C*-algebras associated to graphs, path spaces and equilibrium states

Nadia S. Larsen

University of Oslo

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Joint work with Toke M. Carlsen (Trondheim)
• The formalism of finite quantum systems and the KMS condition
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• The Exel-Laca strategy. Graph algebras as partial crossed products.
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• KMS states on graph algebras and measures on the path space.
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• Examples
Finite quantum systems: a time evolution on $M_n(\mathbb{C})$ is (always) given by a one-parameter group of automorphisms

$$\sigma_t(a) = e^{itH}ae^{-itH},$$

where $t \in \mathbb{R}$, $a \in M_n(\mathbb{C})$ and $H$ is a self-adjoint matrix.
The origins of the KMS condition

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where $t \in \mathbb{R}$, $a \in M_n(\mathbb{C})$ and $H$ is a self-adjoint matrix. The Gibbs state at $\beta > 0$ is $\varphi_G(a) = \frac{\text{Tr}(ae^{-\beta H})}{\text{Tr}(e^{-\beta H})}$. It minimizes a certain quantity (the free energy) and satisfies

$$\varphi_G(ab) = \varphi_G(b\sigma_i \beta(a)),$$  \hfill (1)

for $a, b \in M_n(\mathbb{C})$ analytic, i.e. allowing $\sigma_z(a), \sigma_z(b)$ for $z \in \mathbb{C}$. 

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Haag-Hugenholtz-Winnick (1967) proposed (1), the KMS condition (Kubo-Martin-Schwinger), as the equilibrium for a state on a $C^*$-algebra $A$ with a time evolution $\sigma$. The partition function of $(M_n(\mathbb{C}), \sigma)$ is $\beta \mapsto \text{Tr}(e^{-\beta H})$.

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KMS states

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By analogy with $M_n(\mathbb{C})$ and the Gibbs state, extend the notions of $\text{KMS}_\beta$ state, partition function, inverse temperature. Given $A$ a $C^*$-algebra and a time evolution (dynamics) $\sigma$ from $\mathbb{R}$ to $\text{Aut}(A)$, a $\sigma$-$\text{KMS}$ state at inverse temperature $\beta \in [0, \infty)$ is a state $\varphi$ such that

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

for all $a, b$ in $A$ with $b$ analytic, i.e. $t \rightarrow \sigma_t(b)$ extends to an entire function on $\mathbb{C}$. 
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A state $\varphi$ is a ground state if for all $a, b$ analytic, the function $z \to \varphi(a \sigma_z(b))$ is bounded in the upper-half plane.
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Verifications can often be reduced to $a, b$ in a norm-dense $*$-subalgebra of analytic elements.
First examples

The Toeplitz algebra $\mathcal{T} = C^*(s)$ has dynamics $\sigma_t(s) = e^{it}s$, $t \in \mathbb{R}$. It has a unique KMS$_\beta$ state $\varphi(s^m s^{n*}) = e^{-n\beta}\delta_{n,m}$ at $0 \leq \beta < \infty$. References: Olesen-Pedersen (1978), Evans (1980) for non-periodic case. Similar results are known for $O_n$, cf. later slide. Our interest will be in $C^*$-algebras associated to countable directed graphs.
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The Toeplitz-Cuntz algebra $\mathcal{T}\mathcal{O}_n$ is generated by isometries $s_1, \ldots s_n$ with $\sum_{j=1}^{n} s_j s_j^* < 1$. The periodic gauge action is $\sigma_t(s_j) = e^{it}s_j$, $t \in \mathbb{R}$. Form $s_\mu = s_{i_1} \ldots s_{i_m}$, $m \geq 1$. Then

$$\varphi_\beta(s_\mu s_\nu^*) = e^{-|\mu|\beta}\delta_{\mu,\nu}$$

is the unique $\text{KMS}_\beta$ state for each $\beta \geq \log n$. References: Olesen-Pedersen (1978), Evans (1980) for non-periodic case. Similar results are known for $\mathcal{O}_n$, cf. later slide.
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A directed graph $E = (E^0, E^1, s, r)$ consists of

- $E^0$ (countable) set of vertices $v$;
- $E^1$ (countable) set of edges $e$;
- $s, r : E^1 \to E^0$ the source and range map.
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The graph $C^*$-algebra $C^*(E)$ is the universal $C^*$-algebra with generators $G = \{s_e, p_v\}_{e \in E^1, v \in E^0}$, where $s_e$ are partial isometries with mutually orthogonal range projections and $p_v$ are mutually orthogonal projections, subject to the Cuntz-Krieger relations

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Enomoto-Watatani, Kumjian-Pask-Raeburn-Renault, Fowler-Raeburn, Bates-Hong-Raeburn-Szymański, ...
The *Toeplitz graph algebra* $\mathcal{T} C^*(E)$ is the universal $C^*$-algebra with generators $(\tilde{s}_e, \tilde{p}_v)_{e \in E^1, v \in E^0}$ consisting of

- $\tilde{s}_e$, $e \in E^1$: partial isometries with mutually orthogonal range projections, and
- $\tilde{p}_v$, $v \in E^0$: mutually orthogonal projections satisfying the Toeplitz-Cuntz-Krieger relations
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1. $\tilde{s}_e^* \tilde{s}_e = \tilde{p}_{r(e)}$ for all $e \in E^1$ and

2. $\tilde{p}_v \geq \sum_{e \in F} \tilde{s}_e \tilde{s}_e^*$ for every finite subset $F$ of $vE^1$. 
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\text{1) } & \tilde{s}_e^* \tilde{s}_e = \tilde{p}_{r(e)} \text{ for all } e \in E^1 \text{ and} \\
\text{2) } & \tilde{p}_v \geq \sum_{e \in F} \tilde{s}_e \tilde{s}_e^* \text{ for every finite subset } F \text{ of } vE^1.
\end{align*}

There is a surjective homomorphism $q: \mathcal{T} C^*(E) \to C^*(E)$ which sends $\tilde{s}_e$ to $s_e$ and $\tilde{p}_v$ to $p_v$ for all $e \in E^1$ and $v \in E^0$. 
KMS states on graph $C^*$-algebras

- Basic examples: $\mathcal{T}\mathcal{O}_n$ and $\mathcal{O}_n$ with the gauge action (periodic or not necessarily so) cf. Olesen-Pedersen (1978) and Evans (1980); the Cuntz-Krieger algebra $\mathcal{O}_A$ of an irreducible matrix with the gauge action (periodic or not), cf. Enomoto-Fujii-Watatani (1984) and Zacharias (2000).


- an Huef-Laca-Raeburn-Sims (2012): finite (strongly connected) graphs; proved sharp existence results for both $\mathcal{T}C^*(E)$ and $C^*(E)$.


Goal: analyse KMS states for $\mathcal{T}C^*(E)$ and $C^*(E)$ for countable $E$. 
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**KMS states on \( C^* \)-algebras associated to finite \( E \)**

*Proposition* (an Huef-Laca-Raeburn-Sims, 2012) Let \( E \) be a finite graph, \( E^* \) the set of paths and \( \alpha \) the dynamics on \( TC^*(E) \) determined by the gauge action, i.e. for \( \mu, \nu \in E^* \),

\[
\alpha_t(s_\mu s_\nu^*) = \exp(it(|\mu| - |\nu|))s_\mu s_\nu^*,
\]

where \( \text{span}\{s_\mu s_\nu^*\} \) is a dense subspace of analytic elements.

A state \( \phi \) of \( TC^*(E) \) is KMS at \( \beta \in (0, \infty) \) if and only if

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\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} \exp(-\beta |\mu|)\phi(p_{s(\mu)}).
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KMS states on $C^*$-algebras associated to finite $E$

**Proposition** (an Huef-Laca-Raeburn-Sims, 2012) Let $E$ be a finite graph, $E^*$ the set of paths and $\alpha$ the dynamics on $\mathcal{T}C^*(E)$ determined by the gauge action, i.e. for $\mu, \nu \in E^*$, $\alpha_t(s_\mu s_\nu^*) = \exp(it(|\mu| - |\nu|))s_\mu s_\nu^*$, where $\text{span}\{s_\mu s_\nu^*\}$ is a dense subspace of analytic elements.

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**Theorem** (an Huef-Laca-Raeburn-Sims, 2012)

Let $A$ be the vertex matrix of $E$. Consider $\mathcal{T}C^*(E)$ and $C^*(E)$ with the gauge action.

(a) If $E$ is strongly connected (i.e. $A$ is irreducible), then there is a unique KMS$_\beta$ state at $\ln(\rho(A))$.

(b) In general, for $\beta > \ln(\rho(A))$, the simplex of KMS$_\beta$ states of $C^*(E)$ has dimension equal to the set of sources (sinks.)
Exel-Laca’s strategy

Exel and Laca (2003) studied KMS states for non-periodic dynamics on $C^*$-algebras associated to a countably infinite $0 - 1$ matrix $A$. Of the six algebras $\mathcal{T} \mathcal{O}_A$, $\mathcal{T}_A$, $\mathcal{O}_A$ (and three unitised versions), best results are for $\mathcal{T}_A$ and $A$ irreducible.
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\[ F \bowtie \Omega \]

with $\Omega$ locally compact "path" space; identify scaling measures on $\Omega$ of finite type (supported on finite "paths") and of infinite type (supported on infinite "paths"); introduce partition functions and identify convergence regions depending on critical inverse temperatures:
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<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$(-\infty, \beta'')$</th>
<th>$(\beta'', \beta')$</th>
<th>$(\beta', \beta)$</th>
<th>$(\beta, \infty)$</th>
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<tbody>
<tr>
<td>KMS states</td>
<td>none</td>
<td>infinite type</td>
<td>?</td>
<td>finite type</td>
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</table>
Partial actions of discrete groups

A *partial action* of a discrete group $G$ on a $C^*$-algebra $A$ is

$$\Theta = (\{D_g\}_{g \in G}, \{\theta_g\}_{g \in G}),$$

where $D_g$ is a closed two-sided ideal of $A$ for all $g \in G$ s.t.
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1. $D_e = A$ and $\theta_e = \text{id}_A$. 

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4. $\theta_{gh} = \theta_g \theta_h$ on $D_{(gh)^{-1}} \cap D_{h^{-1}}$, $g, h \in G$.

The crossed product $A \rtimes G$ is the enveloping $C^*$-algebra of the space \{\(f : G \to A\) | \(f(g) \in D_g, g \in G, \sum_g \|f(g)\| < \infty\)\} endowed with suitable convolution and involution. Write \(f = \sum_g a_g \delta_g\) where \(f(g) = a_g \in D_g\).

References: Exel, McClanahan, Quigg-Raeburn.
Partial actions of the free group $\mathbb{F}$

$A \rtimes \mathbb{F}$ has a grading with subspaces $D_g \delta_g$, for $g \in \mathbb{F}$. There is a canonical conditional expectation $E : A \rtimes \mathbb{F} \rightarrow D_e \delta_e$, where $D_e \delta_e \cong A$. 

Let $|g|$ denote the length of $g \in \mathbb{F}$, i.e. the number of generators in the reduced form of $g$. The action is semi-saturated: $D_{gh} \subset D_g$ when $|gh| = |g| + |h|$. 

Orthogonal: $D_x \cap D_y = \{0\}$ when $|x| = |y| = 1$ and $x \neq y$. 

If $A = C_0(X)$ with $X$ a locally compact Hausdorff space, $D_g = C_0(U_g)$ for open and closed subspaces $U_g$ of $X$, semi-saturation and orthogonality take the form (Exel): $U_{gh} \subset U_g$, $U_x \cap U_y = \emptyset$. 

Exel-Laca: $TA = C_0(\Omega) \rtimes \mathbb{F}$ for a semi-saturated orthogonal action.
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1. **semi-saturated**: $D_{gh} \subset D_g$ when $|gh| = |g| + |h|$.
2. **orthogonal**: $D_x \cap D_y = \{0\}$ when $|x| = |y| = 1$ and $x \neq y$. 

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Exel-Laca: $\mathcal{T}_A = C_0(\Omega) \rtimes \mathbb{F}$ for a semi-saturated orthogonal action.
KMS states on partial crossed products

Theorem (Exel-Laca)

Let \((A, \mathcal{F}, \Theta)\) be a semi-saturated orthogonal partial action and \(N : \mathcal{F} \to (1, \infty)\) a homomorphism. There is a unique dynamics \(\sigma^N : \mathbb{R} \to \text{Aut}(A \ltimes \mathcal{F})\) s.t. \(\sigma^N_t(a\delta_g) = N(g)^{it} a\delta_g\) for \(a\delta_g \in D_g\delta_g\). The \(\sigma^N\)-KMS states at inverse temperature \(\beta \in (0, \infty)\) are given by \(\varphi = \psi \circ E\), where \(\psi\) is a state on \(A\) s.t.
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Note: \(\psi\) is a ground state if \(\psi |_{D_g\delta_g} = \{0\}\) for all generators \(x\). On \(C^*\)-algebras of etale principal groupoids, KMS states are given by quasi-invariant probability measures on the unit space via a Radon-Nikodym cocycle \(e^{-\beta c}\), Renault (1980). Neshveyev (2011): non-principal groupoids.
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Given graph $E = (E^0, E^1, s, r)$, call $v \in E^0$ regular if $vE^1 = \{ f \in E^1 | s(f) = v \}$ is non-empty and finite, i.e. $v$ emits at least one and at most finitely many edges. Fix $R$ a subset of the regular vertices $E^0_{\text{reg}}$.
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The relative graph algebra (Muhly-Tomforde) $C^*(E, R)$ is the universal $C^*$-algebra generated by partial isometries $s_e, e \in E^1$ with mutually orthogonal range projections and mutually orthogonal projections $p_v, v \in E^0$, subject to

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C*-algebras associated to graphs revisited

Given graph \( E = (E^0, E^1, s, r) \), call \( v \in E^0 \) regular if \( vE^1 = \{ f \in E^1 \mid s(f) = v \} \) is non-empty and finite, i.e. \( v \) emits at least one and at most finitely many edges. Fix \( R \) a subset of the regular vertices \( E^0_{\text{reg}} \).

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- $C^*(E)$ corresponds to $R = E^0_{\text{reg}}$.
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C*-algebras associated to graphs III

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The total path space $E^{\leq \infty}$ consisting of finite and infinite paths is a totally disconnected locally compact Hausdorff space: if cylinder sets are $Z(u) = \{ x \in E^{\leq \infty} \mid u \leq x \}$ for $u \in E^*$, then a basis of open and compact subsets is given by

$$Z_F(u) = Z(u) \setminus \left( \bigcup_{u' \in F, u \leq u'} Z(u') \right),$$

for $F$ finite subset of the finite paths, see e.g. Webster (2011).

The (relative) boundary path space is

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Note: $\partial_R E = E^{\leq \infty}$ when $R = \emptyset$. 
A partial action of $\mathbb{F}$ on paths

Given $E$ countable, let $\mathbb{F}$ be the free group on $|E^1|$-generators. Let 1 be the identity element in $\mathbb{F}$ and $e \in E^1$. 
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- $U(1) = E^{\leq \infty}$
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and define continuous maps $\{ \phi_g \}_{g \in \mathbb{F}}$ by

$$\phi_1 = \text{id}$$

$$\phi_e(x) = ex, \; \phi_e : U(e^{-1}) \to E^{\leq \infty},$$

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Thus $\phi_e(x)$ ”copies” the edge $e$ to the left in $x$ and $\phi_{e^{-1}}(ex')$ ”deletes” the edge $e$ from $x$. 
A partial action of $F$ on paths

Define $\phi_g : U(g^{-1}) \rightarrow U(g)$ recursively:
for $g = a_ia_{i-1} \cdots a_1$ in $F$ in reduced form let

$$\phi_{a_ia_{i-1} \cdots a_1}(x) = \phi_{a_i}(\phi_{a_{i-1} \cdots a_1}(x))$$

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Thus $\phi_{a_ia_{i-1} \cdots a_1}$ acts on a path $x$ by adding or deleting edges according to the reduced word $a_ia_{i-1} \cdots a_1$. 

Partial actions of $\mathbb{F}$ on path spaces

**Proposition**

The maps $\phi_g$ for $g \in \mathbb{F}$ induce a semi-saturated and orthogonal partial action $\Theta$ of $\mathbb{F}$ on $C_0(\partial_R E)$. 

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**Theorem (Carlsen-L)**

There is an isomorphism $\rho: C^*(E, R) \to C_0(\partial_R E) \rtimes \Theta F$ such that $\rho(p_{v}) = \chi_{Z(v)} \cap \partial_R E$ for $v \in E_0$ and $\rho(s_e) = \delta_e$ for $e \in E_1$.

Taking $R = \emptyset$ and $R = E_0$ reg gives:

1. $C_0(E_{\leq \infty}) \rtimes \Theta F \cong T$.
2. $C_0(\partial E) \rtimes \Theta F \cong C^*(E)$. 
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Taking $R = \emptyset$ and $R = E_{reg}^0$ gives:

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Proposition

Given $E$, $R \subseteq E_{\text{reg}}^0$ and a function $N : E^1 \rightarrow (1, \infty)$, let $\sigma$ be the unique dynamics s.t. $\sigma_t(s_e) = N(e)^t s_e$ and $\sigma_t(p_v) = p_v$. For $\beta \in [0, \infty)$, there are isomorphisms of the convex sets of
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KMS states on $C^*(E, R)$

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1. KMS$_\beta$ states of $C^*(E, R)$;
2. regular Borel probability measures $\mu$ on $\partial_R E$ satisfying the scaling condition

$$\mu(\phi_e(A)) = N(e)^{-\beta}\mu(A)$$

for $e \in E^1$ and $A \subseteq Z(r(e)) \cap \partial_R E$ measurable;
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3. states $\omega$ of $C_0(\partial_R E)$ satisfying

$$\omega(f \circ \phi_e^{-1}) = N(e)^{-\beta}\omega(f)$$

for $e \in E^1$ and $f \in C_0(Z(r(e)) \cap \partial_R E)$. 

C$^*$-algebras associated to graphs, path spaces and equilibrium states

Nadia S. Larsen
To get a characterisation useful in computations, need to exploit the graph a bit more. Suppose $\beta \in [0, \infty)$. Let $C^\beta$ be the collection of functions $m : E^0 \to [0, 1]$ such that

1. $\sum_{v \in E^0} m(v) = 1$;
2. $m(v) \geq \sum_{e \in vE^1} N(e)^{-\beta} m(r(e))$ for all $v \in E^0$;
3. $m(v) = \sum_{e \in vE^1} N(e)^{-\beta} m(r(e))$ if $v \in R$.

Theorem (Carlsen-L) The set of KMS $\beta$ states of $C^*(E, R)$ is isomorphic as a convex set to $C^\beta$. Explicitly, $m \in C^\beta$ corresponds to the measure $\mu$ on $\partial R E$ that satisfies the scaling condition via $m(v) = \mu(Z(v) \cap \partial R E)$, $v \in E^0$. 
KMS states on $C^*(E, R)$

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The set of $KMS_\beta$ states of $C^*(E, R)$ is isomorphic as a convex set to $C^\beta$.

Explicitly, $m \in C^\beta$ corresponds to the measure $\mu$ on $\partial_R E$ that satisfies the scaling condition via

$$m(v) = \mu(Z(v) \cap \partial_R E), \ v \in E^0.$$
$\beta$-regular and critical points

Question: which $m \in C^\beta$ are extreme points?
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For $v \in E^0$ consider sets of finite paths

- $E^*v_{aper} =$ paths that end properly in $v$ (no finite initial subpath ends in $v$),
- $vE^*v_{aper} =$ proper loops at $v$ (no subpath ends in $v$).
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Define *partition functions with fixed target* (a’la Exel-Laca)

- $Z_v(\beta) = \sum_{u \in E^\ast v_{aper}} N(u)^{-\beta}$,
- $Z_{vv}(\beta) = \sum_{u \in vE^\ast v_{aper}} N(u)^{-\beta}$,
β-regular and critical points

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\end{itemize}
and define
\begin{itemize}
  \item \( v \) is \( \beta \)-regular iff \( Z_v(\beta) < \infty \) and \( Z_{vv}(\beta) < 1 \).
  \item \( v \) is \( \beta \)-critical iff \( Z_v(\beta) < \infty \) and \( Z_{vv}(\beta) = 1 \).
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For \( v \in E^0 \) consider sets of finite paths

- \( E^* v_{aper} = \) paths that end properly in \( v \) (no finite initial subpath ends in \( v \)),
- \( vE^* v_{aper} = \) proper loops at \( v \) (no subpath ends in \( v \)).

Define *partition functions with fixed target* (a’la Exel-Laca)

\[
Z_v(\beta) = \sum_{u \in E^* v_{aper}} N(u)^{-\beta},
\]
\[
Z_{vv}(\beta) = \sum_{u \in vE^* v_{aper}} N(u)^{-\beta},
\]

and define

- \( v \) is \( \beta \)-regular iff \( Z_v(\beta) < \infty \) and \( Z_{vv}(\beta) < 1 \).
- \( v \) is \( \beta \)-critical iff \( Z_v(\beta) < \infty \) and \( Z_{vv}(\beta) = 1 \).

Let \( E_{\beta,r}^0 \) and \( E_{\beta,c}^0 \) be the sets of \( \beta \)-regular and \( \beta \)-critical vertices. Fact: if \( m \in C^\beta \) with \( m(v) \neq 0 \), then \( v \in E_{\beta,r}^0 \cup E_{\beta,c}^0 \).
Main results

For $v \in E_{\beta,r} \sqcup E_{\beta,c}^0$ define $m_v^\beta : E^0 \to [0, 1]$ by

$$m_v^\beta(v') = \frac{1}{Z_v(\beta)} \sum_{u \in v'E^*v_{aper}} N(u)^{-\beta}, \ v' \in E^0.$$
Main results

For \( v \in E_{\beta,r}^0 \sqcup E_{\beta,c}^0 \) define \( m^\beta_v : E^0 \to [0, 1] \) by

\[
m^\beta_v(v') = \frac{1}{Z_v(\beta)} \sum_{u \in v' E^*_v \text{aper}} N(u)^{-\beta}, \quad v' \in E^0.
\]

Theorem (Carlsen-L)

Let \( C_f^\beta \) consist of \( m \in C^\beta \) such the associated measure \( \mu \) is supported on the finite paths in \( \partial_R E \). Define

\[
\psi(m)(v) = \frac{Z_v(\beta)}{1 - Z_{vv}(\beta)} (m(v) - \sum_{e \in vE^1} N(e)^{-\beta} m(r(e))).
\]

Then \( \psi : C_f^\beta \to \{ \epsilon : E_{\beta,r}^0 \to [0, 1] \mid \sum \epsilon(v) = 1 \} \) is a convex isomorphism such that \( \psi(m^\beta_v) = \delta_v \) for \( v \in E_{\beta,r}^0 \).
Main results

Given vertices \( v_1 \neq v_2 \), write \( v_1 \sim v_2 \) if there exist finite paths between \( v_1 \) and \( v_2 \). Then \( m^\beta_{v_1} = m^\beta_{v_2} \) if and only if \( v_1, v_2 \in E^{0}_{\beta,c} \) and \( v_1 \sim v_2 \). Let \([v]\) denote the equivalence class of \( v \) with respect to \( \sim \).
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Define \( E^\infty_{\text{rec}} \) to be the collection of infinite paths that meet a vertex of \( E \) infinitely often (\textit{recurrent} infinite paths).
Given vertices $v_1 \neq v_2$, write $v_1 \sim v_2$ if there exist finite paths between $v_1$ and $v_2$. Then $m_{v_1}^\beta = m_{v_2}^\beta$ if and only if $v_1, v_2 \in E_0^{\beta,c}$ and $v_1 \sim v_2$. Let $[v]$ denote the equivalence class of $v$ with respect to $\sim$.

Define $E_\infty^{\text{rec}}$ to be the collection of infinite paths that meet a vertex of $E$ infinitely often (recurrent infinite paths).

**Theorem (Carlsen-L)**

Let $C_{\inf,a}^\beta$ consist of $m \in C^\beta$ such the associated measure $\mu$ is supported on $E_\infty^{\text{rec}}$. The map

$$\phi(m)(v) = m(v)Z_v(\beta)$$

is a well-defined isomorphism of convex sets from $C_{\inf,a}^\beta$ to $\{\epsilon : E_0^{\beta,c}/\sim \rightarrow [0, 1] \mid \sum[v] \epsilon([v]) = 1\}$. Moreover, $\phi(m_v^\beta) = \delta_v$ for all $v \in [v]$. 
Examples

In case of $\mathcal{T}O_n$ we have $E^0 = \{v\}$ and $E^1 = \{e_1, e_2, \ldots, e_n\}$, where $s(e_i) = r(e_i) = v$ for all $i = 1, \ldots, n$. 
In case of $\mathcal{T}O_n$ we have $E^0 = \{v\}$ and $E^1 = \{e_1, e_2, \ldots, e_n\}$, where $s(e_i) = r(e_i) = v$ for all $i = 1, \ldots, n$. For $n = 3$, the graph looks like

\begin{center}
\begin{tikzpicture}
  \node (v1) at (0,0) {$v_1$};
  \node (e1) at (-1,1) {$e_1$};
  \node (e2) at (1,1) {$e_2$};
  \node (e3) at (0,-1) {$e_3$};
  \draw[->] (v1) to (e1);
  \draw[->] (v1) to (e2);
  \draw[->] (v1) to (e3);
\end{tikzpicture}
\end{center}
Examples

In case of $TO_n$ we have $E^0 = \{v\}$ and $E^1 = \{e_1, e_2, \ldots, e_n\}$, where $s(e_i) = r(e_i) = v$ for all $i = 1, \ldots, n$. For $n = 3$, the graph looks like

\[ 
\begin{array}{c}
\text{v}_1 \\
\text{e}_1 \\
\text{e}_2 \\
\text{e}_3
\end{array}
\]

The $\beta$-regular and critical points and the KMS states are as follows:
Examples

In case of $\mathcal{T}O_n$ we have $E^0 = \{v\}$ and $E^1 = \{e_1, e_2, \ldots, e_n\}$, where $s(e_i) = r(e_i) = v$ for all $i = 1, \ldots, n$. For $n = 3$, the graph looks like

![Graph](image)

The $\beta$-regular and critical points and the KMS states are as follows:

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$(0, \log n)$</th>
<th>$\log n$</th>
<th>$(\log n, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^0_{\beta,r}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$E^0 = {v_1}$</td>
</tr>
<tr>
<td>$E^0_{\beta,c}$</td>
<td>$\emptyset$</td>
<td>${v_1}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>KMS$_\beta$</td>
<td>$\emptyset$</td>
<td>$C^\beta_{\inf,a}$</td>
<td>$C^\beta_f$</td>
</tr>
</tbody>
</table>
Examples

Consider the graph

\[
\begin{array}{ccc}
  v_0 \xrightarrow{e_0} & v_1 \xrightarrow{e_1} & v_2 \xrightarrow{e_2} & \ldots \\
  f_0 \downarrow & f_1 \downarrow & f_2 \downarrow & \ldots
\end{array}
\]

For \( \mathcal{T} C^* (E) \) and the gauge-action:

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>0</th>
<th>((0, \ln 2))</th>
<th>([\ln 2, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E^0_{\beta, r} )</td>
<td>( \emptyset )</td>
<td>( E^0 )</td>
<td>( E^0 )</td>
</tr>
<tr>
<td>( E^0_{\beta, c} )</td>
<td>( {v_0} )</td>
<td>( \emptyset )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>

KMS\( _\beta \)

\[
C^\beta_{\text{inf}, a} = \{m_{v_0}\} \quad C^\beta_f \sqcup m^\beta_{\text{inf}, b} \quad C^\beta_f
\]

Note: \( m^\beta_{\text{inf}, b} \) has support on the \textit{non-recurrent} path \( f_0 f_1 \ldots \).

Such paths do not occur in the case of finite graphs.