# $C^*$-algebras associated to graphs, path spaces and equilibrium states

Nadia S. Larsen

University of Oslo

"Classifying Structures for Operator Algebras and Dynamical Systems"
Aberystwyth, 16-20 September 2013
Joint work with Toke M. Carlsen (Trondheim)

# Outline

- The formalism of finite quantum systems and the KMS condition

# Outline

- The formalism of finite quantum systems and the KMS condition
- Early examples. Graph algebras.

# Outline

- The formalism of finite quantum systems and the KMS condition
- Early examples. Graph algebras.
- The Exel-Laca strategy. Graph algebras as partial crossed products.

# Outline

- The formalism of finite quantum systems and the KMS condition
- Early examples. Graph algebras.
- The Exel-Laca strategy. Graph algebras as partial crossed products.
- KMS states on graph algebras and measures on the path space.

# Outline

- The formalism of finite quantum systems and the KMS condition
- Early examples. Graph algebras.
- The Exel-Laca strategy. Graph algebras as partial crossed products.
- KMS states on graph algebras and measures on the path space.
- Examples

# The origins of the KMS condition

Finite quantum systems: a time evolution on $M_n(\mathbb{C})$ is (always) given by a one-parameter group of automorphisms

$$\sigma_t(a) = e^{itH} a e^{-itH},$$

where $t \in \mathbb{R}$, $a \in M_n(\mathbb{C})$ and $H$ is a self-adjoint matrix.

# The origins of the KMS condition

Finite quantum systems: a time evolution on $M_n(\mathbb{C})$ is (always) given by a one-parameter group of automorphisms

$$\sigma_t(a) = e^{itH} a e^{-itH},$$

where $t \in \mathbb{R}$, $a \in M_n(\mathbb{C})$ and $H$ is a self-adjoint matrix.
The *Gibbs state* at $\beta > 0$ is $\varphi_G(a) = \frac{\mathrm{Tr}(ae^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$. It minimizes a certain quantity (the free energy) and satisfies

$$\varphi_G(ab) = \varphi_G(b\sigma_{i\beta}(a)), \quad (1)$$

for $a, b \in M_n(\mathbb{C})$ analytic, i.e. allowing $\sigma_z(a)$, $\sigma_z(b)$ for $z \in \mathbb{C}$.

# The origins of the KMS condition

Finite quantum systems: a time evolution on $M_n(\mathbb{C})$ is (always) given by a one-parameter group of automorphisms

$$\sigma_t(a) = e^{itH} a e^{-itH},$$

where $t \in \mathbb{R}$, $a \in M_n(\mathbb{C})$ and $H$ is a self-adjoint matrix.
The *Gibbs state* at $\beta > 0$ is $\varphi_G(a) = \frac{\mathrm{Tr}(ae^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$. It minimizes a certain quantity (the free energy) and satisfies

$$\varphi_G(ab) = \varphi_G(b\sigma_{i\beta}(a)), \tag{1}$$

for $a, b \in M_n(\mathbb{C})$ analytic, i.e. allowing $\sigma_z(a)$, $\sigma_z(b)$ for $z \in \mathbb{C}$.
Haag-Hugenholtz-Winnick (1967) proposed (1), the *KMS condition* (Kubo-Martin-Schwinger), as the equilibrium for a state on a $C^*$-algebra $A$ with a time evolution $\sigma$. The *partition function* of $(M_n(\mathbb{C}), \sigma)$ is $\beta \mapsto \mathrm{Tr}(e^{-\beta H})$.
References: Bratteli-Robinson, Pedersen.

# KMS states

By analogy with $M_n(\mathbb{C})$ and the Gibbs state, extend the notions of KMS$_\beta$ state, partition function, inverse temperature.

# KMS states

By analogy with $M_n(\mathbb{C})$ and the Gibbs state, extend the notions of $\mathrm{KMS}_\beta$ state, partition function, inverse temperature. Given $A$ a $C^*$-algebra and a time evolution (dynamics) $\sigma$ from $\mathbb{R}$ to $\mathrm{Aut}(A)$, a $\sigma$-KMS state at inverse temperature $\beta \in [0, \infty)$ is a state $\varphi$ such that

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

for all $a, b$ in $A$ with $b$ analytic, i.e. $t \to \sigma_t(b)$ extends to an entire function on $\mathbb{C}$.

# KMS states

By analogy with $M_n(\mathbb{C})$ and the Gibbs state, extend the notions of $\text{KMS}_\beta$ state, partition function, inverse temperature. Given $A$ a $C^*$-algebra and a time evolution (dynamics) $\sigma$ from $\mathbb{R}$ to $\text{Aut}(A)$, a $\sigma$-KMS state at inverse temperature $\beta \in [0, \infty)$ is a state $\varphi$ such that

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

for all $a, b$ in $A$ with $b$ analytic, i.e. $t \to \sigma_t(b)$ extends to an entire function on $\mathbb{C}$.

A state $\varphi$ is a ground state if for all $a, b$ analytic, the function $z \to \varphi(a\sigma_z(b))$ is bounded in the upper-half plane.

# KMS states

By analogy with $M_n(\mathbb{C})$ and the Gibbs state, extend the notions of $\text{KMS}_\beta$ state, partition function, inverse temperature.
Given $A$ a $C^*$-algebra and a time evolution (dynamics) $\sigma$ from $\mathbb{R}$ to $\text{Aut}(A)$, a $\sigma$-KMS state at inverse temperature $\beta \in [0, \infty)$ is a state $\varphi$ such that

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

for all $a, b$ in $A$ with $b$ analytic, i.e. $t \to \sigma_t(b)$ extends to an entire function on $\mathbb{C}$.

A state $\varphi$ is a ground state if for all $a, b$ analytic, the function $z \to \varphi(a\sigma_z(b))$ is bounded in the upper-half plane.

Verifications can often be reduced to $a, b$ in a norm-dense $*$-subalgebra of analytic elements.

# First examples

The Toeplitz algebra $\mathcal{T} = C^*(s)$ has dynamics $\sigma_t(s) = e^{it}s$, $t \in \mathbb{R}$. It has a unique KMS$_\beta$ state $\varphi(s^m s^{*n}) = e^{-n\beta}\delta_{n,m}$ at $0 \leq \beta < \infty$.

# First examples

The Toeplitz algebra $\mathcal{T} = C^*(s)$ has dynamics $\sigma_t(s) = e^{it}s$, $t \in \mathbb{R}$. It has a unique KMS$_\beta$ state $\varphi(s^m s^{*n}) = e^{-n\beta}\delta_{n,m}$ at $0 \leq \beta < \infty$.

The Toeplitz-Cuntz algebra $\mathcal{T}\mathcal{O}_n$ is generated by isometries $s_1, \ldots s_n$ with $\sum_{j=1}^n s_j s_j^* < 1$. The periodic gauge action is $\sigma_t(s_j) = e^{it}s_j$, $t \in \mathbb{R}$. Form $s_\mu = s_{i_1} \ldots s_{i_m}$, $m \geq 1$. Then

$$\varphi_\beta(s_\mu s_\nu^*) = e^{-|\mu|\beta}\delta_{\mu,\nu}$$

is the unique *KMS$_\beta$ state* for each $\beta \geq \log n$. References: Olesen-Pedersen (1978), Evans (1980) for non-periodic case. Similar results are known for $\mathcal{O}_n$, cf. later slide.

# First examples

The Toeplitz algebra $\mathcal{T} = C^*(s)$ has dynamics $\sigma_t(s) = e^{it}s$, $t \in \mathbb{R}$. It has a unique KMS$_\beta$ state $\varphi(s^m s^{*n}) = e^{-n\beta}\delta_{n,m}$ at $0 \leq \beta < \infty$.

The Toeplitz-Cuntz algebra $\mathcal{TO}_n$ is generated by isometries $s_1, \ldots s_n$ with $\sum_{j=1}^n s_j s_j^* < 1$. The periodic gauge action is $\sigma_t(s_j) = e^{it}s_j$, $t \in \mathbb{R}$. Form $s_\mu = s_{i_1} \ldots s_{i_m}$, $m \geq 1$. Then

$$\varphi_\beta(s_\mu s_\nu^*) = e^{-|\mu|\beta}\delta_{\mu,\nu}$$

is the unique *KMS$_\beta$ state* for each $\beta \geq \log n$. References: Olesen-Pedersen (1978), Evans (1980) for non-periodic case. Similar results are known for $\mathcal{O}_n$, cf. later slide.

Our interest will be in $C^*$-algebras associated to countable directed graphs.

# $C^*$-algebras associated to countable graphs I

A directed *graph* $E = (E^0, E^1, s, r)$ consists of

- $E^0$ (countable) set of vertices $v$;
- $E^1$ (countable) set of edges $e$;
- $s, r : E^1 \to E^0$ the *source* and *range* map.

# $C^*$-algebras associated to countable graphs I

A directed *graph* $E = (E^0, E^1, s, r)$ consists of

- $E^0$ (countable) set of vertices $v$;
- $E^1$ (countable) set of edges $e$;
- $s, r : E^1 \to E^0$ the *source* and *range* map.

The *graph $C^*$-algebra* $C^*(E)$ is the universal $C^*$-algebra with generators $\mathcal{G} = \{s_e, p_v\}_{e \in E^1, v \in E^0}$, where $s_e$ are partial isometries with mutually orthogonal range projections and $p_v$ are mutually orthogonal projections, subject to the Cuntz-Krieger relations

- $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,

# $C^*$-algebras associated to countable graphs I

A directed *graph* $E = (E^0, E^1, s, r)$ consists of

- $E^0$ (countable) set of vertices $v$;
- $E^1$ (countable) set of edges $e$;
- $s, r : E^1 \to E^0$ the *source* and *range* map.

The *graph $C^*$-algebra* $C^*(E)$ is the universal $C^*$-algebra with generators $\mathcal{G} = \{s_e, p_v\}_{e \in E^1, v \in E^0}$, where $s_e$ are partial isometries with mutually orthogonal range projections and $p_v$ are mutually orthogonal projections, subject to the Cuntz-Krieger relations

- $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
- $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$,

# $C^*$-algebras associated to countable graphs I

A directed *graph* $E = (E^0, E^1, s, r)$ consists of

- $E^0$ (countable) set of vertices $v$;
- $E^1$ (countable) set of edges $e$;
- $s, r : E^1 \to E^0$ the *source* and *range* map.

The *graph $C^*$-algebra* $C^*(E)$ is the universal $C^*$-algebra with generators $\mathcal{G} = \{s_e, p_v\}_{e \in E^1, v \in E^0}$, where $s_e$ are partial isometries with mutually orthogonal range projections and $p_v$ are mutually orthogonal projections, subject to the Cuntz-Krieger relations

- $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
- $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$,
- $p_v = \sum_{e \in vE^1} s_e s_e^*$ when $vE^1 = \{f \in E^1 \mid s(f) = v\}$ is non-empty and finite.

# $C^*$-algebras associated to countable graphs I

A directed *graph* $E = (E^0, E^1, s, r)$ consists of

- $E^0$ (countable) set of vertices $v$;
- $E^1$ (countable) set of edges $e$;
- $s, r : E^1 \to E^0$ the *source* and *range* map.

The *graph $C^*$-algebra* $C^*(E)$ is the universal $C^*$-algebra with generators $\mathcal{G} = \{s_e, p_v\}_{e \in E^1, v \in E^0}$, where $s_e$ are partial isometries with mutually orthogonal range projections and $p_v$ are mutually orthogonal projections, subject to the Cuntz-Krieger relations

- $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
- $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$,
- $p_v = \sum_{e \in vE^1} s_e s_e^*$ when $vE^1 = \{f \in E^1 \mid s(f) = v\}$ is non-empty and finite.

Enomoto-Watatani, Kumjian-Pask-Raeburn-Renault, Fowler-Raeburn, Bates-Hong-Raeburn-Szymański, ...

# $C^*$-algebras associated to countable graphs II

The *Toeplitz graph algebra* $\mathcal{T}C^*(E)$ is the universal $C^*$-algebra with generators $(\tilde{s}_e, \tilde{p}_v)_{e\in E^1, v\in E^0}$ consisting of

- $\tilde{s}_e, e \in E^1$: partial isometries with mutually orthogonal range projections, and
- $\tilde{p}_v, v \in E^0$: mutually orthogonal projections

satisfying the Toeplitz-Cuntz-Krieger relations

# $C^*$-algebras associated to countable graphs II

The *Toeplitz graph algebra* $\mathcal{T}C^*(E)$ is the universal $C^*$-algebra with generators $(\tilde{s}_e, \tilde{p}_v)_{e\in E^1, v\in E^0}$ consisting of

- $\tilde{s}_e, e \in E^1$: partial isometries with mutually orthogonal range projections, and
- $\tilde{p}_v, v \in E^0$: mutually orthogonal projections

satisfying the Toeplitz-Cuntz-Krieger relations

(T1) $\tilde{s}_e^*\tilde{s}_e = \tilde{p}_{r(e)}$ for all $e \in E^1$ and

(T2) $\tilde{p}_v \geq \sum_{e\in F} \tilde{s}_e\tilde{s}_e^*$ for every finite subset $F$ of $vE^1$.

# $C^*$-algebras associated to countable graphs II

The *Toeplitz graph algebra* $\mathcal{T}C^*(E)$ is the universal $C^*$-algebra with generators $(\tilde{s}_e, \tilde{p}_v)_{e \in E^1, v \in E^0}$ consisting of

- $\tilde{s}_e, e \in E^1$: partial isometries with mutually orthogonal range projections, and
- $\tilde{p}_v, v \in E^0$: mutually orthogonal projections

satisfying the Toeplitz-Cuntz-Krieger relations

(T1) $\tilde{s}_e^* \tilde{s}_e = \tilde{p}_{r(e)}$ for all $e \in E^1$ and

(T2) $\tilde{p}_v \geq \sum_{e \in F} \tilde{s}_e \tilde{s}_e^*$ for every finite subset $F$ of $vE^1$.

There is a surjective homomorphism $q : \mathcal{T}C^*(E) \to C^*(E)$ which sends $\tilde{s}_e$ to $s_e$ and $\tilde{p}_v$ to $p_v$ for all $e \in E^1$ and $v \in E^0$.

# KMS states on graph $C^*$-algebras

- Basic examples: $\mathcal{TO}_n$ and $\mathcal{O}_n$ with the gauge action (periodic or not necessarily so) cf. Olesen-Pedersen (1978) and Evans (1980); the Cuntz-Krieger algebra $\mathcal{O}_A$ of an irreducible matrix with the gauge action (periodic or not), cf. Enomoto-Fujii-Watatani (1984) and Zacharias (2000).
- Laca-Neshveyev (2004): for graphs with no sinks (vertices that emit no edge) using Toeplitz and Cuntz-Pimsner algebras of bimodules.
- Kajiwara-Watatani (2010): finite graphs with sinks (or sources).
- an Huef-Laca-Raeburn-Sims (2012): finite (strongly connected) graphs; proved sharp existence results for both $\mathcal{T}C^*(E)$ and $C^*(E)$.
- de Castro-Mortari (2012): non-periodic dynamics on $C^*(E)$ for countable $E$.

# KMS states on graph $C^*$-algebras

- Basic examples: $\mathcal{TO}_n$ and $\mathcal{O}_n$ with the gauge action (periodic or not necessarily so) cf. Olesen-Pedersen (1978) and Evans (1980); the Cuntz-Krieger algebra $\mathcal{O}_A$ of an irreducible matrix with the gauge action (periodic or not), cf. Enomoto-Fujii-Watatani (1984) and Zacharias (2000).
- Laca-Neshveyev (2004): for graphs with no sinks (vertices that emit no edge) using Toeplitz and Cuntz-Pimsner algebras of bimodules.
- Kajiwara-Watatani (2010): finite graphs with sinks (or sources).
- an Huef-Laca-Raeburn-Sims (2012): finite (strongly connected) graphs; proved sharp existence results for both $\mathcal{T}C^*(E)$ and $C^*(E)$.
- de Castro-Mortari (2012): non-periodic dynamics on $C^*(E)$ for countable $E$.

Goal: analyse KMS states for $\mathcal{T}C^*(E)$ and $C^*(E)$ for countable $E$.

# KMS states on $C^*$-algebras associated to finite $E$

*Proposition* (an Huef-Laca-Raeburn-Sims, 2012) Let $E$ be a finite graph, $E^*$ the set of paths and $\alpha$ the dynamics on $\mathcal{T}C^*(E)$ determined by the gauge action, i.e. for $\mu, \nu \in E^*$, $\alpha_t(s_\mu s_\nu^*) = \exp(it(|\mu| - |\nu|))s_\mu s_\nu^*$, where $\operatorname{span}\{s_\mu s_\nu^*\}$ is a dense subspace of analytic elements.
A state $\phi$ of $\mathcal{T}C^*(E)$ is KMS at $\beta \in (0, \infty)$ if and only if

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} \exp(-\beta|\mu|)\phi(p_{s(\mu)}).$$

# KMS states on $C^*$-algebras associated to finite $E$

*Proposition* (an Huef-Laca-Raeburn-Sims, 2012) Let $E$ be a finite graph, $E^*$ the set of paths and $\alpha$ the dynamics on $\mathcal{T}C^*(E)$ determined by the gauge action, i.e. for $\mu, \nu \in E^*$, $\alpha_t(s_\mu s_\nu^*) = \exp(it(|\mu| - |\nu|))s_\mu s_\nu^*$, where $\mathrm{span}\{s_\mu s_\nu^*\}$ is a dense subspace of analytic elements.
A state $\phi$ of $\mathcal{T}C^*(E)$ is KMS at $\beta \in (0, \infty)$ if and only if

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} \exp(-\beta|\mu|)\phi(p_{s(\mu)}).$$

**Theorem (an Huef-Laca-Raeburn-Sims, 2012)**

*Let $A$ be the vertex matrix of $E$. Consider $\mathcal{T}C^*(E)$ and $C^*(E)$ with the gauge action.*
(a) *If $E$ is strongly connected (i.e. $A$ is irreducible), then there is a unique $KMS_\beta$ state at* $\ln(\rho(A))$.
(b) *In general, for $\beta > \ln(\rho(A))$, the simplex of $KMS_\beta$ states of $C^*(E)$ has dimension equal to the set of sources (sinks.)*

# Exel-Laca's strategy

Exel and Laca (2003) studied KMS states for non-periodic dynamics on $C^*$-algebras associated to a countably infinite $0-1$ matrix $A$. Of the six algebras $\mathcal{TO}_A$, $\mathcal{T}_A$, $\mathcal{O}_A$ (and three unitised versions), best results are for $\mathcal{T}_A$ and $A$ irreducible.

# Exel-Laca's strategy

Exel and Laca (2003) studied KMS states for non-periodic dynamics on $C^*$-algebras associated to a countably infinite $0-1$ matrix $A$. Of the six algebras $\mathcal{TO}_A$, $\mathcal{T}_A$, $\mathcal{O}_A$ (and three unitised versions), best results are for $\mathcal{T}_A$ and $A$ irreducible. Key is: obtain $\mathcal{T}_A$ from a *partial action*

$$\mathbb{F} \curvearrowright \Omega$$

with $\Omega$ locally compact "path" space; identify scaling measures on $\Omega$ of *finite type* (supported on finite "paths") and of *infinite type* (supported on infinite "paths"); introduce partition functions and identify convergence regions depending on critical inverse temperatures:

# Exel-Laca's strategy

Exel and Laca (2003) studied KMS states for non-periodic dynamics on $C^*$-algebras associated to a countably infinite $0-1$ matrix $A$. Of the six algebras $\mathcal{TO}_A$, $\mathcal{T}_A$, $\mathcal{O}_A$ (and three unitised versions), best results are for $\mathcal{T}_A$ and $A$ irreducible. Key is: obtain $\mathcal{T}_A$ from a *partial action*

$$\mathbb{F} \curvearrowright \Omega$$

with $\Omega$ locally compact "path" space; identify scaling measures on $\Omega$ of *finite type* (supported on finite "paths") and of *infinite type* (supported on infinite "paths"); introduce partition functions and identify convergence regions depending on critical inverse temperatures:

| $\beta$ | $(-\infty, \beta'')$ | $(\beta'', \beta')$ | $(\beta', \beta)$ | $(\beta, \infty)$ |
|---|---|---|---|---|
| KMS states | none | infinite type | ? | finite type |

# Partial actions of discrete groups

A *partial action* of a discrete group $G$ on a $C^*$-algebra $A$ is

$$\Theta = \big(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big),$$

where $D_g$ is a closed two-sided ideal of $A$ for all $g \in G$ s.t.

# Partial actions of discrete groups

A *partial action* of a discrete group $G$ on a $C^*$-algebra $A$ is

$$\Theta = \big(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big),$$

where $D_g$ is a closed two-sided ideal of $A$ for all $g \in G$ s.t.

1. $D_e = A$ and $\theta_e = \mathrm{id}_A$.

# Partial actions of discrete groups

A *partial action* of a discrete group $G$ on a $C^*$-algebra $A$ is

$$\Theta = \big(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big),$$

where $D_g$ is a closed two-sided ideal of $A$ for all $g \in G$ s.t.

1. $D_e = A$ and $\theta_e = \mathrm{id}_A$.
2. $\theta_g : D_{g^{-1}} \to D_g$ is an isomorphism;

# Partial actions of discrete groups

A *partial action* of a discrete group $G$ on a $C^*$-algebra $A$ is

$$\Theta = \big(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big),$$

where $D_g$ is a closed two-sided ideal of $A$ for all $g \in G$ s.t.

1. $D_e = A$ and $\theta_e = \mathrm{id}_A$.
2. $\theta_g : D_{g^{-1}} \to D_g$ is an isomorphism;
3. $\theta_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$ and

# Partial actions of discrete groups

A *partial action* of a discrete group $G$ on a $C^*$-algebra $A$ is

$$\Theta = \big(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big),$$

where $D_g$ is a closed two-sided ideal of $A$ for all $g \in G$ s.t.

1. $D_e = A$ and $\theta_e = \mathrm{id}_A$.
2. $\theta_g : D_{g^{-1}} \to D_g$ is an isomorphism;
3. $\theta_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$ and
4. $\theta_{gh} = \theta_g\theta_h$ on $D_{(gh)^{-1}} \cap D_{h^{-1}}$, $g, h \in G$.

The crossed product $A \rtimes G$ is the enveloping $C^*$-algebra of the space $\{f : G \to A \mid f(g) \in D_g, g \in G, \sum_g \|f(g)\| < \infty\}$ endowed with suitable convolution and involution. Write $f = \sum_g a_g\delta_g$ where $f(g) = a_g \in D_g$.
References: Exel, McClanahan, Quigg-Raeburn.

# Partial actions of the free group $\mathbb{F}$

$A \rtimes \mathbb{F}$ has a grading with subspaces $D_g\delta_g$, for $g \in \mathbb{F}$. There is a canonical conditional expectation $E : A \rtimes \mathbb{F} \to D_e\delta_e$, where $D_e\delta_e \cong A$.

# Partial actions of the free group $\mathbb{F}$

$A \rtimes \mathbb{F}$ has a grading with subspaces $D_g\delta_g$, for $g \in \mathbb{F}$. There is a canonical conditional expectation $E : A \rtimes \mathbb{F} \to D_e\delta_e$, where $D_e\delta_e \cong A$.

Let $|g|$ denote the length of $g \in \mathbb{F}$, i.e. the number of generators in the reduced form of $g$. The action is

# Partial actions of the free group $\mathbb{F}$

$A \rtimes \mathbb{F}$ has a grading with subspaces $D_g\delta_g$, for $g \in \mathbb{F}$. There is a canonical conditional expectation $E : A \rtimes \mathbb{F} \to D_e\delta_e$, where $D_e\delta_e \cong A$.

Let $|g|$ denote the length of $g \in \mathbb{F}$, i.e. the number of generators in the reduced form of $g$. The action is

1. *semi-saturated*: $D_{gh} \subset D_g$ when $|gh| = |g| + |h|$.

# Partial actions of the free group $\mathbb{F}$

$A \rtimes \mathbb{F}$ has a grading with subspaces $D_g\delta_g$, for $g \in \mathbb{F}$. There is a canonical conditional expectation $E : A \rtimes \mathbb{F} \to D_e\delta_e$, where $D_e\delta_e \cong A$.

Let $|g|$ denote the length of $g \in \mathbb{F}$, i.e. the number of generators in the reduced form of $g$. The action is

1. *semi-saturated*: $D_{gh} \subset D_g$ when $|gh| = |g| + |h|$.
2. *orthogonal*: $D_x \cap D_y = \{0\}$ when $|x| = |y| = 1$ and $x \neq y$.

# Partial actions of the free group $\mathbb{F}$

$A \rtimes \mathbb{F}$ has a grading with subspaces $D_g\delta_g$, for $g \in \mathbb{F}$. There is a canonical conditional expectation $E : A \rtimes \mathbb{F} \to D_e\delta_e$, where $D_e\delta_e \cong A$.

Let $|g|$ denote the length of $g \in \mathbb{F}$, i.e. the number of generators in the reduced form of $g$. The action is

1. *semi-saturated*: $D_{gh} \subset D_g$ when $|gh| = |g| + |h|$.
2. *orthogonal*: $D_x \cap D_y = \{0\}$ when $|x| = |y| = 1$ and $x \neq y$.

If $A = C_0(X)$ with $X$ a locally compact Hausdorff space, $D_g = C_0(U_g)$ for open and closed subspaces $U_g$ of $X$, semi-saturation and orthogonality take the form (Exel):

$$U_{gh} \subset U_g, \; U_x \cap U_y = \emptyset.$$

# Partial actions of the free group $\mathbb{F}$

$A \rtimes \mathbb{F}$ has a grading with subspaces $D_g\delta_g$, for $g \in \mathbb{F}$. There is a canonical conditional expectation $E : A \rtimes \mathbb{F} \to D_e\delta_e$, where $D_e\delta_e \cong A$.

Let $|g|$ denote the length of $g \in \mathbb{F}$, i.e. the number of generators in the reduced form of $g$. The action is

1. *semi-saturated*: $D_{gh} \subset D_g$ when $|gh| = |g| + |h|$.
2. *orthogonal*: $D_x \cap D_y = \{0\}$ when $|x| = |y| = 1$ and $x \neq y$.

If $A = C_0(X)$ with $X$ a locally compact Hausdorff space, $D_g = C_0(U_g)$ for open and closed subspaces $U_g$ of $X$, semi-saturation and orthogonality take the form (Exel):

$$U_{gh} \subset U_g,\ U_x \cap U_y = \emptyset.$$

Exel-Laca: $\mathcal{T}_A = C_0(\Omega) \rtimes \mathbb{F}$ for a semi-saturated orthogonal action.

# KMS states on partial crossed products

**Theorem (Exel-Laca)**

*Let $(A, \mathbb{F}, \Theta)$ be a semi-saturated orthogonal partial action and $N : \mathbb{F} \to (1, \infty)$ a homomorphism. There is a unique dynamics $\sigma^N : \mathbb{R} \to \mathrm{Aut}(A \rtimes \mathbb{F})$ s.t. $\sigma_t^N(a\delta_g) = N(g)^{it} a\delta_g$ for $a\delta_g \in D_g\delta_g$. The $\sigma^N$-KMS states at inverse temperature $\beta \in (0, \infty)$ are given by $\varphi = \psi \circ E$, where $\psi$ is a state on $A$ s.t.*

# KMS states on partial crossed products

## Theorem (Exel-Laca)

*Let $(A, \mathbb{F}, \Theta)$ be a semi-saturated orthogonal partial action and $N : \mathbb{F} \to (1, \infty)$ a homomorphism. There is a unique dynamics $\sigma^N : \mathbb{R} \to \mathrm{Aut}(A \rtimes \mathbb{F})$ s.t. $\sigma^N_t(a\delta_g) = N(g)^{it} a\delta_g$ for $a\delta_g \in D_g\delta_g$. The $\sigma^N$-KMS states at inverse temperature $\beta \in (0, \infty)$ are given by $\varphi = \psi \circ E$, where $\psi$ is a state on $A$ s.t.*

1. *$\psi$ is a trace on $A$,*

# KMS states on partial crossed products

### Theorem (Exel-Laca)

*Let $(A, \mathbb{F}, \Theta)$ be a semi-saturated orthogonal partial action and $N : \mathbb{F} \to (1, \infty)$ a homomorphism. There is a unique dynamics $\sigma^N : \mathbb{R} \to \mathrm{Aut}(A \rtimes \mathbb{F})$ s.t. $\sigma_t^N(a\delta_g) = N(g)^{it} a\delta_g$ for $a\delta_g \in D_g\delta_g$. The $\sigma^N$-KMS states at inverse temperature $\beta \in (0, \infty)$ are given by $\varphi = \psi \circ E$, where $\psi$ is a state on $A$ s.t.*

1. *$\psi$ is a trace on $A$,*
2. *$\psi(\theta_x(f)) = N(x)^{-\beta}\psi(f)$ for $x \in \mathbb{F}$, $|x| = 1$, $f \in D_{x^{-1}}$.*

# KMS states on partial crossed products

## Theorem (Exel-Laca)

*Let $(A, \mathbb{F}, \Theta)$ be a semi-saturated orthogonal partial action and $N : \mathbb{F} \to (1, \infty)$ a homomorphism. There is a unique dynamics $\sigma^N : \mathbb{R} \to \mathrm{Aut}(A \rtimes \mathbb{F})$ s.t. $\sigma^N_t(a\delta_g) = N(g)^{it} a\delta_g$ for $a\delta_g \in D_g\delta_g$. The $\sigma^N$-KMS states at inverse temperature $\beta \in (0, \infty)$ are given by $\varphi = \psi \circ E$, where $\psi$ is a state on $A$ s.t.*

1. *$\psi$ is a trace on $A$,*
2. *$\psi(\theta_x(f)) = N(x)^{-\beta}\psi(f)$ for $x \in \mathbb{F}$, $|x| = 1$, $f \in D_{x^{-1}}$.*

*Note: $\varphi$ is a ground state if $\varphi|_{D_x} = \{0\}$ for all generators $x$.*

# KMS states on partial crossed products

## Theorem (Exel-Laca)

*Let $(A, \mathbb{F}, \Theta)$ be a semi-saturated orthogonal partial action and $N : \mathbb{F} \to (1, \infty)$ a homomorphism. There is a unique dynamics $\sigma^N : \mathbb{R} \to \mathrm{Aut}(A \rtimes \mathbb{F})$ s.t. $\sigma_t^N(a\delta_g) = N(g)^{it} a\delta_g$ for $a\delta_g \in D_g\delta_g$. The $\sigma^N$-KMS states at inverse temperature $\beta \in (0, \infty)$ are given by $\varphi = \psi \circ E$, where $\psi$ is a state on $A$ s.t.*

1. *$\psi$ is a trace on $A$,*
2. *$\psi(\theta_x(f)) = N(x)^{-\beta}\psi(f)$ for $x \in \mathbb{F}$, $|x| = 1$, $f \in D_{x^{-1}}$.*

*Note: $\varphi$ is a ground state if $\varphi|_{D_x} = \{0\}$ for all generators $x$.*

On $C^*$-algebras of etale principal groupoids, KMS states are given by quasi-invariant probability measures on the unit space via a Radon-Nikodym cocycle $e^{-\beta c}$, Renault (1980). Neshveyev (2011): non-principal groupoids.

# $C^*$-algebras associated to graphs revisited

Given *graph* $E = (E^0, E^1, s, r)$, call $v \in E^0$ *regular* if $vE^1 = \{f \in E^1 \mid s(f) = v\}$ is non-empty and finite, i.e. $v$ emits at least one and at most finitely many edges. Fix $R$ a subset of the regular vertices $E^0_{\text{reg}}$.

# $C^*$-algebras associated to graphs revisited

Given *graph* $E = (E^0, E^1, s, r)$, call $v \in E^0$ *regular* if $vE^1 = \{f \in E^1 \mid s(f) = v\}$ is non-empty and finite, i.e. $v$ emits at least one and at most finitely many edges. Fix $R$ a subset of the regular vertices $E^0_{\text{reg}}$.

The *relative graph algebra* (Muhly-Tomforde) $C^*(E, R)$ is the universal $C^*$-algebra generated by partial isometries $s_e$, $e \in E^1$ with mutually orthogonal range projections and mutually orthogonal projections $p_v$, $v \in E^0$, subject to

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,

# $C^*$-algebras associated to graphs revisited

Given *graph* $E = (E^0, E^1, s, r)$, call $v \in E^0$ *regular* if $vE^1 = \{f \in E^1 \mid s(f) = v\}$ is non-empty and finite, i.e. $v$ emits at least one and at most finitely many edges. Fix $R$ a subset of the regular vertices $E^0_{\mathrm{reg}}$.

The *relative graph algebra* (Muhly-Tomforde) $C^*(E, R)$ is the universal $C^*$-algebra generated by partial isometries $s_e$, $e \in E^1$ with mutually orthogonal range projections and mutually orthogonal projections $p_v$, $v \in E^0$, subject to

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
2. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$,

# $C^*$-algebras associated to graphs revisited

Given *graph* $E = (E^0, E^1, s, r)$, call $v \in E^0$ *regular* if $vE^1 = \{f \in E^1 \mid s(f) = v\}$ is non-empty and finite, i.e. $v$ emits at least one and at most finitely many edges. Fix $R$ a subset of the regular vertices $E^0_{\text{reg}}$.

The *relative graph algebra* (Muhly-Tomforde) $C^*(E, R)$ is the universal $C^*$-algebra generated by partial isometries $s_e$, $e \in E^1$ with mutually orthogonal range projections and mutually orthogonal projections $p_v$, $v \in E^0$, subject to

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
2. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$,
3. $p_v = \sum_{e \in vE^1} s_e s_e^*$ when $v \in R$.

# $C^*$-algebras associated to graphs revisited

Given *graph* $E = (E^0, E^1, s, r)$, call $v \in E^0$ *regular* if $vE^1 = \{f \in E^1 \mid s(f) = v\}$ is non-empty and finite, i.e. $v$ emits at least one and at most finitely many edges. Fix $R$ a subset of the regular vertices $E^0_{\text{reg}}$.

The *relative graph algebra* (Muhly-Tomforde) $C^*(E, R)$ is the universal $C^*$-algebra generated by partial isometries $s_e$, $e \in E^1$ with mutually orthogonal range projections and mutually orthogonal projections $p_v$, $v \in E^0$, subject to

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
2. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$,
3. $p_v = \sum_{e \in vE^1} s_e s_e^*$ when $v \in R$.

Note:

- $C^*(E)$ corresponds to $R = E^0_{\text{reg}}$.
- $\mathcal{T}C^*(E)$ corresponds to $R = \emptyset$.

# $C^*$-algebras associated to graphs III

$E = (E^0, E^1, s, r)$ is a countable graph and $R \subset E^0_{\mathrm{reg}}$.

# $C^*$-algebras associated to graphs III

$E = (E^0, E^1, s, r)$ is a countable graph and $R \subset E^0_{\mathrm{reg}}$. A path of length $n \geq 1$ in $E^*$ is $u = e_1 e_2 \ldots e_n$ where $s(e_{j+1}) = r(e_j)$ for $j = 1, \ldots, n-1$.

# $C^*$-algebras associated to graphs III

$E = (E^0, E^1, s, r)$ is a countable graph and $R \subset E^0_{\text{reg}}$. A path of length $n \geq 1$ in $E^*$ is $u = e_1 e_2 \ldots e_n$ where $s(e_{j+1}) = r(e_j)$ for $j = 1, \ldots, n-1$. Infinite paths in $E^\infty$ have form $e_1 e_2 \ldots$ with $s(e_{j+1}) = r(e_j)$.

# $C^*$-algebras associated to graphs III

$E = (E^0, E^1, s, r)$ is a countable graph and $R \subset E^0_{\text{reg}}$. A path of length $n \geq 1$ in $E^*$ is $u = e_1 e_2 \ldots e_n$ where $s(e_{j+1}) = r(e_j)$ for $j = 1, \ldots, n-1$. Infinite paths in $E^\infty$ have form $e_1 e_2 \ldots$ with $s(e_{j+1}) = r(e_j)$.

The total *path space* $E^{\leq\infty}$ consisting of finite and infinite paths is a totally disconnected locally compact Hausdorff space: if *cylinder sets* are $Z(u) = \{x \in E^{\leq\infty} \mid u \leq x\}$ for $u \in E^*$, then a basis of open and compact subsets is given by

$$Z_F(u) = Z(u) \setminus \Big( \bigcup_{u' \in F, u \leq u'} Z(u') \Big),$$

for $F$ finite subset of the finite paths, see e.g. Webster (2011).

The (relative) *boundary path space* is

$$\partial_R E = E^\infty \cup \{u \in E^* \mid r(u) \notin R\}.$$

# $C^*$-algebras associated to graphs III

$E = (E^0, E^1, s, r)$ is a countable graph and $R \subset E^0_{\text{reg}}$. A path of length $n \geq 1$ in $E^*$ is $u = e_1 e_2 \dots e_n$ where $s(e_{j+1}) = r(e_j)$ for $j = 1, \dots, n-1$. Infinite paths in $E^\infty$ have form $e_1 e_2 \dots$ with $s(e_{j+1}) = r(e_j)$.

The total *path space* $E^{\leq\infty}$ consisting of finite and infinite paths is a totally disconnected locally compact Hausdorff space: if *cylinder sets* are $Z(u) = \{x \in E^{\leq\infty} \mid u \leq x\}$ for $u \in E^*$, then a basis of open and compact subsets is given by

$$Z_F(u) = Z(u) \setminus \Big( \bigcup_{u' \in F, u \leq u'} Z(u') \Big),$$

for $F$ finite subset of the finite paths, see e.g. Webster (2011).

The (relative) *boundary path space* is

$$\partial_R E = E^\infty \cup \{u \in E^* \mid r(u) \notin R\}.$$

# $C^*$-algebras associated to graphs III

$E = (E^0, E^1, s, r)$ is a countable graph and $R \subset E^0_{\text{reg}}$. A path of length $n \geq 1$ in $E^*$ is $u = e_1 e_2 \dots e_n$ where $s(e_{j+1}) = r(e_j)$ for $j = 1, \dots, n-1$. Infinite paths in $E^\infty$ have form $e_1 e_2 \dots$ with $s(e_{j+1}) = r(e_j)$.

The total *path space* $E^{\leq\infty}$ consisting of finite and infinite paths is a totally disconnected locally compact Hausdorff space: if *cylinder sets* are $Z(u) = \{x \in E^{\leq\infty} \mid u \leq x\}$ for $u \in E^*$, then a basis of open and compact subsets is given by

$$Z_F(u) = Z(u) \setminus \Big( \bigcup_{u' \in F, u \leq u'} Z(u') \Big),$$

for $F$ finite subset of the finite paths, see e.g. Webster (2011). The (relative) *boundary path space* is

$$\partial_R E = E^\infty \cup \{u \in E^* \mid r(u) \notin R\}.$$

Note: $\partial_R E = E^{\leq\infty}$ when $R = \emptyset$.

# A partial action of $\mathbb{F}$ on paths

Given $E$ countable, let $\mathbb{F}$ be the free group on $|E^1|$-generators. Let 1 be the identity element in $\mathbb{F}$ and $e \in E^1$.

# A partial action of $\mathbb{F}$ on paths

Given $E$ countable, let $\mathbb{F}$ be the free group on $|E^1|$-generators. Let 1 be the identity element in $\mathbb{F}$ and $e \in E^1$. Define open and compact subsets $U(g^{-1})$ and $U(g)$ of $E^{\leq\infty}$ (or of $\partial_R E$ upon intersecting with it) as follows: for $e, e^{-1} \in \mathbb{F}$,

$$U(1) = E^{\leq\infty}$$

$$U(e) = Z(e) = \{x \in E^{\leq\infty} \mid x = ex', x' \text{ path}\},$$

$$U(e^{-1}) = Z(r(e)) = \{x \in E^{\leq\infty} \mid ex \in E^{\leq\infty}\},$$

# A partial action of $\mathbb{F}$ on paths

Given $E$ countable, let $\mathbb{F}$ be the free group on $|E^1|$-generators. Let 1 be the identity element in $\mathbb{F}$ and $e \in E^1$. Define open and compact subsets $U(g^{-1})$ and $U(g)$ of $E^{\leq\infty}$ (or of $\partial_R E$ upon intersecting with it) as follows: for $e, e^{-1} \in \mathbb{F}$,

$$U(1) = E^{\leq\infty}$$

$$U(e) = Z(e) = \{x \in E^{\leq\infty} \mid x = ex', x' \text{ path}\},$$

$$U(e^{-1}) = Z(r(e)) = \{x \in E^{\leq\infty} \mid ex \in E^{\leq\infty}\},$$

and define continuous maps $\{\phi_g\}_{g\in\mathbb{F}}$ by

$$\phi_1 = \text{id}$$

$$\phi_e(x) = ex,\ \phi_e : U(e^{-1}) \to E^{\leq\infty},$$

$$\phi_{e^{-1}}(ex') = x', \phi_{e^{-1}} : U(e) \to E^{\leq\infty}.$$

# A partial action of $\mathbb{F}$ on paths

Given $E$ countable, let $\mathbb{F}$ be the free group on $|E^1|$-generators. Let 1 be the identity element in $\mathbb{F}$ and $e \in E^1$. Define open and compact subsets $U(g^{-1})$ and $U(g)$ of $E^{\leq\infty}$ (or of $\partial_R E$ upon intersecting with it) as follows: for $e, e^{-1} \in \mathbb{F}$,

$$U(1) = E^{\leq\infty}$$

$$U(e) = Z(e) = \{x \in E^{\leq\infty} \mid x = ex', x' \text{ path}\},$$

$$U(e^{-1}) = Z(r(e)) = \{x \in E^{\leq\infty} \mid ex \in E^{\leq\infty}\},$$

and define continuous maps $\{\phi_g\}_{g\in\mathbb{F}}$ by

$$\phi_1 = \mathrm{id}$$

$$\phi_e(x) = ex, \ \phi_e : U(e^{-1}) \to E^{\leq\infty},$$

$$\phi_{e^{-1}}(ex') = x', \phi_{e^{-1}} : U(e) \to E^{\leq\infty}.$$

Thus $\phi_e(x)$ "copies" the edge $e$ to the left in $x$ and $\phi_{e^{-1}}(ex')$ "deletes" the edge $e$ from $x$.

# A partial action of $\mathbb{F}$ on paths

Define $\phi_g : U(g^{-1}) \to U(g)$ recursively:
for $g = a_i a_{i-1} \cdots a_1$ in $\mathbb{F}$ in reduced form let

$$\phi_{a_i a_{i-1} \cdots a_1}(x) = \phi_{a_i}(\phi_{a_{i-1} \cdots a_1}(x))$$

on $U((a_i a_{i-1} \cdots a_1)^{-1}) = \phi^{-1}_{a_{i-1} \cdots a_1}(U(a_i^{-1}))$.

# A partial action of $\mathbb{F}$ on paths

Define $\phi_g : U(g^{-1}) \to U(g)$ recursively:
for $g = a_i a_{i-1} \cdots a_1$ in $\mathbb{F}$ in reduced form let

$$\phi_{a_i a_{i-1} \cdots a_1}(x) = \phi_{a_i}(\phi_{a_{i-1} \cdots a_1}(x))$$

on $U((a_i a_{i-1} \cdots a_1)^{-1}) = \phi^{-1}_{a_{i-1} \cdots a_1}(U(a_i^{-1}))$.

Thus $\phi_{a_i a_{i-1} \cdots a_1}$ acts on a path $x$ by adding or deleting edges according to the reduced word $a_i a_{i-1} \cdots a_1$.

# Partial actions of $\mathbb{F}$ on path spaces

## Proposition

*The maps $\phi_g$ for $g \in \mathbb{F}$ induce a semi-saturated and orthogonal partial action $\Theta$ of $\mathbb{F}$ on $C_0(\partial_R E)$.*

# Partial actions of $\mathbb{F}$ on path spaces

## Proposition

*The maps $\phi_g$ for $g \in \mathbb{F}$ induce a semi-saturated and orthogonal partial action $\Theta$ of $\mathbb{F}$ on $C_0(\partial_R E)$.*

## Theorem (Carlsen-L)

*There is an isomorphism $\rho : C^*(E, R) \to C_0(\partial_R E) \rtimes_\Theta \mathbb{F}$ such that*

$$\rho(p_v) = \chi_{Z(v)\cap\partial_R E} \text{ for } v \in E^0$$
$$\rho(s_e) = \delta_e \text{ for } e \in E^1.$$

# Partial actions of $\mathbb{F}$ on path spaces

## Proposition

*The maps $\phi_g$ for $g \in \mathbb{F}$ induce a semi-saturated and orthogonal partial action $\Theta$ of $\mathbb{F}$ on $C_0(\partial_R E)$.*

## Theorem (Carlsen-L)

*There is an isomorphism $\rho : C^*(E, R) \to C_0(\partial_R E) \rtimes_\Theta \mathbb{F}$ such that*

$$\rho(p_v) = \chi_{Z(v)\cap\partial_R E} \text{ for } v \in E^0$$
$$\rho(s_e) = \delta_e \text{ for } e \in E^1.$$

Taking $R = \emptyset$ and $R = E^0_{\mathrm{reg}}$ gives:

1. $C_0(E^{\leq\infty}) \rtimes_\Theta \mathbb{F} \cong \mathcal{T}C^*(E)$.
2. $C_0(\partial E) \rtimes_\Theta \mathbb{F} \cong C^*(E)$.

# KMS states on $C^*(E, R)$

## Proposition

*Given $E$, $R \subseteq E^0_{\mathrm{reg}}$ and a function $N : E^1 \to (1, \infty)$, let $\sigma$ be the unique dynamics s.t. $\sigma_t(s_e) = N(e)^{it} s_e$ and $\sigma_t(p_v) = p_v$. For $\beta \in [0, \infty)$, there are isomorphisms of the convex sets of*

# KMS states on $C^*(E, R)$

## Proposition

*Given $E$, $R \subseteq E^0_{\text{reg}}$ and a function $N : E^1 \to (1, \infty)$, let $\sigma$ be the unique dynamics s.t. $\sigma_t(s_e) = N(e)^{it} s_e$ and $\sigma_t(p_v) = p_v$. For $\beta \in [0, \infty)$, there are isomorphisms of the convex sets of*

1. *$KMS_\beta$ states of $C^*(E, R)$;*

# KMS states on $C^*(E, R)$

## Proposition

*Given $E$, $R \subseteq E^0_{\text{reg}}$ and a function $N : E^1 \to (1, \infty)$, let $\sigma$ be the unique dynamics s.t. $\sigma_t(s_e) = N(e)^{it} s_e$ and $\sigma_t(p_v) = p_v$. For $\beta \in [0, \infty)$, there are isomorphisms of the convex sets of*

1. *$KMS_\beta$ states of $C^*(E, R)$;*
2. *regular Borel probability measures $\mu$ on $\partial_R E$ satisfying the* scaling condition

$$\mu(\phi_e(A)) = N(e)^{-\beta} \mu(A)$$

*for $e \in E^1$ and $A \subseteq Z(r(e)) \cap \partial_R E$ measurable;*

# KMS states on $C^*(E,R)$

## Proposition

*Given $E$, $R \subseteq E^0_{\mathrm{reg}}$ and a function $N : E^1 \to (1,\infty)$, let $\sigma$ be the unique dynamics s.t. $\sigma_t(s_e) = N(e)^{it} s_e$ and $\sigma_t(p_v) = p_v$. For $\beta \in [0,\infty)$, there are isomorphisms of the convex sets of*

1. *$KMS_\beta$ states of $C^*(E,R)$;*
2. *regular Borel probability measures $\mu$ on $\partial_R E$ satisfying the* scaling condition

$$\mu(\phi_e(A)) = N(e)^{-\beta}\mu(A)$$

*for $e \in E^1$ and $A \subseteq Z(r(e)) \cap \partial_R E$ measurable;*

3. *states $\omega$ of $C_0(\partial_R E)$ satisfying*

$$\omega(f \circ \phi_e^{-1}) = N(e)^{-\beta}\omega(f)$$

*for $e \in E^1$ and $f \in C_0(Z(r(e)) \cap \partial_R E)$.*

# KMS states on $C^*(E,R)$

To get a characterisation useful in computations, need to exploit the graph a bit more. Suppose $\beta \in [0,\infty)$.
Let $\mathcal{C}^\beta$ be the collection of functions $m : E^0 \to [0,1]$ such that

1. $\sum_{v\in E^0} m(v) = 1$;
2. $m(v) \geq \sum_{e\in vE^1} N(e)^{-\beta} m(r(e))$ for all $v \in E^0$;
3. $m(v) = \sum_{e\in vE^1} N(e)^{-\beta} m(r(e))$ if $v \in R$.

# KMS states on $C^*(E,R)$

To get a characterisation useful in computations, need to exploit the graph a bit more. Suppose $\beta \in [0,\infty)$.
Let $\mathcal{C}^\beta$ be the collection of functions $m : E^0 \to [0,1]$ such that

1. $\sum_{v \in E^0} m(v) = 1$;
2. $m(v) \geq \sum_{e \in vE^1} N(e)^{-\beta} m(r(e))$ for all $v \in E^0$;
3. $m(v) = \sum_{e \in vE^1} N(e)^{-\beta} m(r(e))$ if $v \in R$.

## Theorem (Carlsen-L)

*The set of $KMS_\beta$ states of $C^*(E,R)$ is isomorphic as a convex set to $\mathcal{C}^\beta$.*

# KMS states on $C^*(E, R)$

To get a characterisation useful in computations, need to exploit the graph a bit more. Suppose $\beta \in [0, \infty)$.
Let $\mathcal{C}^\beta$ be the collection of functions $m : E^0 \to [0, 1]$ such that

1. $\sum_{v \in E^0} m(v) = 1$;
2. $m(v) \geq \sum_{e \in vE^1} N(e)^{-\beta} m(r(e))$ for all $v \in E^0$;
3. $m(v) = \sum_{e \in vE^1} N(e)^{-\beta} m(r(e))$ if $v \in R$.

## Theorem (Carlsen-L)

*The set of KMS$_\beta$ states of $C^*(E, R)$ is isomorphic as a convex set to $\mathcal{C}^\beta$.*

Explicitly, $m \in \mathcal{C}^\beta$ corresponds to the measure $\mu$ on $\partial_R E$ that satisfies the scaling condition via

$$m(v) = \mu(Z(v) \cap \partial_R E),\ v \in E^0.$$

# $\beta$-regular and critical points

Question: which $m \in \mathcal{C}^\beta$ are extreme points?

# $\beta$-regular and critical points

Question: which $m \in \mathcal{C}^\beta$ are extreme points?
For $v \in E^0$ consider sets of finite paths

- $E^* v_{\text{aper}}$ = paths that end properly in $v$ (no finite initial subpath ends in $v$),
- $vE^* v_{\text{aper}}$ = proper loops at $v$ (no subpath ends in $v$).

# $\beta$-regular and critical points

Question: which $m \in \mathcal{C}^\beta$ are extreme points?
For $v \in E^0$ consider sets of finite paths

- $E^* v_{\text{aper}} =$ paths that end properly in $v$ (no finite initial subpath ends in $v$),
- $vE^* v_{\text{aper}} =$ proper loops at $v$ (no subpath ends in $v$).

Define *partition functions with fixed target* (a'la Exel-Laca)

- $Z_v(\beta) = \sum_{u \in E^* v_{\text{aper}}} N(u)^{-\beta}$,
- $Z_{vv}(\beta) = \sum_{u \in vE^* v_{\text{aper}}} N(u)^{-\beta}$,

# $\beta$-regular and critical points

Question: which $m \in \mathcal{C}^\beta$ are extreme points?
For $v \in E^0$ consider sets of finite paths

- $E^* v_{\text{aper}}$ = paths that end properly in $v$ (no finite initial subpath ends in $v$),
- $vE^* v_{\text{aper}}$ = proper loops at $v$ (no subpath ends in $v$).

Define *partition functions with fixed target* (a'la Exel-Laca)

- $Z_v(\beta) = \sum_{u \in E^* v_{\text{aper}}} N(u)^{-\beta}$,
- $Z_{vv}(\beta) = \sum_{u \in vE^* v_{\text{aper}}} N(u)^{-\beta}$,

and define

- $v$ is $\beta$-regular iff $Z_v(\beta) < \infty$ and $Z_{vv}(\beta) < 1$.
- $v$ is $\beta$-critical iff $Z_v(\beta) < \infty$ and $Z_{vv}(\beta) = 1$.

# $\beta$-regular and critical points

Question: which $m \in \mathcal{C}^\beta$ are extreme points?
For $v \in E^0$ consider sets of finite paths

- $E^* v_{\text{aper}}$ = paths that end properly in $v$ (no finite initial subpath ends in $v$),
- $vE^* v_{\text{aper}}$ = proper loops at $v$ (no subpath ends in $v$).

Define *partition functions with fixed target* (a'la Exel-Laca)

- $Z_v(\beta) = \sum_{u \in E^* v_{\text{aper}}} N(u)^{-\beta}$,
- $Z_{vv}(\beta) = \sum_{u \in vE^* v_{\text{aper}}} N(u)^{-\beta}$,

and define

- $v$ is $\beta$-regular iff $Z_v(\beta) < \infty$ and $Z_{vv}(\beta) < 1$.
- $v$ is $\beta$-critical iff $Z_v(\beta) < \infty$ and $Z_{vv}(\beta) = 1$.

Let $E^0_{\beta,\text{r}}$ and $E^0_{\beta,\text{c}}$ be the sets of $\beta$-regular and $\beta$-critical vertices. Fact: if $m \in \mathcal{C}^\beta$ with $m(v) \neq 0$, then $v \in E^0_{\beta,\text{r}} \sqcup E^0_{\beta,\text{c}}$.

# Main results

For $v \in E^0_{\beta,\mathrm{r}} \sqcup E^0_{\beta,\mathrm{c}}$ define $m_v^\beta : E^0 \to [0,1]$ by

$$m_v^\beta(v') = \frac{1}{Z_v(\beta)} \sum_{u \in v' E^* v_{\mathrm{aper}}} N(u)^{-\beta},\ v' \in E^0.$$

# Main results

For $v \in E^0_{\beta,\mathrm{r}} \sqcup E^0_{\beta,\mathrm{c}}$ define $m_v^\beta : E^0 \to [0,1]$ by

$$m_v^\beta(v') = \frac{1}{Z_v(\beta)} \sum_{u \in v' E^* v_{\mathrm{aper}}} N(u)^{-\beta}, \ v' \in E^0.$$

### Theorem (Carlsen-L)

*Let $\mathcal{C}_f^\beta$ consist of $m \in \mathcal{C}^\beta$ such the associated measure $\mu$ is supported on the finite paths in $\partial_R E$. Define*

$$\psi(m)(v) = \frac{Z_v(\beta)}{1 - Z_{vv}(\beta)}(m(v) - \sum_{e \in vE^1} N(e)^{-\beta} m(r(e))).$$

*Then $\psi : \mathcal{C}_f^\beta \to \{\epsilon : E^0_{\beta,\mathrm{r}} \to [0,1] \mid \sum \epsilon(v) = 1\}$ is a convex isomorphism such that $\psi(m_v^\beta) = \delta_v$ for $v \in E^0_{\beta,\mathrm{r}}$.*

# Main results

Given vertices $v_1 \neq v_2$, write $v_1 \sim v_2$ if there exist finite paths between $v_1$ and $v_2$. Then $m^\beta_{v_1} = m^\beta_{v_2}$ if and only if $v_1, v_2 \in E^0_{\beta,\mathrm{c}}$ and $v_1 \sim v_2$. Let $[v]$ denote the equivalence class of $v$ with respect to $\sim$.

## Main results

Given vertices $v_1 \neq v_2$, write $v_1 \sim v_2$ if there exist finite paths between $v_1$ and $v_2$. Then $m^\beta_{v_1} = m^\beta_{v_2}$ if and only if $v_1, v_2 \in E^0_{\beta,\mathrm{c}}$ and $v_1 \sim v_2$. Let $[v]$ denote the equivalence class of $v$ with respect to $\sim$.

Define $E^\infty_{\mathrm{rec}}$ to be the collection of infinite paths that meet a vertex of $E$ infinitely often (*recurrent* infinite paths).

# Main results

Given vertices $v_1 \neq v_2$, write $v_1 \sim v_2$ if there exist finite paths between $v_1$ and $v_2$. Then $m^\beta_{v_1} = m^\beta_{v_2}$ if and only if $v_1, v_2 \in E^0_{\beta,\mathrm{c}}$ and $v_1 \sim v_2$. Let $[v]$ denote the equivalence class of $v$ with respect to $\sim$.

Define $E^\infty_{\mathrm{rec}}$ to be the collection of infinite paths that meet a vertex of $E$ infinitely often (*recurrent* infinite paths).

## Theorem (Carlsen-L)

*Let $\mathcal{C}^\beta_{\mathrm{inf,a}}$ consist of $m \in \mathcal{C}^\beta$ such the associated measure $\mu$ is supported on $E^\infty_{\mathrm{rec}}$. The map*

$$\phi(m)(v) = m(v) Z_v(\beta)$$

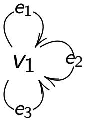
*is a well-defined isomorphism of convex sets from $\mathcal{C}^\beta_{\mathrm{inf,a}}$ to $\{\epsilon : E^0_{\beta,\mathrm{c}}/\sim \to [0,1] \mid \sum_{[v]} \epsilon([v]) = 1\}$. Moreover, $\phi(m^\beta_v) = \delta_v$ for all $v \in [v]$.*

# Examples

In case of $\mathcal{TO}_n$ we have $E^0 = \{v\}$ and $E^1 = \{e_1, e_2, \ldots, e_n\}$, where $s(e_i) = r(e_i) = v$ for all $i = 1, \ldots, n$.

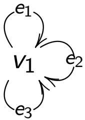
# Examples

In case of $\mathcal{TO}_n$ we have $E^0 = \{v\}$ and $E^1 = \{e_1, e_2, \ldots, e_n\}$, where $s(e_i) = r(e_i) = v$ for all $i = 1, \ldots, n$. For $n = 3$, the graph looks like



# Examples

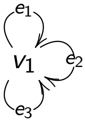
In case of $\mathcal{T}\mathcal{O}_n$ we have $E^0 = \{v\}$ and $E^1 = \{e_1, e_2, \ldots, e_n\}$, where $s(e_i) = r(e_i) = v$ for all $i = 1, \ldots, n$. For $n = 3$, the graph looks like




The $\beta$-regular and critical points and the KMS states are as follows:

# Examples

In case of $\mathcal{T}\mathcal{O}_n$ we have $E^0 = \{v\}$ and $E^1 = \{e_1, e_2, \dots, e_n\}$, where $s(e_i) = r(e_i) = v$ for all $i = 1, \dots, n$. For $n = 3$, the graph looks like

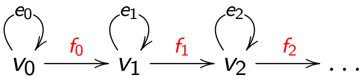



The $\beta$-regular and critical points and the KMS states are as follows:

| $\beta$ | $(0, \log n)$ | $\log n$ | $(\log n, \infty)$ |
|---|---|---|---|
| $E^0_{\beta,\mathrm{r}}$ | $\emptyset$ | $\emptyset$ | $E^0 = \{v_1\}$ |
| $E^0_{\beta,\mathrm{c}}$ | $\emptyset$ | $\{v_1\}$ | $\emptyset$ |
| $\mathrm{KMS}_\beta$ | $\emptyset$ | $\mathcal{C}^\beta_{\mathrm{inf,a}}$ | $\mathcal{C}^\beta_f$ |

# Examples

Consider the graph




For $\mathcal{T}C^*(E)$ and the gauge-action:

| $\beta$ | $0$ | $(0, \ln 2)$ | $[\ln 2, \infty)$ |
|---|---|---|---|
| $E^0_{\beta,\mathrm{r}}$ | $\emptyset$ | $E^0$ | $E^0$ |
| $E^0_{\beta,\mathrm{c}}$ | $\{v_0\}$ | $\emptyset$ | $\emptyset$ |
| $\mathrm{KMS}_\beta$ | $\mathcal{C}^\beta_{\mathrm{inf,a}} = \{m_{v_0}\}$ | $\mathcal{C}^\beta_f \sqcup m^\beta_{\mathrm{inf,b}}$ | $\mathcal{C}^\beta_f$ |

Note: $m^\beta_{\mathrm{inf,b}}$ has support on the *non-recurrent* path $f_0 f_1 \ldots$.
Such paths do not occur in the case of finite graphs.