Approximately inner flows, quasi-diagonal flows, and MF flows

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I have been working on approximately inner flows for some time but without much success. So what I will do is to give definitions and some simple consequences and leave obvious problems unanswered.

Let A be a C*-algebra and let $\mathrm{Aut}(A)$ be the automorphism group of A. We call a homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ a *flow* if $t \mapsto \alpha_t(x)$ is continuous for all $x \in A$.

Definition 1 *Let α be a flow on A. We say that α is* **approximately inner** *or* **AI** *if for any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there is a $h \in A_{sa}$ such that*

$$\|\alpha_t(x) - \mathrm{Ad}\, e^{ith}(x)\| \leq \epsilon \|x\|, \quad x \in \mathcal{F},\ t \in [-1, 1].$$

If α is a flow on a separable A, then there is a sequence (h_n) in A_{sa} such that

$$\alpha_t(x) = \lim_n \mathrm{Ad}\, e^{ith_n}(x), \quad x \in A,$$

uniformly in t on every bounded interval of $\mathbb{R}$.

Given a flow α we denote by δ_α the generator of α, which is a closed derivation, i.e., a closed $*$-linear map from a dense $*$-subalgebra $D(\delta_\alpha)$ of A into A such that

$$\delta_\alpha(xy) = \delta_\alpha(x)y + x\delta_\alpha(y).$$

If (h_n) is as given as above then δ_α is the graph limit of $\mathrm{ad}\, ih_n$, i.e., $x \in D(\delta_\alpha)$ iff there is a sequence (x_n) in A such that

$$x_n \to x, \qquad \mathrm{ad}\, ih_n(x_n) = i[h_n, x_n] \to \delta_\alpha(x).$$

However we choose (h_n) the set $D = \{x \in D(\delta_\alpha);\ \lim_n \mathrm{ad}\, ih_n(x) = \delta_\alpha(x)\}$ is strictly smaller than $D(\delta_\alpha)$ under a mild condition (e.g., if A is unital and simple and the Connes spectrum of α is not trivial) [2].

Proposition 2 *[6] Let A be a separable C^*-algebra. Then A is antiliminary iff there is an AI flow on A which has full Connes spectrum.*

The AI flow α constructed for the proof of 'only if' part is as follows: There is a bounded sequence (h_n) in A_{sa} and $t_0 > 0$ such that the set of elements $x \in A$

$$\sum_{n=0}^{\infty} \sum_{k_1,k_2,\ldots,k_n} \frac{t_0^n}{n!} \|\mathrm{ad} ih_{k_1} \mathrm{ad} ih_{k_2} \cdots \mathrm{ad} ih_{k_n}(x)\| < \infty$$

is dense and

$$\alpha_t(x) = \lim_n \mathrm{Ad} e^{it \sum_{i=1}^n h_i}(x), \quad x \in A.$$

This seems to be quite special among the AI flows; the D defined for the sequence $H_n = \sum_{i=1}^n h_i$ is a core for δ_α. Thus α satisfies at least the following property:

Definition 3 *We call a flow α on a C^*-algebra is* **continuously AI** *if there is a continuous function h from $[0, \infty)$ into A_{sa} such that*

$$\alpha_t(x) = \lim_{s \to \infty} \mathrm{Ad}\, e^{ith(s)}(x), \quad x \in A.$$

I believe continuous AI is strictly stronger than AI but so far without proof.

Remark 4 *[5] There is a family of simple separable C^*-algebras which have non-AI flows, e.g, the class of $A\mathbb{T}$-algebras of real rank zero with unique tracial state and $K_1(A) \neq \{0\}, \mathbb{Z}$.*

Definition 5 *[9] Let α be a flow on A. We say that α is* **quasi-diagonal** *or* **QD** *if for any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there is a $k \in \mathbb{N}$, and a flow β on M_k, and a completely positive (or CP) contraction ϕ of A into M_k such that*

$$\|\phi(x)\| \geq (1-\epsilon)\|x\|, \quad x \in \mathcal{F},$$
$$\|\phi(xy) - \phi(x)\phi(y)\| \leq \epsilon \|x\| \|y\|, \quad x, y \in \mathcal{F},$$
$$\|\phi\alpha_t - \beta_t \phi\| \leq \epsilon, \quad t \in [-1, 1].$$

Proposition 6 *Given a flow α on A α is QD iff for any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there is a covariant representation (π, U) of (A, α) and a finite-rank operator E on $\mathcal{H}_\pi$ such that*

$$\|E\pi(x)E\| \geq (1-\epsilon)\|x\|, \quad x \in \mathcal{F},$$
$$\|[E, \pi(x)]\| \leq \epsilon\|x\|, \quad x \in \mathcal{F},$$
$$\|[E, U_t]\| < \epsilon, \quad t \in [-1, 1].$$

If one (π, U) satisfies that for any pair $(\mathcal{F}, \epsilon)$ and any finite-rank projection F on $\mathcal{H}_\pi$ there is a finite-rank projection E such that $E \geq F$ and the second and third conditions of the above proposition hold, then we say that (π, U) is QD.

Proposition 7 *Suppose that (π, U) is QD. Then there is an increasing sequence (P_n) of finite-rank projections on $\mathcal{H}_\pi$ such that $\|[P_n, \pi(x)]\| \to 0$ for $x \in A$ and $\|[P_n, H_U]\| \to 0$ and $\lim_n P_n = 1$, where $U_t = e^{itH_U}$.*

Since H_U is unbounded in general, $[P_n, H_U]$ means that $P_n D(H_U) \subset D(H_U)$ and is defined as the closure of the operator $P_n H_U - H_U P_n$ on $D(H_U)$

Proposition 8 *Let A be a unital simple separable exact C^*-algebra and let α be a flow on A such that α has full Connes spectrum. Then the following conditions are equivalent:*

1. *α is QD.*
2. *There exists a covariant irreducible representation (π, U) such that $\pi \times U$ is faithful and essential and (π, U) is QD.*
3. *For any covariant representation (π, U) if $\pi \times U$ is faithful and essential then (π, U) is QD.*
4. *For any covariant representation (π, U) of (A, α) if $\pi \times U$ is faithful and essential there exists a sequence (ϕ_n) of unital CP maps of A into $\mathcal{B}(\mathcal{H}_\pi)$, a sequence (h_n) in A_{sa}, and a sequence (H_n) of self-adjoint operators on $\mathcal{H}_\pi$ such that $\phi_n(A)$ is a finite-dimensional C^*-algebra,*

$$\mathrm{Ad}\, e^{itH_n}\phi_n = \phi_n\alpha_t^{(h_n)},$$
$$\|h_n\| \to 0,$$
$$\|H_U - H_n\| \to 0,$$
$$\|\pi(x) - \phi_n(x)\| \to 0, \quad x \in A,$$

and $H_U + \pi(h_n) - H_n$ is compact.

An idea of the proof of deriving Condition (3) based on [4].

Lemma 9 *Let α be a flow on a unital C^*-algebra A. Let H be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ and let $\gamma_t = \mathrm{Ad}\, e^{itH}$ on $\mathcal{B}(\mathcal{H})$. Let $\mathcal{B}(\mathcal{H})_\gamma$ be the maximal C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ on which γ is strongly norm-continuous. Let ϕ be a unital CP map of A into $\mathcal{B}(\mathcal{H})_\gamma$ such that*

$$\gamma_{-t}\phi\alpha_t \le e^{\epsilon|t|}\phi, \quad t \in \mathbb{R}.$$

Then there exists a covariant representation (π, U) of (A, α) and an isometry W of $\mathcal{H}$ into $\mathcal{H}_\pi$ with $P = WW^*$ such that $\pi(A)P\mathcal{H}_\pi$ is dense in $\mathcal{H}_\pi$,

$$\phi(\,\cdot\,) = W^*\pi(\,\cdot\,)W,$$
$$\|[P, H_U]\| \le \epsilon/2,$$
$$W^* H_U W = H.$$

This is a covariant version of the Stinespring theorem.

A CP map satisfying the above condition may be obtained from any CP map $\psi : A \to \mathcal{B}(\mathcal{H})$ by

$$\phi = \frac{\epsilon}{2}\int e^{-\epsilon|t|}\gamma_{-t}\psi\alpha_t dt.$$

We call α an RF flow if there is a faithful family of α-covariant finite-dimensional representations.

Lemma 10 *Let α be an RF flow on a unital separable C^*-algebra A and let (π, U) be a covariant nuclear representation of (A, α). Let $\gamma_t = \mathrm{Ad}\, U_t$ on $\mathcal{B}(\mathcal{H}_\pi)$ and let $\mathcal{B}(\mathcal{H}_\pi)_\gamma$ be the maximal C^*-subalgebra of $\mathcal{B}(\mathcal{H}_\pi)$ on which $t \mapsto \gamma_t(x)$ is norm-continuous. Then for any decreasing sequence (ϵ_n) of positive numbers converging to zero there exists a sequence (r_n) in $\mathbb{N}$ and a flow β_n on M_{r_n}, unital CP maps*

$$\sigma_n : A \to M_{r_n}, \quad \tau_n : M_{r_n} \to \mathcal{B}(\mathcal{H}_\pi)_\gamma$$

such that

$$\beta_{n,-t}\sigma_n\alpha_t \le e^{\epsilon_n|t|}\sigma_n, \quad \gamma_{-t}\tau_n\beta_{n,t} \le e^{\epsilon_n|t|}\tau_n, \tag{1}$$
$$\|\tau_n\sigma_n(x) - \pi(x)\| \to 0, \quad x \in A. \tag{2}$$

Proof. Without the condition (1) (and with $\mathcal{B}(\mathcal{H})$ in place of $\mathcal{B}(\mathcal{H})_\gamma$) this follows from π being nuclear. We may further suppose that σ_n is α-covariant, i.e., there is a covariant representation (π_n, U_n) with a finite-rank projection P_n on the representation Hilbert space $\mathcal{H}_n$ such that $P_n\mathcal{H}_n$ is in the domain of the self-adjoint generator H_n of U_n and σ_n can be identified with $P_n\pi_n(\,\cdot\,)P_n$. By adding a covariant representation to (π_n, U_n) if necessary we may suppose that $\pi_n \times U_n$ is faithful and essential.

Since (π_n, U_n) is QD, for any $\delta > 0$ there is a finite-rank projection E on $\mathcal{H}_n$ such that

$$P_n \leq E, \quad \|[E, H_n]\| < \delta/2.$$

Denote by V the unitary flow on $E\mathcal{H}_n$ generated by EH_nE and set $\beta_t = \mathrm{Ad}\, V_t$ on $\mathcal{B}(E\mathcal{H}_n)$. Then

$$\|(U_t - V_t)E\| \leq \delta|t|/2.$$

We fix n and set $\sigma'(x) = E\pi_n(x)E$, which is a unital CP map of $A \to \mathcal{B}(E\mathcal{H}_{\pi_n}) = M_r$ with $r = \dim E\mathcal{H}_{\pi_n}$ satisfying

$$\|\beta_{-t}\sigma'\alpha_t - \sigma'\| \leq \delta|t|.$$

We replace σ' by the unital CP map $\sigma'' : A \to M_r$ defined by

$$\sigma''(x) = \frac{\epsilon}{2}\int e^{-\epsilon|t|}\beta_{-t}\sigma'\alpha_t(x)dt.$$

Then it follows that $\|\sigma'' - \sigma'\| \leq \delta/\epsilon$.

We define a CP map τ' of $\mathcal{B}(E\mathcal{H}_n)$ into $\mathcal{B}(\mathcal{H}_\pi)$ by $\tau'(Q) = \tau_n(P_nQP_n)$, which satisfies that

$$\tau'\sigma'(x) = \tau_n\sigma_n(x).$$

Then we define a CP map $\tau'' : \mathcal{B}(E\mathcal{H}_n) \to \mathcal{B}(\mathcal{H}_\pi)_\gamma$ by

$$\tau''(x) = \frac{\epsilon}{2}\int e^{-\epsilon|t|}\gamma_{-t}\tau'\beta_t dt.$$

Since

$$\|\gamma_{-t}\tau'\beta_t\sigma'(x) - \pi(x)\| \leq \delta|t|\|x\| + \|\tau_n\sigma_n\alpha_t(x) - \pi\alpha_t(x)\|,$$

it follows

$$\|\tau''\sigma'(x) - \pi(x)\| \leq \delta/\epsilon\|x\| + \frac{\epsilon}{2}\int e^{-\epsilon|t|}\|\tau_n\sigma_n\alpha_t(x) - \pi\alpha_t(x)\|dt.$$

Then note:

$$\|\tau''\sigma''(x)-\pi(x)\| \le \delta/\epsilon\|x\| + \|\tau''\sigma'(x)-\pi(x)\|$$

and

$$\beta_{-t}\sigma''\alpha_t \le e^{\epsilon|t|}\sigma'', \quad \gamma_{-t}\tau''\beta_t \le e^{\epsilon|t|}\tau''.$$

We obtain the desired σ, τ as σ'', τ'' by setting $\delta = \epsilon^2/4$ and choosing a sufficiently large n. QED

The proof of Proposition (1)⇒(3) for RF flows:

Since α is an RF flow we can assume that there is an ideal I_n with $\sigma_n(I_n) = 0$ and $\dim A/I_n < \infty$. Note $\gamma_{-t}\phi_n\alpha_t \le e^{2\epsilon_n|t|}\phi_n$ for $\phi_n = \tau_n\sigma_n$.

By the Stinespring Theorem there is a covariant representation (ρ_n, V_n) and an isometry $W_n : \mathcal{H}_\pi \to \mathcal{H}_n$ such that $\phi_n(\,\cdot\,) = W_n^*\rho_n(\,\cdot\,)W_n$, $H_U = W_n^* H_n W_n$ etc., where $\dim \rho_n(A) < \infty$. Then setting $P_n = W_n W_n^*$, Voiculescu's theorem shows ρ_n is close to π:

$$\begin{aligned}\rho_n &\sim P_n\rho_n P_n \oplus (1-P_n)\rho_n(1-P_n)\\ &\sim \pi \oplus (1-P_n)\rho_n(1-P_n)\\ &\sim \pi \oplus \pi \oplus (1-P_n)\rho_n(1-P_n)\\ &\sim \pi \oplus \rho_n\\ &\sim \pi.\end{aligned}$$

Following the definition and results on MF C*-algebras [1]:

Definition 11 *Let α be a flow on A. We say that α is* **MF** *if for any finite subset $\mathcal{F}$ of $D(\delta_\alpha)$ and $\epsilon > 0$ there is a $k \in \mathbb{N}$, a flow β on M_k, and a bounded $*$-linear map of A into M_k such that*

$$\begin{aligned}(1-\epsilon)\|x\| \le \|\phi(x)\| &\le (1+\epsilon)\|x\|, \;\; x \in \mathcal{F},\\ \|\phi(xy)-\phi(x)\phi(y)\| &\le \epsilon\|x\|\|y\|, \;\; x, y \in \mathcal{F},\\ \|\phi\alpha_t(x) - \beta_t\phi(x)\| &\le \epsilon\|x\|, \;\; x \in \mathcal{F}, \; t \in [-1,1],\\ \|\phi\alpha_s(x) - \phi\alpha_t(x)\| &\le \epsilon\|x\| + (1+\epsilon)\|\alpha_s(x)-\alpha_t(x)\|, \;\; x, y \in \mathcal{F}.\end{aligned}$$

Compare with the definition of QD:

α is QD if for any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there is a $k \in \mathbb{N}$, and a flow β on M_k, and a CP contraction ϕ of A into M_k such that

$$\begin{aligned}\|\phi(x)\| &\ge (1-\epsilon)\|x\|, \;\; x \in \mathcal{F},\\ \|\phi(xy)-\phi(x)\phi(y)\| &\le \epsilon\|x\|\|y\|, \;\; x, y \in \mathcal{F},\\ \|\phi\alpha_t - \beta_t\phi\| &\le \epsilon, \;\; t \in [-1,1].\end{aligned}$$

Proposition 12 *Let A be a separable C^*-algebra and α a flow on A. Then α is MF iff it can be embedded into the flow $\beta = \prod_{n=1}^{\infty} \beta_n$ on*

$$(\prod_n M_{k_n})_\beta / \bigoplus_n M_{k_n}$$

for some sequence (k_n) in $\mathbb{N}$ and some flow β_n on the matrix algebra M_{k_n}, where $(\prod_n M_{k_n})_\beta$ denotes the maximal C^*-subalgebra on which β is continuous.

The flow α is quasi-diagonal iff this embedding can be derived from a CP contraction from B into $(\prod_n M_{k_n})_\beta$. (Hence if B is nuclear then MF flows are automatically QD.)

Instead of (M_{k_n}, β_n) we may take the C^*-algebra $\mathcal{K}$ of compact operators on $L^2(\mathbb{R})$ and the flow $\gamma : t \mapsto \mathrm{Ad}\, U_t$ for all n, where U_t is the unitary induced by translation by t. That is, with $\gamma_t^\infty = \prod_n \gamma_t$, (A, α) is MD iff it is realized as a subsystem of

$$((\prod_n \mathcal{K})_{\gamma^\infty} / \bigoplus \mathcal{K}, \gamma^\infty)$$

An outline of the proof.

Suppose that there is an embedding ϕ of A into $(\prod_{n=1}^{\infty} M_{k_n})_\beta / \bigoplus_{n=1}^{\infty} M_{k_n}$ such that $\phi\alpha_t = \beta_t\phi$, where β also denotes the induced flow on the quotient.

Then there is a map

$$\phi' : A \to (\prod_{n=1}^{\infty} M_{k_n})_\beta \quad \text{with} \quad Q\phi' = \phi$$

where Q is the quotient map of $(\prod_{n=1}^{\infty} M_{k_n})_\beta / \bigoplus_{n=1}^{\infty} M_{k_n}$. We may suppose that ϕ' is a $*$-linear map. We may also suppose that $\phi'(D(\delta_\alpha)) \subset D(\delta_\beta)$.

We write $\phi' = (\phi_n)$, where ϕ_n is a $*$-linear map of A into M_{k_n}. We may assume

$$\lim_n \|\phi_n(a)\| = \|a\|, \quad a \in A.$$

Since ϕ is a homomorphism with $\phi\alpha_t = \beta_t\phi$ we have that

$$\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \to 0, \quad a, b \in A$$

and

$$\|\phi_n\alpha_t(a) - \beta_{n,t}\phi_n(a)\| \to 0, \quad t \in \mathbb{R}, \quad a \in A.$$

Let V be a finite-dimensional $*$-linear subspace of $D(\delta_\alpha)$ and $\epsilon > 0$. Set

$$D = \dim V, \quad E = \max\{\|\delta_\alpha|V\|, \|\delta_\beta\phi'|V\|\}.$$

We choose an $N \in \mathbb{N}$ such that

$$\frac{E(1+\epsilon)(2N+1)^{1/2}D^{1/2}}{N} < \epsilon/3.$$

Let V_1 be the finite-dimensional subspace generated by

$$\{\alpha_t(x) \mid x \in V,\ t \in [-1,1] \cap \mathbb{Z}/N\}.$$

Then V_1 is $*$-linear and $\dim V_1 \leq (2N+1)D$. Noting that $V_1^* = V_1$ let P be a $*$-linear projection from A onto V_1 such that $\|P\| \leq \sqrt{(2N+1)D}$. (The existence follows from Pisier [10].)

For a sufficiently large m, $\phi_m : A \to M_{k_m}$ satisfies that

$$(1-\epsilon)\|x\| \leq \|\phi_m(x)\| \leq (1+\epsilon)\|x\|, \quad x \in V_1, \tag{3}$$

$$\|\phi_m(x)\phi_m(y) - \phi_m(xy)\| \leq \epsilon\|x\|\|y\|, \quad x, y \in V_1 \tag{4}$$

$$\|\phi_m\alpha_t(x) - \beta_t\phi_m(x)\| \leq \epsilon\|x\|/2, \quad t \in [-1,1] \cap \mathbb{Z}/N, \quad x \in V. \tag{5}$$

Define a $*$-linear map $\phi : A \to M_{k_m}$ by

$$\phi = \phi_m P.$$

which satisfies $\|\phi\| \leq (1+\epsilon)\sqrt{(2N+1)D}$, which implies that $\|\phi\|E/N < \epsilon/3$.

The first two conditions ((3) and (4) above) are satisfied since $\phi(x) = \phi_m(x)$ for $x \in V$.

Let $t \in [-1,1]$ and $x \in V$. Since there is a $t_0 \in [-1,1] \cap \mathbb{Z}/N$ such that $|t - t_0| \leq (2N)^{-1}$, we have that

$$\begin{aligned}\|\phi\alpha_t(x) - \beta_{m,t}\phi(x)\| &\leq \|\phi\alpha_t(x) - \phi\alpha_{t_0}(x)\| + \|\beta_t\phi(x) - \beta_{t_0}\phi(x)\| + \|\phi\alpha_{t_0}(x) - \beta_{t_0}\phi(x)\| \\ &\leq \|\phi\|\|x\|E/2N + \|x\|E/2N + \epsilon\|x\|/2 \\ &\leq \epsilon\|x\|.\end{aligned}$$

Thus the third condition ((5) for $t \in [-1,1]$) is also satisfied. The fourth condition (continuity of $t \mapsto \phi\alpha_t(x),\ \ x \in V$, independent of ϕ) will be shown similarly.

Suppose α is an MF flow. Let S be a countable dense subset of A with $S^* = S \subset D(\delta_\alpha)$. Let S_1 be the union of all $\alpha_t(S)$, $t \in \mathbb{Q}$ and let S_2 be the union of $S_1, S_1 \cdot S_1, S_1 \cdot S_1 \cdot S_1, \ldots$. Note that S_2 is a countable subset of $D(\delta_\alpha)$ and the linear span B of S_2 is a $*$-subalgebra of A, left invariant under α_t, $t \in \mathbb{Q}$. Let (b_i) be an enumeration of S_2.

For the linear subspace generated by $b_1, b_2, \ldots, b_m$ and $\epsilon = 1/m$ we obtain $k_m \in \mathbb{N}$ and a bounded $*$-linear map of A into M_{k_m} and a flow β_m on M_{k_m} such that the conditions in the definition are satisfied. Then we can form a $*$-linear map $\phi = (\phi_m)$ of A into $\prod_{m=1}^\infty M_{k_m} / \bigoplus_{m=1}^\infty M_{k_m}$. Let $\beta_t = \prod_{m=1}^\infty \beta_{m,t}$ on $\prod_{m=1}^\infty M_{k_m}$ and let $(\prod_{m=1}^\infty M_{k_m})_\beta$ be the maximal C*-subalgebra on which β is continuous. Then it follows that $\phi|B$ is an isometric isomorphism into $(\prod_{m=1}^\infty M_{k_m})_\beta / \bigoplus_{m=1}^\infty M_{k_m}$ such that $\phi\alpha_t(x) = \beta_t\phi(x)$ for $x \in B$ and $t \in \mathbb{R}$.

Thus $\phi|B$ extends to an isomorphism $\bar{\phi}$ of A into $(\prod_{m=1}^\infty M_{k_m})_\beta / \bigoplus_{m=1}^\infty M_{k_m}$. Since $\phi\alpha_t(x) = \beta_t\phi(x)$ for $x \in B$ and $s, t \in \mathbb{Q}$ we deduce that $\bar{\phi}\alpha_t(x) = \beta_t\bar{\phi}(x)$ for all $x \in A$ and $t \in \mathbb{R}$.

Unfortunately we do not have a non-trivial example which shows approximate innerness is really stronger in case the C*-algebra is simple. See [12] and [1] for quasidiagonal C*-algebras and MF C*-algebras respectively.

Theorem 13

1. *QD implies MF for flows.*
2. *If A is a QD C*-algebra then AI implies QD for flows.*
3. *If A is an MF (separable) C*-algebra then AI implies MF for flows.*
4. *If A is amenable (separable) then QD and MF are equivalent for flows.*
5. *If α is an MF flow on a unital C*-algebra, then α has KMS states for all inverse temperatures.*

Let (A_n, α_n) be a continuous field of flows over $\mathbb{N} \cup \{\infty\}$, i.e., $A_n, n \in \mathbb{N} \cup \{\infty\}$, is a continuous field of C*-algebras; α_n is a flow on A_n; and if $n \mapsto x_n \in A_n$ is a continuous field then $n \mapsto \alpha_{n,t}(x_n)$ is for all $t \in \mathbb{R}$. Thus the C*-algebra A generated by continuous fields has a flow α defined through $\alpha_n, n \in \mathbb{N} \cup \{\infty\}$.

Proposition 14 *In the above situation if α_n is an MF flow for $n \in \mathbb{N}$ then α_∞ is.*

If A_n is finite-dimensional for $n \in \mathbb{N}$ then α_∞ is an MF flow.

A basic ingredient of the proof of all these is a generalization of Voiculescu's Weyl-von Neumann theorem [11]. For example,

Lemma 15 *Let π and ρ be representations of $A \times_\alpha \mathbb{R}$. Suppose that $\rho^{-1}(\mathcal{K}(\mathcal{H}_\rho)) \subset$* $\mathrm{Ker}(\pi)$. *Then there is a sequence (W_n) of isometries of $\mathcal{H}_\pi$ into $\mathcal{H}_\rho$ such that*

1. *$W_n W_n^*$ are mutually orthogonal,*

2. *$\pi(x) - W_n^* \rho(x) W_n$ is compact and $\|\pi(x) - W_n^* \rho(x) W_n\| \to 0$ for $x \in A \cup A \times_\alpha \mathbb{R}$,*

3. *$H_U - W_n^* H_V W_n$ is compact and $\|H_U - W_n^* H_V W_n\| \to 0$,*

4. *$[W_n W_n^*, H_V]$ is compact and $\|[W_n W_n^*, H_V]\| \to 0$.*

Lemma 16 *Let ρ be a faithful representation of $A \times_\alpha \mathbb{R}$ and let $I = \{x \in A \times_\alpha \mathbb{R} \mid (\rho \times U)(x) \in \mathcal{K}(\mathcal{H}_\rho)\}$. Let π be a representation of $A \times_\alpha \mathbb{R}$ such that $\pi(I) = 0$. Then there is a sequence of unitaries $W_k : \mathcal{H}_\rho \oplus \mathcal{H}_\pi \to \mathcal{H}_\rho$*

1. *$\rho(x) - W_k^*(\rho(x) \oplus \pi(x)) W_k$ is compact and $\|\rho(x) - W_k^*(\rho(x) \oplus \pi(x)) W_k\| \to 0$ for $x \in A \cup A \times_\alpha \mathbb{R}$,*

2. *$\rho(H) - W_k^*(\rho(H) \oplus \pi(H)) W_k$ is compact and $\|\rho(H) - W_k^*(\rho(H) \oplus \pi(H)) W_k\| \to 0$.*

The above text is obtained by assembling all the slides presented in the conference. The questions and comments I received after suggest I should have added the following: The condition that a flow α on A is QD is strictly stronger than the crossed product $A \times_\alpha \mathbb{R}$ being QD; an example is given by the translation flow γ on $C_0(\mathbb{R})$, where γ is not QD but $C_0(\mathbb{R}) \times_\gamma \mathbb{R}$ is QD by being the compact operators. If the group $\mathbb{R}$ is replaced by a discrete group G, then a natural concept for an action of G to be QD may be that the crossed product is QD.

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