Quantum Symmetric States on Free Product C^*-Algebras

Claus Köstler
School of Mathematical Sciences
University College Cork

Joint work with
Ken Dykema & John Williams
arXiv:1305.7293

WIMCS-LMS Conference
Classifying Structures for Operator Algebras and Dynamical Systems
Aberystwyth University

16 - 20 September 2013

What is the free probability counterpart to the following 'semi-classical' result?

Theorem (Størmer 1969)

Let A be a unital C*-algebra with state space $\mathsf{S}(A)$. Then

$$\partial_e \mathsf{SS}(A) = \{\bigotimes_1^\infty \psi \mid \psi \in \mathsf{S}(A)\}.$$

Here $\mathsf{SS}(A)$ denotes the set of all symmetric states on $\bigotimes_1^\infty A$.

Remarks

- $\mathsf{SS}(A)$ is a Choquet simplex and $\mathsf{SS}(A) \overset{1:1}{\longleftrightarrow} M_1(\mathsf{S}(A))$
- Hewitt & Savage's result (1955) on symmetric measures on Cartesian products is recovered for $A = C(X)$, with X compact Hausdorff space.

Symmetric states

Definition

Given a unital C*-algebra A, a C*-algebraic probability space $(\mathfrak{B}, \phi)$ and unital *-homomorphisms $\lambda_i \colon A \to \mathfrak{B}$ $(i \in \mathbb{N})$, the sequence $(\lambda_i)_i$ is said to be **exchangeable w.r.t.** ϕ or the state ϕ is said to be **symmetric w.r.t.** $(\lambda_i)_i$ if

$$\phi(\lambda_{i_1}(a_1) \cdots \lambda_{i_n}(a_n)) = \phi(\lambda_{\sigma(i_1)}(a_1) \cdots \lambda_{\sigma(i_k))}(a_n)).$$

for all $n \in \mathbb{N}$, all $a_1, \ldots, a_n \in A$, all permutations σ (on $\mathbb{N}$).

Discussion of Exchangeability

- **Tensor independence:** Størmer's result is about the special case $\mathfrak{B} = \bigotimes_1^\infty A$, where λ_i is the canonical embedding of A into the i-th tensor product factor.
- Exchangeability **implies** noncommutative independence in terms of Popa's **commuting squares** (K 2010, K & Gohm 2009).
- Exchangeability is **too weak as a distributional symmetry** to single out **free independence** among all these different forms of noncommutative independence!
- **New idea** (K & Speicher 2009): Strengthen exchangeability to **quantum exchangeability**:

permutation group $\rightsquigarrow$ **quantum permutation group**

Quantum Permutation Groups

Definition and Theorem (Wang 1998)

The **quantum permutation group** $A_s(k)$ is the universal unital C^*-algebra generated by e_{ij} $(i, j = 1, \ldots, k$ subject to the relations

1. $e_{ij}^2 = e_{ij} = e_{ij}^*$ for all $i, j = 1, \ldots, k$
2. each column and row of $\begin{bmatrix} e_{11} & \cdots & e_{1k} \\ \vdots & \ddots & \vdots \\ e_{k1} & \cdots & e_{kk} \end{bmatrix}$ is a partition of unity

$A_s(k)$ is a **compact quantum group** in the sense of Woronowicz, in particular a Hopf C*-algebra.

The **abelianization** of $A_s(k)$ is $C(\mathbb{S}_k)$, the continuous functions on the symmetric group $\mathbb{S}_k$.

Quantum Exchangeability/Quantum Symmetric States

Definition (K & Speicher '09 / Dykema & K & Williams '13)

Given a unital C*-algebra A, a C*-algebraic probability space $(\mathfrak{B}, \phi)$ and unital *-homomorphisms $\lambda_i \colon A \to \mathfrak{B}$ ($i \in \mathbb{N}$), the sequence $(\lambda_i)_i$ is said to be **quantum exchangeable w.r.t.** ϕ or the state ϕ is said to be **quantum symmetric w.r.t.** $(\lambda_i)_i$ if

$$\mathbb{1}_{A_s(k)}\phi\big(\lambda_{i_1}(a_1)\cdots\lambda_{i_n}(a_n)\big) = \sum_{j_1,\ldots,j_n=1}^{k} e_{i_1 j_1}\cdots e_{i_n j_n}\phi\big(\lambda_{j_1}(a_1)\cdots\lambda_{j_n}(a_n)\big)$$

for all $n \in \mathbb{N}$, all $i_1, \ldots, i_n \in \{1, \ldots, k\}$, all $a_1, \ldots, a_n \in A$, all $k \times k$-matrices $(e_{ij})_{ij}$ satisfying defining relations for $A_s(k)$.

Remark

One has the coaction $\sum_{j=1}^{k} e_{ij} \otimes \lambda_j \colon A \to A_s(k) \otimes \mathfrak{B}$

Quantum symmetric states arising from freeness

Proposition

Let $\mathfrak{D} \subset \mathfrak{B}$ be unital C*-algebras with conditional expectation $E\colon \mathfrak{B} \to \mathfrak{D}$ (i.e. a projection of norm one onto $\mathfrak{D}$). Suppose that

$$\lambda_i\colon A \to \mathfrak{B} \qquad (i \in \mathbb{N})$$

are *-homomorphisms such that $E \circ \lambda_i$ are the same for all i and $\lambda_i(A))_{i=1}^{\infty}$ is free w.r.t. E. Let $\mathfrak{A} = *_1^{\infty} A$ and

$$\lambda = *_{i=1}^{\infty} \lambda_i\colon \mathfrak{A} \to \mathfrak{B}$$

be the resulting free product *-homomorphism. For a state ρ on $\mathfrak{D}$ consider the state $\psi = \rho \circ E \circ \lambda$. Then ψ is **quantum symmetric**.

[The proof uses Speicher's free $\mathfrak{D}$-valued cumulants and the defining properties of the projections e_{ij} from Wang's quantum permutation group.]

Tail algebras of (quantum) symmetric states

Let ψ be a state on $\mathfrak{A} = *_1^\infty A$. Passing to the GNS representation $(\mathcal{H}_\psi, \pi_\psi, \Omega_\psi)$ of $(\mathfrak{A}, \psi)$, put

$$\mathcal{M}_\psi = \pi_\psi(\mathfrak{A})'', \qquad \widehat{\psi} = \langle \Omega_\psi, \bullet\, \Omega_\psi \rangle.$$

The **tail algebra** (of $(\lambda_i)_i$) for ψ is the von Neumann subalgebra

$$\mathcal{T}_\psi = \bigcap_{n=1}^\infty W^*\Big(\bigcup_{i=n}^\infty \pi_\psi(\lambda_i(A)) \Big) \qquad \subset \mathcal{M}_\psi.$$

Proposition (Dykema & K & Williams 2013)

Suppose ψ is symmetric. Then there exists a normal $\widehat{\psi}$-preserving conditional expectation E_ψ from $\mathcal{M}_\psi$ onto $\mathcal{T}_\psi$.

Quantum symmetric distributions imply freeness with amalgamation

Theorem (Dykema & K & Williams)

Let ψ be a quantum symmetric state on $\mathfrak{A} = *_1^\infty A$ and put

$$\mathcal{B}_i := W^*\Big(\pi_\psi(\lambda_i(A)) \cup \mathcal{T}_\psi\Big).$$

Then $(\mathcal{B}_i)_{i=1}^\infty$ is free with respect to E_ψ.

[The proof is modeled along the one of K & Speicher (2009), but starts now in a C*-algebraic setting and does not assume faithfulness of a state as required for W*-probability spaces.]

What tail algebras can appear?

In Størmer's setting of symmetric states on $\bigotimes_1^\infty A$ only abelian tail algebras can arise. But in our setting of quantum symmetric states one has:

Theorem (Dykema & K 2012)

Let $\mathcal{N}$ be a countable generated von Neumann algebra. Then there exists a unital C*-algebra A and a quantum symmetric state ψ on $\mathfrak{A} = *_1^\infty A$ such that $\mathcal{T}_\psi \simeq \mathcal{N}$.

Remark

Størmer's approach does **not** use tail algebras; the machinery of ergodic decompositions of states and Choquet theory is available.

Description of quantum symmetric states QSS(A)

For a unital C*-algebra A, let $\mathcal{V}(A)$ be the set (of all equivalence classes) of quintuples $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$ such that

(i) $\mathcal{B}$ is a von Neumann algebra,

(ii) $\mathcal{D}$ is a unital von Neumann subalgebra of $\mathcal{B}$,

(iii) $E\colon \mathcal{B} \to \mathcal{D}$ is a normal conditional expectation onto $\mathcal{D}$,

(iv) $\sigma\colon A \to \mathcal{B}$ is a unital $*$–homomorphism,

(v) ρ is a normal state on $\mathcal{D}$,

(vi) the GNS rep. of $\rho \circ E$ is a faithful rep. of $\mathcal{B}$,

(vii) $\mathcal{B}$ is generated as a von Neumann algebra by $\sigma(A) \cup \mathcal{D}$,

(viii) $\mathcal{D}$ is the smallest unital von Neumann subalgebra of $\mathcal{B}$ that satisfies

$$E\big(d_0\sigma(a_1)d_1 \cdots \sigma(a_n)d_n\big) \in \mathcal{D}$$

whenever $n \in \mathbb{N}$, $d_0, \ldots, d_n \in \mathcal{D}$ and $a_1, \ldots, a_n \in A$.

Theorem (Dykema & K & Williams)

There is a bijection $\mathcal{V}(A) \to \mathrm{QSS}(A)$ that assigns to $W = (\mathcal{B}, \mathcal{D}, E, \sigma, \rho) \in \mathcal{V}(A)$ the state $\psi = \psi_W$ as follows. Let

$$(\mathcal{M}, \widetilde{E}) = (*_{\mathcal{D}})_{i=1}^{\infty}(\mathcal{B}, E)$$

be the amalgamated free product of von Neumann algebras, let

$$\pi_i \colon A \overset{\sigma}{\to} \mathcal{B} \overset{i\text{-th comp.}}{\hookrightarrow} \mathcal{M}, \qquad \pi := *_{i=1}^{\infty}\pi_i : \mathfrak{A} \to \mathcal{M}$$

free product *-homomorphism

and set $\psi = \rho \circ \widetilde{E} \circ \pi$.

Under this correspondence, the following identifications of objects and resulting constructions are naturally made:

from GNS construction	$\mathcal{T}_\psi$	$\mathcal{M}_\psi$	π_ψ	$\widehat{\psi}$	E_ψ
from quintuple W	$\mathcal{D}$	$\mathcal{M}$	π	$\rho \circ \widetilde{E}$	$\widetilde{E}$

N-pure states on von Neumann algebras

A state ψ on a C*-algebra is **pure** if whenever ρ is a state on this C*-algebra with $t\rho \leq \psi$ for some $0 < t < 1$, it follows already $\rho = \psi$.

Proposition

Let ρ be a normal state on the von Neumann algebra $\mathcal{D}$. TFAE:

(i) the support projection of ρ is a minimal projection in $\mathcal{D}$,

(ii) ρ is pure.

To emphasize this support property, a normal pure state ρ on $\mathcal{D}$ is called an **n-pure state**.

Remark

Von Neumann algebras without discrete type *I* parts possesss **no** n-pure states, but such von Neumann algebras **may** appear as tail algebra for a quantum symmetric state.

Extreme quantum symmetric states

Theorem (Dykema & K & Williams 2013)

Let $\psi \in \mathrm{QSS}(A)$ and let $W = (\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$ be the quintuple corresponding to ψ (under the bijection as indicated in the previous theorem).

$$\psi \in \partial_e \mathrm{QSS}(A) \quad \Longleftrightarrow \quad \rho \text{ is a n-pure state on } \mathcal{D}$$

Remarks

- Though having an n-pure state is a restriction on the tail algebra and forces it to have a discrete type I part, the tail algebra can still be quite complicated.
- For various examples of quantum symmetric states with 'exotic tail algebras' see our preprint.

Central quantum symmetric states

$$\begin{aligned} \mathrm{ZQSS}(A) &:= \{\psi \in \mathrm{QSS}(A) \,|\, \mathcal{T}_\psi \subset \mathcal{Z}(\pi_\psi(A)'')\} \\ \mathrm{TQSS}(A) &:= \{\psi \in \mathrm{QSS}(A) \,|\, \psi \text{ is tracial}\} \\ \mathrm{ZTQSS}(A) &:= \mathrm{ZQSS(A)} \cap \mathrm{TQSS(A)} \end{aligned}$$

Theorem (Dykema & K & Williams)

ZQSS(A) and ZTQSS(A) are compact, convex subsets of QSS(A) and both are Choquet simplices, whose extreme points are the free product states and free product tracial states, respectively:

$$\begin{aligned} \partial_e \,\mathrm{ZQSS}(A) &= \{*_1^\infty \psi \,|\, \psi \in \mathrm{S}(A)\} \\ \partial_e \,\mathrm{ZTQSS}(A) &= \{*_1^\infty \tau \,|\, \tau \in \mathrm{TS}(A)\} \end{aligned}$$

Remark

ZQSS(A) is the part of QSS(A) which is in analogy to Størmer's result on symmetric states on the minimal tensor product $\bigotimes_1^\infty A$.

What about TQQS(A)?!

. . . as just announced for the 2014 AMS-MAA Joint Mathematics Meeting in Baltimore . . .

Theorem (Dabrowski & Dykema & Mukjherjee & Williams)

TQQS(A) is a Choquet simplex.

Thank You for your attention!