Universal C*-algebras and K-theory for Conformal Nets

based on joint works with S. Carpi, R. Conti, Y. Kawahigashi, R. Longo, M. Weiner and F. Xu

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Introduction

- Goal in past: mathematically rigorous approach to quantum field theory.
- Idea in past: use operator algebras modelling the observables, ending up with so-called Haag-Kastler nets, or in more special cases (like here) local conformal nets.

- Goal today: understand how this relates to K-theory, dynamical systems, spectral triples and noncommutative geometry, and how the two sides can enrich one another.
- Notice: there are established relations between the latter and conformal field theory without nets and with partly different interpretations.

Local conformal nets

Let $\mathcal{I}$ = set of proper open intervals of circle S^1.

Definition

A local conformal net over S^1 is a family $\mathcal{A} = \{\mathcal{A}(I) \subset B(\mathcal{H}) : I \in \mathcal{I}\}$ of von Neumann algebras on separable $\mathcal{H}$ satisfying

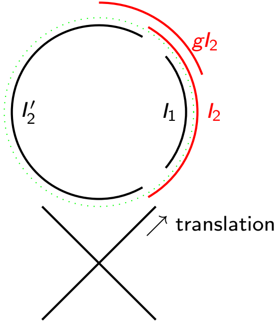
(i) $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$, if $I_1 \subset I_2$, $I_1, I_2 \in \mathcal{I}$,

(ii) $\mathcal{A}(I') = \mathcal{A}(I)'$, $I \in \mathcal{I}$,

(iii) $\mathrm{PSL}(2,\mathbb{R})$-covariance on S^1, namely $U : \mathrm{PSL}(2,\mathbb{R}) \to B(\mathcal{H})$ s.th. $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(g.I)$,

(iv) the conformal Hamiltonian L_0 generating the rotation subgroup of $\mathrm{PSL}(2,\mathbb{R})$ has positive spectrum,

(v) there exists a unique vacuum vector $\xi \in \mathcal{H}$ (i.e., $\mathrm{PSL}(2,\mathbb{R})$-invariant and cyclic for $\mathcal{A}$).



Properties of conformal nets

- Cyclicity. ξ cyclic for $\mathcal{A}$; cyclic and separating for every $\mathcal{A}(I)$, $I \in \mathcal{I}$.
- Factoriality and irreducibility. $\mathcal{A}(I) =$ (hyperfinite) III_1 factor, $\mathcal{A} = B(\mathcal{H})$.
- Geometric modularity. Locally $\Delta_{\mathbb{R}_+}^{\mathrm{i}t} = U(\Lambda(2\pi t))$, for $t \in \mathbb{R}$, and

$$U(\Lambda(2\pi s))\, T_t\, U(\Lambda(2\pi s))^* = T_{e^s t}.$$

- Often: (strong) additivity. $\bigvee_k \mathcal{A}(I_k) = \mathcal{A}(I)$ if $\bigcup_k I_k = I$.
- Often: split property. If $\bar{I}_1 \cap \bar{I}_2 = \emptyset$:

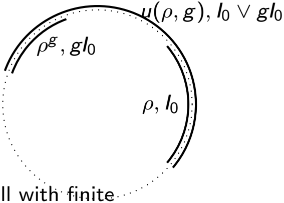
$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \simeq \mathcal{A}(I_1) \otimes \mathcal{A}(I_2).$$

Example

Let G be a compact, simply-connected, simply-laced Lie group, $\pi_{l,0}$ a level l and highest weight 0 unitary irreducible representation of the corresponding loop group.

$$\mathcal{A}_G(I) := \{\pi_{l,0}(x) : x \in C^\infty(S^1, G), x|_{I'} = 1\}'', \quad I \in \mathcal{I}.$$

Representations and endomorphism



- (Locally normal) representation of $\mathcal{A}$: covariant family $\pi = \{\pi_I : \mathcal{A}(I) \to B(\mathcal{H}_\pi) : I \in \mathcal{I}\}$ of (normal) rep's on common separable $\mathcal{H}_\pi$ such that

$$\pi_{I_2}|_{\mathcal{A}(I_1)} = \pi_{I_1} \text{ if } I_1 \subset I_2.$$

- Statistical dimension: $d(\pi)^2 := [\pi_I(\mathcal{A}(I))' : \pi_{I'}(\mathcal{A}(I'))]$.
- Rational net (roughly): finitely many classes of irrep's, all with finite statistical dimension
- Localised endomorphisms: every representation unitarily equivalent to injective localised (unital) *-endomorphism of $\mathcal{A}$ with any given localisation interval $I_0 \in \mathcal{I}$.
- Fusion semiring structure (for rational $\mathcal{A}$). Composition and addition

$$[\rho] + [\sigma] = [v_1\rho(\cdot)v_1^* + v_2\sigma(\cdot)v_2^*], \quad [\rho]\cdot[\sigma] = [\rho\circ\sigma]$$

giving fusion rules for rational net, with $N_{ij}^k \in \mathbb{N}_0$:

$$[\rho_i]\cdot[\rho_j] = \sum_k N_{ij}^k [\rho_k]$$

- Further keywords: $\mathcal{A}(I) - \mathcal{A}(I)$ Hilbert bimodules, subfactors, braid group

Universal C*-algebras

Definition (Fredenhagen-Rehren-Schroer '92, Guido-Longo '92)

The universal C*-algebra of $\mathcal{A}$ is the unique (up to isomorphism) unital C*-algebra $C^*(\mathcal{A})$ such that

- $C^*(\mathcal{A})$ is generated by $\mathcal{A}(I), I \in \mathcal{I}$;
- (universal property) for every representation π of $\mathcal{A}$ on $\mathcal{H}_\pi$, there is a unique representation $\hat{\pi} : C^*(\mathcal{A}) \to B(\mathcal{H}_\pi)$ such that $\pi_I = \hat{\pi} \circ \iota_I, \quad I \in \mathcal{I}$.

- Representations and endomorphisms of $C^*(\mathcal{A})$ corresponding to representations and endomorphisms of $\mathcal{A}$.

Theorem (Carpi-Conti-RH-Weiner '12)

The universal C-algebra $C^*(\mathcal{A})$ is unital, properly infinite, has stable rank ∞ and it is generated by projections. It is neither separable, nor simple, nor exact, nor purely infinite.*

- Usually need universal property only for locally normal representations. Let $(\pi_{\text{ln}}, \mathcal{H}_{\text{ln}})$ be the universal locally normal representation of $C^*(\mathcal{A})$. Define the locally normal universal C*-algebra of $\mathcal{A}$ as the image $C^*_{\text{ln}}(\mathcal{A}) := \pi_{\text{ln}}(C^*(\mathcal{A}))$.

Universal C*-algebras for rational nets

Suppose $\mathcal{A}$ rational. Write ρ_i, with $i = 1, \dots n$, for a suitable such family of representatives. They satisfy the fusion rules

$$[\rho_i] \cdot [\rho_j] = \sum_k N_{ij}^k [\rho_k].$$

Write $\mathcal{R}_{\mathcal{A}}$ for the generated (commutative) fusion semiring.

Theorem (Carpi-Conti-RH-Weiner '12)

- *$C^*_{\mathrm{ln}}(\mathcal{A}) \simeq B(\mathcal{H})^{\oplus n}$.*
- *Define $\mathfrak{K}_{\mathcal{A}} := \mathcal{K}(\mathcal{H}_{\mathrm{ln}}) \cap C^*_{\mathrm{ln}}(\mathcal{A})$.*
- *Every finite-statistic locally normal endomorphism ρ of $\mathcal{A}$ gives rise to a unique normal faithful endomorphism $\hat{\rho}$ of $C^*_{\mathrm{ln}}(\mathcal{A})$ preserving $\mathfrak{K}_{\mathcal{A}}$.*

K-theory for rational net

Easy observation:

- $\mathcal{A}(I)$ and $C^*_{\text{ln}}(\mathcal{A})$ have trivial K-theory.
- $K_0(\mathfrak{K}_{\mathcal{A}}) = \mathbb{Z}^n$ and $K_1(\mathfrak{K}_{\mathcal{A}}) = 0$.

Theorem (Carpi-Conti-RH-Weiner '12/'13)

- *The push-forward of $*$-homomorphisms of $\mathfrak{K}_{\mathcal{A}}$ gives rise to a faithful semiring action of the fusion semiring $\mathcal{R}_{\mathcal{A}}$ on $K_0(\mathfrak{K}_{\mathcal{A}})$, namely an injective semiring homomorphism $\eta : \mathcal{R}_{\mathcal{A}} \to \mathrm{End}(K_0(\mathfrak{K}_{\mathcal{A}}))$ satisfying $\eta_{[\rho]} = (\hat{\rho}|_{\mathfrak{K}_{\mathcal{A}}})_*$ for every localized covariant endomorphism ρ of $C^*(\mathcal{A})$.*
- *It corresponds to the regular representation of the fusion algebra associated to $\mathcal{R}_{\mathcal{A}}$.*
- *It can be expressed as a Kasparov product $KK(\mathbb{C}, \mathfrak{K}_{\mathcal{A}}) \times KK(\mathfrak{K}_{\mathcal{A}}, \mathfrak{K}_{\mathcal{A}}) \to KK(\mathbb{C}, \mathfrak{K}_{\mathcal{A}})$:*

$$a \times j([\rho]) = \eta_{[\rho]}(a), \quad a \in K_0(\mathfrak{K}_{\mathcal{A}}),\ [\rho] \in \mathcal{R}_{\mathcal{A}},$$

with a natural injective semiring homomorphism $j : \mathcal{R}_{\mathcal{A}} \to KK(\mathfrak{K}_{\mathcal{A}}, \mathfrak{K}_{\mathcal{A}})$.

Spectral triples

Definition

A θ-summable spectral triple $(A, (\pi, \mathcal{H}), Q)$ consists of

- a *-algebra A
- a separable Hilbert space $\mathcal{H}$ and a *-representation $\pi : A \to B(\mathcal{H})$;
- a selfadjoint operator Q on $\mathcal{H}$ such that e^{-tQ^2} is trace-class, for $t \in \mathbb{R}_+$, and such that $\pi(A) \subset \mathrm{dom}(\delta)$, with δ the derivation on $B(\mathcal{H})$ induced by Q.

The spectral triple is called *even* if there is a grading Γ on $\mathcal{H}$ such that $[\Gamma, \pi(A)] = 0$ and $\Gamma Q \Gamma = -Q$. Otherwise, it is called *odd*.

Associated a canonical entire cyclic cocycle τ, the JLO cocycle. Details

Our application: graded-local conformal net $\mathcal{A}$ (supersymmetry).

Graded-Local conformal nets

Let $\mathcal{I}$ = set of proper open intervals of circle S^1.

Definition

A graded-local conformal net over S^1 is a family $\mathcal{A} = \{\mathcal{A}(I) \subset B(\mathcal{H}) : I \in \mathcal{I}\}$ of von Neumann algebras on separable $\mathcal{H}$ graded by selfadjoint unitary $\Gamma \in B(\mathcal{H})$ satisfying

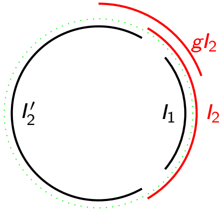
(i) $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$, if $I_1 \subset I_2$, $I_1, I_2 \in \mathcal{I}$,

(ii) $\mathcal{A}(I') = Z\mathcal{A}(I)'Z^*$, $I \in \mathcal{I}$, $Z = \frac{1+i\Gamma}{1+i}$,

(iii) $\mathrm{PSL}(2,\mathbb{R})$-covariance on S^1, namely $U : \mathrm{PSL}(2,\mathbb{R}) \to B(\mathcal{H})$ commuting with Γ s.th. $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(g.I)$,

(iv) the conformal Hamiltonian L_0 generating the rotation subgroup of $\mathrm{PSL}(2,\mathbb{R})$ has positive spectrum,

(v) there exists a unique vacuum vector $\xi \in \mathcal{H}$ (i.e., $\mathrm{PSL}(2,\mathbb{R})$-invariant, Γ-invariant, and cyclic for $\mathcal{A}$).



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The spectral triple is called *even* if there is a grading Γ on $\mathcal{H}$ such that $[\Gamma, \pi(A)] = 0$ and $\Gamma Q \Gamma = -Q$. Otherwise, it is called *odd*.

Associated a canonical entire cyclic cocycle τ, the JLO cocycle.

Our application: graded-local conformal net $\mathcal{A}$, supersymmetry.
[Carpi-RH-Kawahigashi-Longo '10]
Then $(\mathfrak{A}, (\pi_R, \mathcal{H}_R), G_0^{\pi_R})$, where π_R Ramond representation, $G_0^{\pi_R}$ super-Virasoro algebra generator such that $G_0^{\pi_R} G_0^{\pi_R} = L_0^{\pi_R} - \frac{c}{24}$, and $\mathfrak{A}$ to be constructed in suitable non-trivial manner...

Differentiable algebra

Given a unital family of localised endomorphisms Δ of $C^*(\mathcal{A})$, define the corresponding differentiable algebra

$$\mathfrak{A}_\Delta := \{x \in C^*(\mathcal{A}) : \ (\forall \rho \in \Delta)\ \pi_R \circ \rho(x) \in \mathrm{dom}(\delta)\}$$

and $\mathfrak{A}_\Delta(I) := \mathfrak{A}_\Delta \cap \mathcal{A}(I)$. Endowed with the family of norms

$$\| \cdot \|_\rho := \| \cdot \| + \|\delta(\pi_R \circ \rho(\cdot))\|_{\mathcal{H}_R}, \quad \rho \text{ smooth},$$

it becomes a locally convex *-algebra.

Index pairing with $K_*(\mathfrak{A}_\Delta)$

Theorem (Carpi-RH-Longo '13)

Given $\mathcal{A}$, π_R and Δ, then $(\mathfrak{A}_\Delta, (\pi_R\rho, \mathcal{H}_R), Q)_{\rho\in\Delta}$ is a family of even/odd θ-summable spectral triples and the associated even/odd JLO cocycles are $\rho^*\tau$ and have the following properties:

(1) *Suppose that, for fixed automorphism $\sigma \in \Delta$ and all $\rho \in \Delta$ with $\rho \neq \sigma$, $\pi_R\rho$ and $\pi_R\sigma$ are disjoint. Then for every $\rho \in \Delta$ with $\rho \neq \sigma$, we have $[\rho^*\tau] \neq [\sigma^*\tau]$.*

(2) *Suppose Δ consists only of differentiable automorphisms. Then for $\rho, \sigma \in \Delta$,*

$$[\rho] = [\sigma] \quad \textit{iff} \quad [\rho^*\tau] = [\sigma^*\tau].$$

In either case, the two non-equivalent even/odd cocycles are separated by nondegenerate explicit pairing with $K_0(\mathfrak{A}_\Delta)$ / $K_1(\mathfrak{A}_\Delta)$.

- Essential tools in further classification and representation-theoretic description of higher supersymmetric conformal nets [Carpi-RH-Kawahigashi-Longo-Xu preprint].

KMS states for conformal nets

Consider restriction $\mathcal{A}|_\mathbb{R}$ with translation group action α of $\mathbb{R}$.

Definition

Let ϕ be a state on $C^*(\mathcal{A}|_\mathbb{R})$ and $\beta > 0$. ϕ satisfies the KMS condition at inverse temperature β if, for all $x, y \in C^*(\mathcal{A}|_\mathbb{R})$, there is a continuous function

$$F_{xy} : \{z \in \mathbb{C} : \; 0 \leq \Im z \leq \beta\} \to \mathbb{C},$$

analytic on the interior and satisfying

$$F_{xy}(t) = \phi(\alpha_t(x)y), \quad F_{xy}(t + \mathrm{i}\,\beta) = \phi(y\alpha_t(x)), \quad t \in \mathbb{R}.$$

Theorem (Camassa-Longo-Tanimoto-Weiner '11)

Every completely rational nets has precisely one KMS state (explicit).

In general there are more than one. Applications:

- Unruh effect: vacuum state for accelerated observer is KMS w.r.t. translation.
- Weyl expansion for conformal Hamiltonian giving NC area and Euler characteristic [Kawahigashi-Longo '05] Details:

$$t\,\mathrm{tr}(e^{-2\pi t L_{0,\rho}}) \sim \frac{\pi c}{12} + t \log \frac{d(\rho)}{\sqrt{\mu}}, \quad t \to 0^+,$$

increments in representations corresponding to increments in free energy. KMS for rotation here. Inspiring treatment of KMS states for translations

Super-KMS states for graded conformal nets

Assume $\mathcal{A}$ is graded-local and consider restriction $\mathcal{A}|_{\mathbb{R}}$ with translation group action α of $\mathbb{R}$.

Definition (tentative)

Let ϕ be a state on $C^*(\mathcal{A}|_{\mathbb{R}})$ and $\beta > 0$. ϕ satisfies the KMS condition at inverse temperature β if, for all $x, y \in C^*(\mathcal{A}|_{\mathbb{R}})$, there is a continuous function

$$F_{xy} : \{z \in \mathbb{C} : \ 0 \leq \Im z \leq \beta\} \to \mathbb{C},$$

analytic on the interior and satisfying

$$F_{xy}(t) = \phi(\alpha_t(x)y), \quad F_{xy}(t + \mathrm{i}\,\beta) = \phi(\Gamma y \Gamma \alpha_t(x)), \quad t \in \mathbb{R}.$$

- Necessarily unbounded and not positive, so amend this condition, taking in consideration domains, denseness, invariance, hermitianity, etc
- Explicit construction of examples possible, general theory difficult at this stage. [RH preprint]
- Physical interpretation: supersymmetric phase transitions.
- Mathematical application: canonical construction of generalised entire cyclic cocycles and thus noncommutative geometric invariants for graded nets $\mathcal{A}|_{\mathbb{R}}$.

Summary

APPENDIX

Fusion ring action as Kasparov product

Theorem

The above action of $\mathcal{R}_{\mathcal{A}}$ corresponds to a Kasparov product. More precisely, there is an injective unital semiring homomorphism $j : \mathcal{R}_{\mathcal{A}} \to KK(\mathfrak{K}_{\mathcal{A}}, \mathfrak{K}_{\mathcal{A}})$ such that

$$x \times j([\rho]) = \eta_{[\rho]}(x) = (\hat{\rho})_*(x), \quad [\rho] \in \mathcal{R}_{\mathcal{A}},\ x \in K_0(\mathfrak{K}_{\mathcal{A}}) = KK(\mathbb{C}, \mathfrak{K}_{\mathcal{A}}).$$

- Basically an application of the identification $K_0(\mathfrak{K}_{\mathcal{A}}) \simeq KK(\mathbb{C}, \mathfrak{K}_{\mathcal{A}})$, definition of the product and functoriality, together with verification of the axioms.
- To do: give a quantum field theoretical meaning to the KK-classes! Invert the procedure!

Return

Jones index – Fredholm index – free energy

Indices:

$$\mathrm{ind}(Q_{\rho,+}) = \frac{d(\rho)}{\sqrt{\mu_{\mathcal{A}}}} \sum_{\nu \in \mathfrak{R}} \Phi_\nu(\varepsilon(\nu,\rho)^* \varepsilon(\rho,\nu)^*) d(\nu)\, \mathrm{null}(\mathcal{H}_\nu)$$

Incremental free energy:

$$\mathrm{d}\, F(\rho|\sigma) = \frac{2\pi}{\kappa}(\log d(\rho) - \log d(\sigma))$$

Return

Entire cyclic cohomology

Return

Definition

(1) Let $(A, (\|\cdot\|_i)_{i\in I})$ be a locally convex unital $*$-algebra and, for any nonnegative integer n, let $C^n(A)$ be the vector space of multilinear maps : $A\times(A/\mathbb{C}\mathbf{1})^n\to\mathbb{C}$ and $C^\bullet(A):=\prod_{k=0}^\infty C^k(A)$. With $\partial=b+B : C^\bullet(A)\to C^\bullet(A)$,

$$(b\phi)_k(a_0,\dots,a_k) := \sum_{j=0}^{k-1}(-1)^j\phi_{k-1}(a_0,\dots,a_ja_{j+1},\dots,a_k) + (-1)^k\phi_{k-1}(a_ka_0,a_1,\dots,a_{k-1}),$$

$$(B\phi)_k(a_0,\dots,a_k) := \sum_{j=0}^{k}(-1)^{jk}\phi_{k+1}(\mathbf{1},a_j,\dots,a_n,a_0,\dots,a_{j-1}),$$

∂, $C^\bullet(A)$ becomes the *cyclic cocomplex* $C^\bullet(A)=(C^e(A),C^o(A))$ over $\mathbb{Z}/2\mathbb{Z}$.
(2) A cochain $\phi=(\phi_k)_{k\in\mathbb{N}_0}\in C^\bullet(A)$ is called *entire* if, for every bounded subset $B\subset A$, there is a constant c_B such that

$$|\phi_k(a_0,\dots,a_k)|\le\frac{1}{\sqrt{k!}}c_B,\quad a_i\in B, k\in\mathbb{N}_0.$$

Correspondingly: *entire cyclic cohomology* $(HE^e(A), HE^o(A))$