Synchronizing Words and Preparability of States

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Abstract: Synchronizing words are a well known topic in the theory of graphs and automata and in symbolic dynamics. They can be reinterpreted as gadgets for the preparation of states on commutative algebras by repeated interactions with another system. This interpretation suggests noncommutative versions which are relevant to the preparation of states in noncommutative algebras and quantum systems.

Joint work with B. Kümmerer, T. Lang, F. Haag.

Road-coloured graphs and synchronizing words

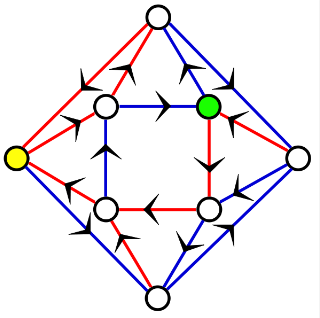

Figure: road-coloured graph, from: Wikimedia Commons

directed (multi-)graph with constant outdegree

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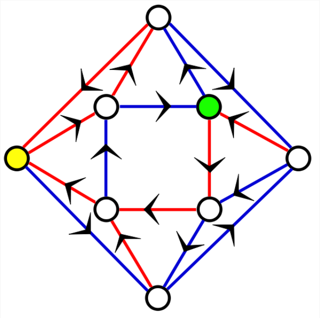

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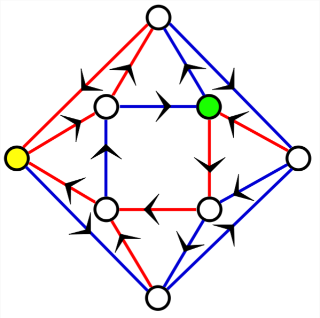

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directed (multi-)graph with constant outdegree
road-colouring = bijection between outgoing arrows (for each vertex) and a set of labels or colours [red (r) and blue (b)]
brrbrrbrr leads from any vertex to the yellow vertex
bbrbbrbbr leads from any vertex to the green vertex
These are examples of **synchronizing words**.

Road-colouring problem

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Computer scientists interpret synchronizing words as reset buttons for finite automata.

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Algebraically:

A set of vertices

C set of colours

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In particular this is a
topological Markov chain / subshift of finite type.

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Probabilistic version. TFAE:

1. There exists a synchronizing word.
2. For any probability measure μ on A there exists $n \in \mathbb{N}$ and a probability measure ν_n on $\prod_1^n C$ such that for any initial probability measure μ_0 on A

$$\mu = \left(\mu_0 \times \nu_n\right) \circ \gamma_n^{-1}$$

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Easy: Prepare the point measure $\mu = \delta_x$ by $\nu_n = \delta_w$ where w is a synchronizing word leading to x. Convex combinations.

Non-commutative topological Markov chain

$\mathcal{A}, \ \mathcal{C} \quad C^*$-algebras

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The corresponding iteration is now

$$J_n \colon \mathcal{A} \to \mathcal{A} \otimes \bigotimes_1^n \mathcal{C}, \quad J_n := \big(J \otimes \mathbb{1}\big) \circ J_{n-1}, \quad J_1 := J$$

We call this a non-commutative topological Markov chain.

Preparability

We say that a state $\rho \in \mathcal{S}(\mathcal{A})$ is **preparable** by $J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ if there exists a sequence (θ_n) with $\theta_n \in \mathcal{S}(\bigotimes_1^n \mathcal{C})$ such that for any initial state $\rho_0 \in \mathcal{S}(\mathcal{A})$

$$\big(\rho_0 \otimes \theta_n\big) \circ J_n \to \rho \quad \text{if } n \to \infty$$

More natural to consider limits here instead of a finite n.

Choice of topology left open for the moment. But

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For commutative algebras from (finite) road-coloured graphs **TFAE**:

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Physical interpretation: Preparing a state (in a quantum system if algebras are non-commutative) by repeated interactions with independent copies of another system.

Attractive feature: The initial state is forgotten.

micromaser experiments

A criterion

Special case: $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\mathcal{C}$ a von Neumann algebra.
Stationarity assumption: There exist faithful normal states $\phi \in \mathcal{S}(\mathcal{A})$ and $\psi \in \mathcal{S}(\mathcal{C})$ such that $(\phi \otimes \psi) \circ J = \phi$.

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$\mathcal{H}_\phi,\ \mathcal{H}_\psi$ GNS-Hilbert spaces w.r.t. $\phi,\ \psi$
$v : \mathcal{H}_\phi \to \mathcal{H}_\phi \otimes \mathcal{H}_\psi$ is the isometry obtained by extending J
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Z is called an **extended transition operator**. Not acting on the original $\mathcal{A} = \mathcal{B}(\mathcal{H})$, in fact $\mathcal{H}_\phi \simeq \mathcal{H} \otimes \mathcal{H}$.

the role of Z

Z^n measures how J_n forgets the $\mathcal{A}$-component of the tensor product:

$$\langle a\Omega_\phi, Z^n(p_\phi)a\Omega_\phi\rangle = \|Q_\phi J_n(a)\|^2_{\psi_n}$$

Ω_ϕ cyclic vector for ϕ, $a \in \mathcal{A} = \mathcal{B}(\mathcal{H})$

p_ϕ the one-dimensional projection onto $\mathbb{C}\Omega_\phi$

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Interpretation: Ergodicity of Z plays a similar role here as the synchronizing word in the easy proof earlier. Indeed, with more work the last property (called asymptotic completeness) makes it possible to prepare arbitrary vector states and then, by convex combinations, arbitrary normal states.