# C*-correspondences related to Dini spaces

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**Notations:**

A function $f\colon X \to [0,\infty)$ is a **Dini function** on a $T_0$ space $X$ if $f$ is lower semi-continuous (=: l.s.c.) and every upward directed net of l.s.c. functions $\{g_\alpha\}$ that converges point-wise to $f$ converges also uniformly to $f$.

A $T_0$ space $X$ is **sober** (also: "tidy", "point-complete", "spectral") if each prime closed subset is the closure of a singleton.

A **Dini space** is a sober second countable $T_0$ space such that the supports of the Dini functions build a base of the topology.
One can show that the Dini spaces are exactly the 2nd countable l.c. sober $T_0$ spaces. (It is a Theorem – not a Definition.)

**Coherent** l.c. spaces $X$:
The *saturation* $\mathrm{Sat}(Y)$ of $Y \subset X$ is the intersection of all open sets $U \subset X$ with $U \supseteq Y$. Then $Y$ is **saturated** if $Y = \mathrm{Sat}(Y)$.
$X$ is **coherent** if $C_1 \cap C_2$ is compact of any saturated compact subsets $C_1$ and $C_2$ of $X$.

**Question:** Is every Dini space homeomorphic to the primitive spectrum of a separable amenable C*-algebra?

It turns out that coherent Dini spaces are primitive ideal spaces of separable amenable C*algebras.

We outline the idea of the proof later, and use it then for some other partial results, e.g. in the following theorem and corollaries.

## Theorem

*Let $X$ a Dini space. Then there exists a stable separable amenable $C^*$-algebra $A \cong A \otimes \mathcal{O}_2$ with coherent primitive spectrum $Z := \mathrm{Prim}(A)$ and a *-monomorphism $h\colon A \to \mathcal{M}(A)$ such that $h$ and the l.s.c. action $\Psi := \Psi_h\colon \mathcal{I}(A) \to \mathcal{I}(A)$ of $Z$ on $A$ given by*

$$\Psi_h(J) := h^{-1}(h(A) \cap \mathcal{M}(A, J)) \quad \textit{for } J \in \mathcal{I}(A) \cong \mathbb{O}(Z)$$

*satisfy:*

- (o) *The lattice $\mathbb{O}(X)$ is order isomorphic to the image $\Psi(\mathcal{I}(A))$ of $\Psi$ in the lattice $\mathcal{I}(A)$.*
- (i) *$h(A)A = A$ ($h$ is non-degenerate).*
- (ii) *$h$ is unitarily equivalent in $\mathcal{M}(A)$ to its infinite repeat $\delta_\infty \circ h$ ($= h \oplus h \oplus \cdots$).*
- (iii) *$\mathcal{M}(h) \circ h$ is approximately unitary equivalent to $h$ in $\mathcal{M}(A)$.*
- (iv) *$J \subseteq \Psi(J)$ for all $J \in \mathcal{I}(A)$.*

Properties (o)-(iv) imply also:

(v) $\Psi^2 = \Psi$.

(vi) $\Phi(J) \subseteq J$ for all $J \in \mathcal{I}(A)$, where $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(A)$ is the Galois adjoint of $\Psi$. (Explicitly given by $\Phi(J) := Ah(J)A$.)

We can derive following corollaries:

**Corollary**

*For every Dini space $X$ there exists an at most 1-dimensional locally compact Polish space $P$ and a *-monomorphism*

$$h\colon\ \mathrm{C}_0(P, \mathbb{K}) \to \mathcal{M}(\mathrm{C}_0(P, \mathbb{K})) = \mathrm{C}_{\mathrm{b,st}}(P, \mathcal{M}(\mathbb{K}))$$

*such that $h$ is non-degenerate, approximately unitary equivalent to its infinite repeat $\delta_\infty \circ h$, and to $\mathcal{M}(h) \circ h$, and that $\mathbb{O}(X)$ is order isomorphic to the image of the l.s.c. map the $\Psi_h\colon \mathbb{O}(P) \to \mathbb{O}(P)$.*

We say that (the ideal lattice $\mathcal{I}(B)$ of) a stable C*-algebra $B$ admits an **Abelian factorization** if there exist a locally compact Hausdorff space $X$ and, with $C := \mathrm{C}_0(X, \mathbb{K})$, C*-morphisms $\phi\colon B \to \mathcal{M}(C)$ and $\psi\colon C \to \mathcal{M}(B)$ such that for each $J \in \mathcal{M}(B)$ and each $b \in B$ holds

$$b \in J \quad \Longleftrightarrow \quad \phi(\psi(b)C)B \subseteq J\,. \tag{1}$$

If $B$ is moreover *separable* then one can replace $X$ by a Polish l.c. space and $\phi$ and $\psi$ by non-degenerate *-monomorphisms. Then $\theta := \mathcal{M}(\phi) \circ \psi$ satisfies, for $J \in \mathcal{I}(B)$.

$$J = \Psi_\theta(J) := \theta^{-1}(\theta(B) \cap \mathcal{M}(B, J))\,. \tag{2}$$

The natural generalization of $\mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B))$ for separable C*-algebras $B$ is $\mathbb{O}(Y)$ with Dini space $Y$.

**Definition**

We say that the lattice $\mathbb{O}(Y)$ of a Dini space $Y$ has **Abelian factorization**, if there exists a Polish l.c. space $X$ and non-degenerate l.s.c. actions $\Phi\colon \mathbb{O}(X) \to \mathbb{O}(Y)$ and $\Psi\colon \mathbb{O}(Y) \to \mathbb{O}(X)$ with $\Phi \circ \Psi = \mathrm{id}$ on $\mathbb{O}(Y)$.

It turns out that $B$ is separable and stable, then $B$ has Abelian factorization
if and only if
the Dini space $\mathrm{Prim}(B)$ has Abelian factorization.

Proof of the last statement:
The definitions show that $\mathbb{O}(Y)$ for $Y := \mathrm{Prim}(B)$ has Abelian factorization in sense of the general definition if $B$ has Abelian factorization.
If $\mathbb{O}(Y) \cong \mathcal{I}(B)$ has Abelian factorization via $\Phi$ and $\Psi$ then there exists non-degenerate *-monomorphisms $\phi\colon B \to \mathcal{M}(C)$ and and $\psi\colon C \to \mathcal{M}(B)$ with $C := \mathrm{C}_0(X, \mathbb{K})$ that define $\Phi$ and $\Psi$ by $\Psi(I) := \psi^{-1}\mathcal{M}(C, I)$ and $\Phi(J) := \phi^{-1}\mathcal{M}(B, J)$. Then $\phi$ and $\psi$ satisfy above equation (2).

## Corollary

*Let $A$ and $B$ separable stable C*-algebras and $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ a non-degenerate lower semi-continuous action.*
*Then there is a commutative C*-algebra $C$ and non-degenerate *-homomorphisms $h\colon C \otimes \mathbb{K} \to \mathcal{M}(B)$ and $k\colon A \to \mathcal{M}(C \otimes \mathbb{K})$ such that*

$$\Psi(J) = (\mathcal{M}(h) \circ k)^{-1}\mathcal{M}(B, J)$$

*and every $\Psi$-residually nuclear map $V\colon A \to B$ for separable $A$ is in the uniform closure of the c.p. maps $V_{b,c}\colon A \to B$ given by $V_{b,c}(a) := b^* h(c^* k(a)c)b$ with $c \in C \otimes \mathbb{K}$ and $b \in B$.*

## Corollary

*All stable separable C*-algebras $B$ and Dini space $X$ have Abelian factorization.*

H.Harnisch + E.K. + M.Rørdam:

A necessary and sufficient *criterium* for a sober $\mathrm{T}_0$ space $X$ to be homeomorphic to $\mathrm{Prim}(A)$ for a separable amenable (=nuclear) C*-algebra $A$ is the following:

*The lattice $\mathbb{O}(X)$ is order isomorphic to a* sup- *and* inf-*invariant sub-lattice of the lattice $\mathbb{O}(P)$ for a Polish l.c. space $P$.*

Then we say that the Dini space $X$ has an "amenable realization".

There are equivalent topological characterizations that do not use lattices.

The Polish space $P$ can be taken always (at most) 1-dimensional.

There is no natural choice of $P$.

The criterium can be used to show that $\mathbb{O}(X)$ has an Abelian decomposition if $X$ is homeomorphic to the primitive ideal space of an amenable separable C*-algebra.

It is *not* known if the Dini spaces $X := \mathrm{Prim}(B)$ for (not necessarily exact) separable C*-algebras $B$ have amenable realizions.

Mixed topological and C*-algebraic arguments show the permanence of the *class of Dini spaces that have an "amenable realization"* under the following operations (1)–(3):

(1) Passage to open or closed subsets (obvious).

(2) If $X$ is a Dini space (not a general top. space !) and there exists $U \subset X$ open such that $U \cong \mathrm{Prim}(B)$ and $X \setminus U \cong \mathrm{Prim}(C)$ for some separable amenable C*-algebras $B$ and $C$, then $X$ has an amenable realization.
("*Solution of the extension problem*" for amenable realizable Dini spaces.)

(3) If $X_1, X_2, \ldots$ is a sequence of amenable realizable Dini spaces and $\Psi_n \colon \mathbb{O}(X_{n+1}) \to \mathbb{O}(X_n)$ are non-degenerate lower s.c. and monotone upper s.c. actions, then the topological space $Y$ with $\mathbb{O}(Y)$ corresponding to the algebraic projective limit of $\mathbb{O}(X_1) \leftarrow \mathbb{O}(X_2) \leftarrow \cdots$ is again amenable realizable.

Define a family of very particular Dini spaces:

Let $X$ a closed subset of the Hilbert cube $Q := [0,1]^\infty$ (closed with respect to the Tikhonov product topology of the $[0,1]$ with Hausdorff topology), but later equipped with the $\mathrm{T}_0$ l.s.c. -topology induced from Tikhonov product $Q_{\mathrm{lsc}} = ([0,1]_{\mathrm{lsc}})^\infty$, where $[0,1]_{\mathrm{lsc}}$ is the space $[0,1]$ with open sets $\emptyset$, $[0,1]$ and $(t,1]$ with $t \in [0,1)$. We denote $X$ with this (induced) topology by $X_{\mathrm{lsc}}$. The compact metric space $X$ with its Hausdorff topology will be denoted $X_H$. Then $\pi := \mathrm{id}_X \colon X_H \to X_{\mathrm{lsc}}$ defines a bijective continuous epimorphism, and $\pi^{-1} \colon \mathbb{O}(X_{\mathrm{lsc}}) \to \mathbb{O}(X_H)$ is an upper semi-continuous embedding (its image is closed under unions).

We say that $X \subseteq Q$ is **solid** if the image $\pi^{-1}(\mathbb{O}(X_{\mathrm{lsc}})) \subseteq \mathbb{O}(X_H)$ is invariant under forming interiors of intersections.
The solid spaces $X_{\mathrm{lsc}}$ are amenable realizable, because they satisfy the Harnisch-K-Rørdam criterium obviously.

Several geometric observations together show the following observation (5) that implies later that all coherent Dini spaces are amenable realizable:

**Theorem**

*(5) The above defined spaces $X_{\mathrm{lsc}}$ are compact coherent Dini spaces, that can be build with help of the operations (1)-(3) from the subclass of solid spaces $X_{\mathrm{lsc}}$.*

The spaces $X_{\mathrm{lsc}}$ are not solid in general: There are closed subsets $Y$ in $[0,1]^2$ such that $X := Y \times [0,1]^\infty$ is not solid.

(6) One gets immediately from permanences (1)-(3) that a Dini space $X$ is amenable realizable if $X$ has a decomposition series $\emptyset = U_0 \subset U_1 \subset \ldots \subset U_\alpha \subset \cdots U_\beta = X$ indexed by ordinals $\alpha$ such that $U_\gamma = \bigcup_{\alpha<\gamma} U_\alpha$ for limit ordinals $\gamma \leq \beta$ and with $U_{\alpha+1} \setminus U_\alpha$ amenable realizable for $0 \leq \alpha < \beta$.

Let $Y$ (any) Dini space.
We equip the lattice $\mathbb{F}(Y)$ with the lsc.-topology (of E.Michael) generated by the complements $\mathbb{F}(Y) \setminus [\emptyset, F]$ of the intervals.
Notation: $\quad \mathbb{F}(Y)_{\mathrm{lsc}}$ .

The Fell-Vietoris topology on $\mathbb{F}(Y)$ is a Hausdorff topology that can be described equivalently by the natural embedding

$$\lambda \colon \mathbb{F}(Y) \to Q_H$$

given by a dense sequence $f_1, f_2, \ldots$ in the Dini functions $f \colon Y \to [0,1]$ with $\sup f(Y) = 1$ by
$\lambda(F) := (\sup f_1(F), \sup f_2(F), \ldots)$. Here let $\sup f_n(\emptyset) := 0$.
Then $\lambda(\mathbb{F}(Y))$ is a closed subspace of $Q_H$ and $\{\lambda(\emptyset)\} = \{0\}$ is a closed singleton in $Q_{\mathrm{lsc}}$.

In particular, $\mathbb{F}(Y)_H$ is a compact metric space. The map $\lambda$ defines also a homeomorphism from $\mathbb{F}(Y)_{\mathrm{lsc}}$ onto the subspace $\lambda(\mathbb{F}(Y))$ of $Q_{\mathrm{lsc}}$.

We denote by $\eta := \eta_X \colon X \to \mathbb{F}(X)$ the homeomorphism from $X$ onto the subspace $\eta(X)$ of $\mathbb{F}(X)_{\mathrm{lsc}}$ given by $\eta(x) := \overline{\{x\}}$.

If $Z \cup \{\emptyset\}$ is a closed subspace of of $\mathbb{F}(Y)_H$ then $Z_{\mathrm{lsc}}$ is amenable realizable.
One can show that *a Dini space $Z$ is coherent if and only if $Y := \{\emptyset\} \cup \eta(Z)$ is closed in $\mathbb{F}(Z)_H$.*

Since $\mathbb{F}(Z)_H$ is in a natural way a closed subset of the Hilbert cube $Q_H$, and is a topological subspace of $Q_{\mathrm{lsc}}$, it follows that $Y_{\mathrm{lsc}}$ has an amenable realization.
The set $\{\emptyset\}$ is a closed subset $Q_{\mathrm{lsc}}$ and (therefore) of $Y_{\mathrm{lsc}}$. Thus $Z \cong \eta_Z(Z)$ has an amenable realization if $Z$ is coherent.

One can show that lower semi-continuous actions $\Psi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ for stable separable $A$ and $B$ can be realized as $\Psi(J) := h^{-1}(h(B) \cap \mathcal{M}(A, J))$ if one of $A$ or $B$ is amenable. It reduces the proofs of the Theorem and the Corollaries to the construction of the corresponding topological l.s.c. actions.

In particular, it suffices to find for given Dini space $X$ a coherent Dini space $Z$ and non-degenerate l.s.c. actions $\mu\colon \mathbb{O}(Z) \to \mathbb{O}(X)$ and $\nu\colon \mathbb{O}(X) \to \mathbb{O}(Z)$ such that $\mu \circ \nu = \mathrm{id}_{\mathbb{O}(X)}$.

Define $Z \subset \mathbb{F}(X)_{\mathrm{lsc}}$ and the opposite maps $\mu^{op}\colon \mathbb{F}(Z) \to \mathbb{F}(X)$ and $\nu^{op}\colon \mathbb{F}(X) \to \mathbb{F}(Y)$ of $\mu$ and $\nu$ as follows:

$$Z := \overline{\eta(X)}^{H} \setminus \{\emptyset\} \subset \mathbb{F}(X)$$

with topology $Z_{\mathrm{lsc}}$ induced from $\mathbb{F}(X)_{\mathrm{lsc}}$.
Notice that $X \cong \eta(X) \subset Z$ is a topological subspace of $Z$.
Define $\mu^{op}$ by

$$G \in \mathbb{F}(Z) \mapsto \mu^{op}(G) := \eta^{-1}(G \cap \eta(X)),$$

and $\nu^{op}$ by

$$F \in \mathbb{F}(X) \mapsto Z \cap \overline{\eta(F)} \in \mathbb{F}(Z).$$

The maps have the required properties, because $\mu^{op}(G)$ is the same as $\bigvee G \subset X$: Indeed, let $\mathcal{G}$ a closed subset of of $\mathbb{F}(X)_{\mathrm{lsc}}$ and $G := Z \cap \mathcal{G}$. Then

$$\eta(\bigvee \mathcal{G}) = \eta(X) \cap \mathcal{G} = \eta(X) \cap G.$$

Here $\bigvee \mathcal{Y}$ denotes the closure of union $\bigcup_{M \in Y} M$ of all subsets $M \subset X$ with $M \in \mathcal{Y}$ for $\mathcal{Y} \subset 2^X$.

Next frames are optional:

**On the used C*-methods**

Among the used C*-algebra methods are the *embedding theorem*:

(I) If $A$ is an exact and separable C*-algebra, $B$ is strongly purely infinite and

$$\Psi\colon \mathbb{O}(\mathrm{Prim}(B)) \cong \mathcal{I}(B) \to \mathbb{O}(\mathrm{Prim}(A)) \cong \mathcal{I}(A)$$

is a *given* l.s.c. action with the additional properties that it is faithful ($\Psi(\{0\}) = \{0\}$), non-degenerate ($\Psi^{-1}(A) = \{B\}$) and $\Psi$ is also monotone upper s.c.,
i.e., $\Psi$ respects sup's and inf's of (upward or downward) *directed* nets and respects (finite) intersections,
then there exists a nuclear *-monomorphism $h\colon A \hookrightarrow B$ that realizes $\Psi$ via formula

$$\Psi(J) := h^{-1}(J) \quad \forall\, J \in \mathcal{I}(B)\,.$$

(II) If $B$ is stable and separable, $B$ admits an Abelian factorization (see Def. above) and $A$ is separable, then any (faithful and non-degenerate) l.s.c. action $\Psi$ of $\mathrm{Prim}(B)$ on $A$ can be realized by by $h\colon A \to \mathcal{M}(B)$ such that

$$\Psi(J) = h^{-1}(h(A) \cap \mathcal{M}(B, J))$$

and $h$ is $\Psi$-residually nuclear, which means that the c.p. maps $V_b\colon A \to B$, given by

$$V_b(a) := b^* h(a) b\,,$$

have the property that

$$[V_b]_J\colon A/\Psi(J) \to B/J$$

are nuclear for all $J \in \mathcal{I}(B)$.

(III) If $A$ is a (not necessarily locally reflexive) C*-algebra then it can happen that there are nuclear maps $V\colon A \to B$ and ideals $I \in \mathcal{I}(A)$, $J \in \mathcal{I}(B)$ such that $V(I) \subseteq J$ and $[V]_J\colon A/I \to B/J$ is not nuclear.

Examples of such $I \triangleleft A$ and $J \triangleleft B$ can be constructed using the *observation of N. Ozawa* that discrete groups $G$ with weakly exact $vN(G) \subseteq \mathcal{L}(\ell_2(G))$ must be exact.

(If $A$ is exact or only *locally reflexive* then nuclear maps $V\colon A \to B$ are always residual nuclear. Proof: Use tensor product characterizations of loc. reflexivity and of nuclear maps.)

The applicability of the embedding theorem to (the solution of) the extension problem uses that for strongly purely infinite separable stable algebras $B$ the sub-lattice of the countably generated ideals of the corona $\mathcal{M}(B)/B$ are generated by elements that are infinite repeats of elements of $B$ (considered modulo $B$).

## Definition

An Abelian C*-subalgebra $C \subseteq B$ is a regular Abelian C*-subalgebra of $B$, if

$$\Psi\colon J \in \mathcal{I}(B) \mapsto J \cap C \in \mathcal{I}(C)$$

is injective and satisfies

$$\Psi(I + J) = \Psi(I) + \Psi(J)\,.$$

Such $\Psi$ maps $\mathcal{I}(B)$ order isomorphic onto a sup-inf-invariant sub-lattice of $\mathbb{O}(\widehat{C})$. Thus $\mathrm{Prim}(B)$ is amenable realizable.

Separable C*-algebras $B$ with the property that $B \otimes \mathcal{O}_2$ contains a regular Abelian C*-subalgebra $C$ have Abelian factorization.

In particular:
The lattices of all Dini spaces that have amenable realizations have also an Abelian factorization.

If $B$ is any stable, separable and strongly purely infinite C*-algebra then $\mathrm{C_b}([0,\infty),B)/\,\mathrm{C_0}([0,\infty),B)$ contains a separable stable strongly purely C*-subalgebra $D$ with $B \subseteq D$ such that $D$ contains a regular Abelian C*-subalgebra.
This allows to remove the restrictions to $B$ with Abelian factorization in the embedding theorem and for the realization of l.s.c. actions to general strongly p.i. $B$.

It follows:
If $B$ is separable and stable, then $B$ has Abelian factorization
if and only if
the Dini space $\mathrm{Prim}(B)$ has Abelian factorization.

The proof uses that $Y := \mathrm{Prim}(B) = \mathrm{Prim}(B \otimes \mathcal{O}_2)$, $\mathcal{M}(B \otimes \mathcal{O}_2) \subset \mathcal{M}(B)$ is strongly p.i. and that a (non-degenerate) lower s.c. action $\Psi\colon \mathcal{I}(B) \cong \mathbb{O}(Y) \to \mathbb{O}(P)$ of $Y$ on a Polish l.c. space $P$ defines an l.s.c. action of $\mathcal{I}(\mathcal{M}(B \otimes \mathcal{O}_2)) \to \mathbb{O}(P)$ that is monotone upper semi-continuous.

Thus we can apply the embedding theorem to $\mathrm{C}_0(P, \mathbb{K})$ and $\mathcal{M}(B \otimes \mathcal{O}_2)$.