V \subseteq U,$$ is the usual restriction map.
Sheaves of $C^*$-algebras

**Example 2.** *The multiplier sheaf*

A $C^*$-algebra with primitive ideal space $\text{Prim}(A)$;

$$\mathcal{M}_A : \mathcal{O}_{\text{Prim}(A)} \rightarrow C^*_1, \quad \mathcal{M}_A(U) = M(A(U)),$$

where $M(A(U))$ denotes the multiplier algebra of the closed ideal $A(U)$ of $A$ associated to the open subset $U \subseteq \text{Prim}(A)$. $M(A(U)) \rightarrow M(A(V))$, $V \subseteq U$, the restriction homomorphisms.

**Proposition**

*The above functor $\mathcal{M}_A$ defines a sheaf of $C^*$-algebras.*
Sheaves of $C^*$-algebras

Example 3. *The injective envelope sheaf*

Let $I(B)$ denote the *injective envelope* of $B$;

$$
\mathcal{I}_A : \mathcal{O}_{\text{Prim}(A)} \to C_1^*, \quad \mathcal{I}_A(U) = p_U I(A) = I(A(U)),
$$

where $p_U = p_{A(U)}$ denotes the unique central open projection in $I(A)$ such that $p_{A(U)} I(A)$ is the injective envelope of $A(U)$. $I(A(U)) \to I(A(V))$, $V \subseteq U$, given by multiplication by $p_V$ (as $p_V \leq p_U$).

$$
\{ p_U \mid U \in \mathcal{O}_{\text{Prim}(A)} \} 	ext{ is a complete Boolean algebra isomorphic to the Boolean algebra of regular open subsets of Prim}(A), \text{ and it is precisely the set of projections of the AW*-algebra } Z(I(A)).
$$
from sheaves to bundles

**Theorem**

Given a presheaf $\mathcal{A}$ of C*-algebras over $X$, there is a canonically associated upper semicontinuous C*-bundle $(A, \pi, X)$ over $X$.

**Idea:**

$x \in X$, define $A_x := \lim_{\to} \mathcal{A}(U)$ (stalk at $x$)

let $A := \bigsqcup_{x \in X} A_x$ and define a topology on $A$ by

$$V(U, s, \varepsilon) = \{a \in A \mid \pi(a) \in U \text{ and } \|a - s(\pi(a))\| < \varepsilon\}$$

is a basic open set, where $\varepsilon > 0$, $U \in \mathcal{O}_X$, $s \in \mathcal{A}(U)$ and $s(x)$ the image under $\mathcal{A}(U) \to A_x$. 

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Derivations and local multipliers of C*-algebras
Theorem 2 (simplified version)

Let $A$ be a quasi-central separable $C^*$-algebra such that $\text{Prim}(A)$ contains a dense $G_\delta$ subset consisting of closed points. Then $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$. 
Outline of proof of Theorem 2

take \( y \in M(J) \) for some closed essential ideal \( J \) of \( M_{loc}(A) \);
let \( T \in \mathcal{T} \) be such that \( y \in \Gamma_b(T, A_{\overline{J}_A}) \);
\textit{WLOG} \( T \) consists of closed separated points of \( \text{Prim}(A) \).

recall: \( t \in \text{Prim}(A) \) is \textit{separated} if \( t \) and every point \( t' \notin \{t\} \) can be separated by disjoint neighbourhoods.

Dixmier 1968 \( \text{Sep}(A) \), the set of all separated points, dense \( G_\delta \) subset of \( \text{Prim}(A) \) as well as a Polish space;

\textbf{Lemma:} There is \( h \in J \) such that \( h(t) \neq 0 \) for all \( t \in T \).

\textbf{Lemma:} There is a separable \( C^* \)-subalgebra \( B \subseteq J \) with \( AhA \subseteq B \) and \( y \in M(B) \).
Outline of proof of Theorem 2

take countable dense subset \( \{ b_n \mid n \in \mathbb{N} \} \) in \( B \) and \( T_n \in \mathcal{T} \) such that \( b_n \in \Gamma_b(T_n, A_{\mathcal{M}_A}) \); put \( A = A_{\mathcal{M}_A} \);
letting \( T' = \bigcap_n T_n \cap T \in \mathcal{T} \), we have \( B \subseteq \Gamma_b(T', A) \), hence

\[
B_t = \{ b(t) \mid b \in B \} \subseteq A_t \quad (t \in T').
\]

in general, \( \exists \varphi_t : A_t \to M_{\text{loc}}(A/t) \)

\[
\begin{align*}
A \text{ quasicentral} & \Rightarrow A/t \text{ unital} \\
t \text{ closed} & \Rightarrow A/t \text{ simple}
\end{align*}
\]

\[
\Rightarrow M_{\text{loc}}(A/t) = A/t.
\]
Outline of proof of Theorem 2

**Main Lemma:** A quasicentral, \( t \in \text{Prim}(A) \) closed, separated \( \Rightarrow \varphi_t \) isomorphism.

(Rests on existence of local identities in quasicentral \( C^* \)-algebras:

\[
\forall \ t \in \text{Prim}(A) \quad \exists \ U_1 \subseteq \text{Prim}(A) \text{ open, } t \in U_1,
\exists \ z \in Z(A)_+, \|z\| = 1: \ z + A(U_2) = 1_{A/A(U_2)},
\]

where \( U_2 = \text{Prim}(A) \setminus \overline{U_1}. \)) As a consequence,

\[
A_t = A_t h(t) A_t = (A/t)h(t)(A/t) = (AhA)_t \subseteq B_t \subseteq A_t \quad (t \in T').
\]

\[
\Rightarrow \exists \ b_t \in B: b_t(t) = 1_{A_t}
\]

\[
\Rightarrow y(t) = y(t) 1_{A_t} = (yb_t)(t) \in A_t \quad (t \in T').
\]

It follows that \( y \in \Gamma_b(T', A_{\mathfrak{M}_A}) \) with \( T' \subseteq T \), proving

that \( y \in M_{\text{loc}}(A). \) \(\Box\)