Derivations and local multipliers of C^*-algebras

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A C^*-algebra

$d \colon A \to A$ linear, derivation if $d(xy) = x(dy) + (dx)y \quad (x, y \in A)$

Sakai 1960: d bounded

d inner if $dx = xa - ax = [x, a] = d_a(x)$ for all x and some a;

Sakai 1968: d is inner if A simple, unital

in general, d not inner;
e.g., $A = K(H)$, $d = d_{a|K(H)}$ where $a \notin K(H) + \mathbb{C}$;

Sakai 1971: d is inner in $M(A)$ if A simple,
where $M(A)$ *multiplier algebra* of A.

A separable C^*-algebra

Theorem (Akemann–Elliott–Pedersen–Tomiyama 1976/1979)

Every derivation $d\colon A \to A$ is inner in $M(A)$ if and only if A is the direct sum of a continuous trace C^*-algebra and a C^*-algebra with discrete spectrum.

Theorem (Pedersen 1978)

Every derivation $d\colon A \to A$ extends uniquely to a derivation $\bar{d}\colon M_{\mathrm{loc}}(A) \to M_{\mathrm{loc}}(A)$ and there is $a \in M_{\mathrm{loc}}(A)$ such that $\bar{d} = d_a$.

here, $M_{\mathrm{loc}}(A)$ denotes the local multiplier algebra.

1978 Pedersen introduces $M_{\mathrm{loc}}(A)$

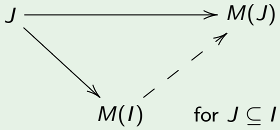
as "algebra of essential multipliers"

Definition

For every C^*-algebra A,

$$M_{\mathrm{loc}}(A) = \varinjlim_{I \in \mathscr{I}_{ce}(A)} M(I),$$

is its *local multiplier algebra*, where

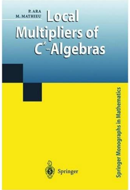


for $J \subseteq I$

$\mathscr{I}_{ce}(A)$ the filter of all closed essential ideals of A;

$M(I) = \{y \in B(H) \mid yI + Iy \subseteq I\}$ *multiplier algebra of* I.

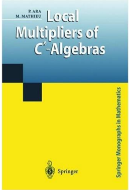
joint work with Pere Ara (Barcelona)



P. Ara and M. Mathieu, Local multipliers of C^*-algebras, Springer-Verlag, London, 2003.

- Automorphisms
- Derivations
- Elementary Operators
- Jordan Homomorphisms
- Lie Derivations, Lie Isomorphisms
- Centralising and Commuting Mappings
- Bi-derivations
- Commutativity Preserving Mapping

joint work with Pere Ara (Barcelona)



Our **algebraic approach** to $M_{\mathrm{loc}}(A)$ enables us to solve complicated operator equations, e.g.,

$$\Big(\big([x,z]y[z,q(x)]-[z,q(x)]y[x,z]\big)\,r\big([x^2,z]y[x,z]-[x,z]y[x^2,z]\big)$$
$$-\big([x^2,z]y[x,z]-[x,z]y[x^2,z]\big)\,r\big([x,z]y[z,q(x)]-[z,q(x)]y[x,z]\big)\Big)\times$$
$$\times\, u\,\big([w^2,v]t[w,v]-[w,v]t[w^2,v]\big)=0$$

for fixed $x, y, z \in A$ and all $r, t, u, v, w \in A$.

joint work with Pere Ara (Barcelona)

P. Ara and M. Mathieu, *A not so simple local multiplier algebra*, J. Funct. Analysis **237** (2006), 721–737.

P. Ara and M. Mathieu, *Maximal C*-algebras of quotients and injective envelopes of C*-algebras*, Houston J. Math. **34** (2008), 827–872.

P. Ara and M. Mathieu, *Sheaves of C*-algebras*, Math. Nachrichten **283** (2010), 21–39.

P. Ara and M. Mathieu, *When is the second local multiplier algebra of a C*-algebra equal to the first?*, Bull. London Math. Soc. **43** (2011), 1167–1180.

Theorem (Pedersen 1978)

Every derivation $d\colon A \to A$ extends uniquely to a derivation $\bar{d}\colon M_{\mathrm{loc}}(A) \to M_{\mathrm{loc}}(A)$.
If A is separable, there is $a \in M_{\mathrm{loc}}(A)$ such that $\bar{d} = d_a$.

Question 1

Is every derivation $d\colon M_{\mathrm{loc}}(A) \to M_{\mathrm{loc}}(A)$ inner?

Question 2

Is $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$ for every C^*-algebra A?

in general, $A \subseteq M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \subseteq \ldots \subseteq I(A)$

Theorem (Ara–Mathieu 2006)

There exist unital, separable, primitive AF-algebras A such that $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \neq M_{\mathrm{loc}}(A)$.

Theorem (Somerset 2000)

Let A be a unital separable C-algebra such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points.*
Then $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$.
Moreover, every derivation on $M_{\mathrm{loc}}(A)$ is inner.

Theorem (Gogič 2013)

Let A be a C-algebra such that every irreducible representation of A is finite dimensional. Then $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$.*
Moreover, every derivation on $M_{\mathrm{loc}}(A)$ is inner.

More on Q2

from Somerset,

A unital, separable, type I $\implies$ $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$.

Argerami, Farenick, Massey 2009:

$M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A)$ if $A = C[0,1] \otimes K(H)$.

Ara–Mathieu 2008:

$M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A)$ if $A = C(X) \otimes B(H)$ where X Stonean with some additional properties.

A dichotomy answer to Pedersen's question

X perfect compact metric space

B separable simple (nuclear) C^*-algebra

(Elliott's programme)

$\Longrightarrow$

$$A = C(X) \otimes B$$

B unital:

$$M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$$

B non-unital:

$$M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \neq M_{\mathrm{loc}}(A)$$

More on Q1

every derivation on $M_{\rm loc}(A)$ is inner if

(i) $M_{\rm loc}(A) = A$ and every derivation on A is inner:
- A von Neumann algebra (Kadison–Sakai);
- A AW^*-algebra (Olesen);
- A simple unital (Sakai).

(ii) $M_{\rm loc}(A) = M(A)$ and every derivation on A is inner in $M(A)$:
- A simple (Sakai).

(iii) $M_{\rm loc}(A)$ simple (possible by Ara–Mathieu 1999!)

(iv) $M_{\rm loc}(A)$ AW^*-algebra:
- A commutative;
- A unital separable type I (Somerset);
- A with all irred. representations finite dimensional (Gogič);

in all these cases $M_{\rm loc}(M_{\rm loc}(A)) = M_{\rm loc}(A)$

Summary

- we have no example in which $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$ and we do not know that every derivation of $M_{\mathrm{loc}}(A)$ is inner;
- we have no example in which $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \neq M_{\mathrm{loc}}(A)$ and we know every derivation of $M_{\mathrm{loc}}(A)$ is inner.

An if-and-only-if condition

Theorem 1 (Ara–MM, 2011)

Let B and C be separable C-algebras and suppose that at least one of them is nuclear. Suppose further that B is simple and non-unital and that $\mathrm{Prim}(C)$ contains a dense G_δ subset consisting of closed points. Let $A = C \otimes B$. Then*

$$M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$$

if and only if $\mathrm{Prim}(A)$ contains a dense subset of isolated points.

A sufficient condition

Theorem 2 (Ara–MM, 2011)

Let A be a quasi-central separable C-algebra such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points. Let B be a C*-subalgebra of $M_{\mathrm{loc}}(A)$ containing A. Then $M_{\mathrm{loc}}(B) \subseteq M_{\mathrm{loc}}(A)$. In particular, $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$.*

A quasi-central if no primitive ideal of A contains $Z(A)$;

e.g., A unital or A commutative

B simple; B quasi-central $\iff$ B unital.

Corollary

Let A be a quasi-central separable C-algebra such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points. Then every derivation of $M_{\mathrm{loc}}(A)$ is inner.*

Outline of the argument

let $d\colon M_{\rm loc}(A) \to M_{\rm loc}(A)$, let $A \subseteq B \subseteq M_{\rm loc}(A)$ separable C^*-subalgebra such that $dB \subseteq B$;
extend $d_{|B}$ uniquely to $d_{M_{\rm loc}(B)}\colon M_{\rm loc}(B) \to M_{\rm loc}(B)$;

next extend both these derivations to the respective injective envelopes, but since
$I(B) = I(M_{\rm loc}(B))$ we have $d_{I(B)} = d_{I(M_{\rm loc}(B))}$;

now extend d to $I(M_{\rm loc}(A))$; since $I(B) = I(A) = I(M_{\rm loc}(A))$,

$$d_{I(M_{\rm loc}(A))} = d_{I(B)} = d_{I(M_{\rm loc}(B))}.$$

$\overset{\text{Pedersen}}{\Longrightarrow}$ $d_{M_{\rm loc}(B)} = d_y$ some $y \in M_{\rm loc}(B) \overset{\text{Theorem 2}}{\subseteq} M_{\rm loc}(A)$;

consequently, $d = d_y$ on $M_{\rm loc}(A)$. $\square$

New Formulas for $M_{\rm loc}(A)$ and $I(A)$

A C^*-algebra

$$M_{\rm loc}(A) = \operatorname{alg}\varinjlim_{T \in \mathcal{T}} \Gamma_b(T, \mathsf{A}_{\mathfrak{M}_A})$$

$$I(A) = \operatorname{alg}\varinjlim_{T \in \mathcal{T}} \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$$

where $\mathsf{A}_{\mathfrak{M}_A}$ and $\mathsf{A}_{\mathfrak{I}_A}$ are the upper semicontinuous C^*-bundles associated to the multiplier sheaf $\mathfrak{M}_A$ and the injective envelope sheaf $\mathfrak{I}_A$ of A, respectively;

$\mathcal{T}$ is the downwards directed family of dense G_δ subsets of $\mathrm{Prim}(A)$;

$\Gamma_b(T, -)$ denotes the bounded continuous local sections on T.

New Formulas for $M_{\text{loc}}(A)$ and $I(A)$

A C^*-algebra

$$M_{\text{loc}}(A) = \operatorname{alg}\varinjlim{}_{T\in\mathcal{T}}\, \Gamma_b(T, \mathsf{A}_{\mathfrak{M}_A})$$

$$I(A) = \operatorname{alg}\varinjlim{}_{T\in\mathcal{T}}\, \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$$

these descriptions are compatible: $\mathsf{A}_{\mathfrak{M}_A} \hookrightarrow \mathsf{A}_{\mathfrak{I}_A}$

Consequence:

$y \in M_{\text{loc}}(M_{\text{loc}}(A)) \subseteq I(A)$ is contained in some C^*-subalgebra $\Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$ and will belong to $M_{\text{loc}}(A)$ once we find $T' \subseteq T$, $T' \in \mathcal{T}$ such that $y \in \Gamma_b(T', \mathsf{A}_{\mathfrak{M}_A})$.

P. Ara, M. Mathieu, *Sheaves of C*-algebras*, Math. Nachrichten **283** (2010), 21–39.

Sheaves of C^*-algebras

X a topological space;
$\mathcal{O}_X$ category of open subsets (with open subsets U as objects and $V \to U$ if and only if $V \subseteq U$).

$\mathcal{C}^*$ category of C^*-algebras.

Definition

A *presheaf of C^*-algebras* is a contravariant functor $\mathfrak{A}\colon \mathcal{O}_X \to \mathcal{C}^*$.

A *sheaf of C^*-algebras* is a presheaf $\mathfrak{A}$ such that $\mathfrak{A}(\emptyset) = 0$ and, for every open subset U of X and every open cover $U = \bigcup_i U_i$, the maps $\mathfrak{A}(U) \to \mathfrak{A}(U_i)$ are the limit of the diagrams $\mathfrak{A}(U_i) \to \mathfrak{A}(U_i \cap U_j)$ for all i, j.

Sheaves of C^*-algebras

Universal Property:

$$\begin{array}{ccccc} \mathfrak{A}(U) & \xrightarrow{\ \rho\ } & \prod_i \mathfrak{A}(U_i) & \overset{\nu}{\underset{\mu}{\rightrightarrows}} & \prod_{i,j} \mathfrak{A}(U_i \cap U_j) \\ \uparrow & \nearrow_{\sigma} & & & \\ B & & & & \end{array}$$

$U_i \cap U_j \longrightarrow U_i$ yields $\rho_{ji} \colon \mathfrak{A}(U_i) \longrightarrow \mathfrak{A}(U_i \cap U_j)$;

similarly, $\rho_i \colon \mathfrak{A}(U) \longrightarrow \mathfrak{A}(U_i)$

requirement $\quad \nu \circ \rho = \mu \circ \rho$;

if (B, σ) has like properties as $(\mathfrak{A}(U), \rho)$ then $\exists!\ B \longrightarrow \mathfrak{A}(U)$.

Sheaves of C^*-algebras

Notation and Terminology:

the C^*-algebra $\mathfrak{A}(U)$ is the *section algebra* over $U \in \mathcal{O}_X$;

by $s_{|V}$, $V \subseteq U$ open, we mean the "restriction" of $s \in \mathfrak{A}(U)$ to V; i.e., the image of s in $\mathfrak{A}(V)$ under $\mathfrak{A}(U) \to \mathfrak{A}(V)$;

the unique gluing property of a sheaf can be expressed as follows:

for each compatible family of sections $s_i \in \mathfrak{A}(U_i)$, i.e., $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for all i, j, there is a unique section $s \in \mathfrak{A}(U)$ such that $s_{|U_i} = s_i$ for all i.

Sheaves of C^*-algebras

Example 1. *Sheaves from bundles*

Let (A, π, X) be an upper semicontinuous C^*-bundle. Then

$$\Gamma_b(-, \mathsf{A})\colon \mathcal{O}_X \to \mathcal{C}_1^*, \quad U \mapsto \Gamma_b(U, \mathsf{A})$$

defines the *sheaf of bounded continuous local sections of* A, where $\mathcal{C}_1^*$ is the category of unital C^*-algebras.

$\Gamma_b(U, \mathsf{A}) \to \Gamma_b(V, \mathsf{A})$, $V \subseteq U$, is the usual restriction map.

Sheaves of C^*-algebras

Example 2. *The multiplier sheaf*

A C^*-algebra with primitive ideal space $\mathrm{Prim}(A)$;

$$\mathfrak{M}_A\colon \mathcal{O}_{\mathrm{Prim}(A)} \to \mathcal{C}_1^*, \quad \mathfrak{M}_A(U) = M(A(U)),$$

where $M(A(U))$ denotes the multiplier algebra of the closed ideal $A(U)$ of A associated to the open subset $U \subseteq \mathrm{Prim}(A)$.
$M(A(U)) \to M(A(V))$, $V \subseteq U$, the restriction homomorphisms.

Proposition

The above functor $\mathfrak{M}_A$ defines a sheaf of C^*-algebras.

Sheaves of C^*-algebras

Example 3. *The injective envelope sheaf*

let $I(B)$ denote the *injective envelope* of B;

$$\mathfrak{I}_A\colon \mathcal{O}_{\mathrm{Prim}(A)} \to \mathcal{C}_1^*, \quad \mathfrak{I}_A(U) = p_U I(A) = I(A(U)),$$

where $p_U = p_{A(U)}$ denotes the unique central open projection in $I(A)$ such that $p_{A(U)} I(A)$ is the injective envelope of $A(U)$. $I(A(U)) \to I(A(V))$, $V \subseteq U$, given by multiplication by p_V (as $p_V \leq p_U$).

$\{p_U \mid U \in \mathcal{O}_{\mathrm{Prim}(A)}\}$ is a complete Boolean algebra isomorphic to the Boolean algebra of regular open subsets of $\mathrm{Prim}(A)$, and it is precisely the set of projections of the AW^*-algebra $Z(I(A))$.

from sheaves to bundles

Theorem

Given a presheaf $\mathfrak{A}$ of C-algebras over X, there is a canonically associated upper semicontinuous C*-bundle (A, π, X) over X.*

Idea:

$x \in X$, define $\mathsf{A}_x := \varinjlim_{x \in U} \mathfrak{A}(U)$ (stalk at x)

let $\mathsf{A} := \bigsqcup_{x \in X} \mathsf{A}_x$ and define a topology on A by

$$V(U, s, \varepsilon) = \{a \in \mathsf{A} \mid \pi(a) \in U \text{ and } \|a - s(\pi(a))\| < \varepsilon\}$$

is a basic open set, where $\varepsilon > 0$, $U \in \mathcal{O}_X$, $s \in \mathfrak{A}(U)$ and $s(x)$ the image under $\mathfrak{A}(U) \to \mathsf{A}_x$.

Theorem 2 (simplified version)

Let A be a quasi-central separable C^*-algebra such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points.
Then $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$.

Outline of proof of Theorem 2

take $y \in M(J)$ for some closed essential ideal J of $M_{\mathrm{loc}}(A)$;
let $T \in \mathcal{T}$ be such that $y \in \Gamma_b(T, \mathsf{A}_{\mathfrak{J}_A})$;
WLOG T consists of closed separated points of $\mathrm{Prim}(A)$.

recall: $t \in \mathrm{Prim}(A)$ is *separated* if t and every point $t' \notin \overline{\{t\}}$ can be separated by disjoint neighbourhoods.

Dixmier 1968 — $\mathrm{Sep}(A)$, the set of all separated points, dense G_δ subset of $\mathrm{Prim}(A)$ as well as a Polish space;

Lemma: There is $h \in J$ such that $h(t) \neq 0$ for all $t \in T$.

Lemma: There is a separable C^*-subalgebra $B \subseteq J$ with $AhA \subseteq B$ and $y \in M(B)$.

Outline of proof of Theorem 2

take countable dense subset $\{b_n \mid n \in \mathbb{N}\}$ in B and $T_n \in \mathcal{T}$ such that $b_n \in \Gamma_b(T_n, \mathsf{A}_{\mathfrak{M}_A})$; put $\mathsf{A} = \mathsf{A}_{\mathfrak{M}_A}$;
letting $T' = \bigcap_n T_n \cap T \in \mathcal{T}$, we have $B \subseteq \Gamma_b(T', \mathsf{A})$, hence

$$B_t = \{b(t) \mid b \in B\} \subseteq \mathsf{A}_t \qquad (t \in T').$$

in general, $\exists\ \varphi_t \colon \mathsf{A}_t \to M_{\mathrm{loc}}(A/t)$

$$\left.\begin{array}{r} A \quad \text{quasicentral} \ \Rightarrow \ A/t \text{ unital} \\ t \ \text{ closed} \ \Rightarrow \ A/t \text{ simple} \end{array}\right\} \Rightarrow \quad M_{\mathrm{loc}}(A/t) = A/t.$$

Outline of proof of Theorem 2

Main Lemma: A quasicentral, $t \in \mathrm{Prim}(A)$ closed, separated $\Rightarrow$ φ_t isomorphism.

(Rests on existence of local identities in quasicentral C^*-algebras:

$$\forall\, t \in \mathrm{Prim}(A) \quad \exists\, U_1 \subseteq \mathrm{Prim}(A) \text{ open}, t \in U_1, \\ \exists\, z \in Z(A)_+,\ \|z\| = 1\colon\ z + A(U_2) = 1_{A/A(U_2)},$$

where $U_2 = \mathrm{Prim}(A) \setminus \overline{U_1}$.) As a consequence,

$$\mathsf{A}_t = \mathsf{A}_t\, h(t)\, \mathsf{A}_t = (A/t)h(t)(A/t) = (AhA)_t \subseteq B_t \subseteq \mathsf{A}_t \quad (t \in T').$$

$$\Rightarrow \exists\, b_t \in B\colon\ b_t(t) = 1_{\mathsf{A}_t}$$
$$\Rightarrow y(t) = y(t)\, 1_{\mathsf{A}_t} = (yb_t)(t) \in \mathsf{A}_t \quad (t \in T').$$

It follows that $y \in \Gamma_b(T', \mathsf{A}_{\mathfrak{M}_A})$ with $T' \subseteq T$, proving that $y \in M_{\mathrm{loc}}(A)$. $\square$