Subfactor Fusion Categories — from Ising to Haagerup

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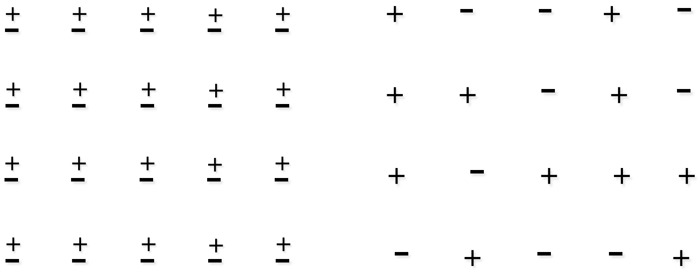
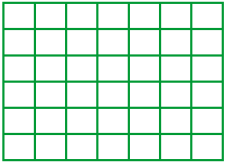
Overview

- subfactors and fusion rings
- Ising model and conformal field theory
- exotic subfactors and exotic modular tensor categories





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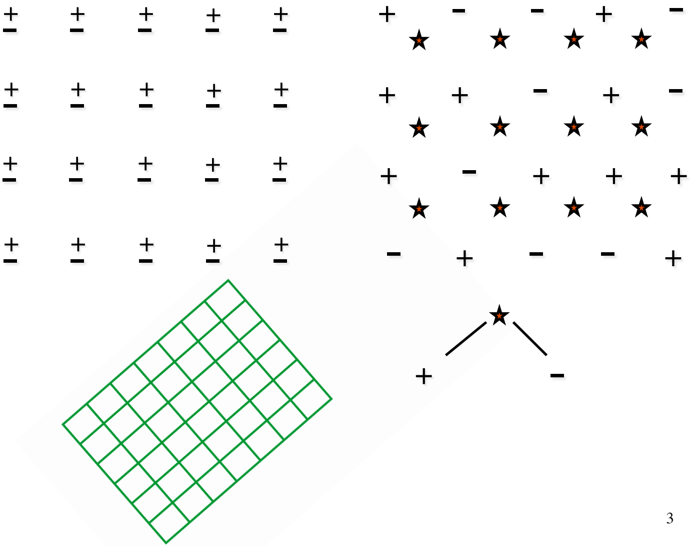
Ising



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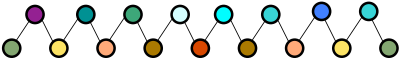
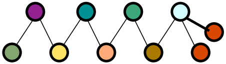
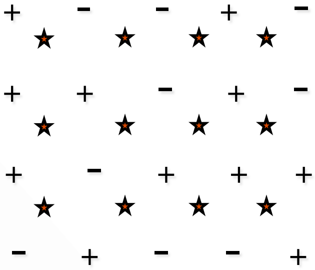
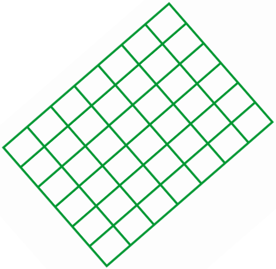
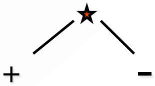
SU(n) models

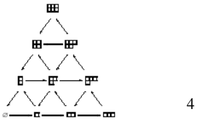












Subfactors

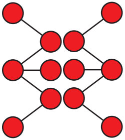
$N \subset M$

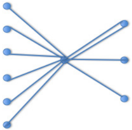
$\rho : M \to N, \quad \bar\rho : N \to M \qquad \rho\bar\rho \succeq id_N$

ρ generates N-N, N-M, M-N, M-M sectors

$\rho\bar\rho\,,\ \rho\bar\rho\rho\,,\ \bar\rho\rho\bar\rho$ etc

fusion graphs — principal graphs, dual principal graphs etc





Ising

Ising $H(\sigma) = \Sigma_{\alpha,\beta} J \sigma_\alpha \sigma_\beta$

$Z = \Sigma_\sigma exp(-H(\sigma)) = \Sigma \;\; \Pi$ Boltzmann weights

$$T \;=\; V^{1/2}\, W \;V^{1/2} = e^{-\mathcal{H}}$$

$V = exp\, K \Sigma \sigma_j^x \sigma_{j+1}^x \qquad\qquad W = exp\, L^* \Sigma \sigma_j^z$

$$\sigma^x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad\qquad \sigma^z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Ising

$$\alpha_t = T^{it}(\ \ )T^{-it} = Ad\, e^{i\mathcal{H}t} \text{ on } M_2 \otimes M_2 \otimes \dots$$

$$\phi_0^+ = \otimes_{\mathbb{Z}} \omega_{\binom{1}{0}}$$

$$\phi_0^- = \otimes_{\mathbb{Z}} \omega_{\binom{0}{1}}$$

$$\phi_\infty^+ = \otimes_{\mathbb{Z}} \omega_{\binom{1/\sqrt{2}}{1/\sqrt{2}}}$$

$$\omega_\zeta A = \langle A\zeta, \zeta\rangle$$

$$\phi_0^+ = \phi_\infty \nu$$

$$\nu : \sigma_j^x \sigma_{j+1}^x \leftrightarrow \sigma_j^z$$

$\nu^2 =$ shift on even algebra $\sigma_j^z \to \sigma_{j+1}^z \ \sigma_j^x \sigma_{j+1}^x \to \sigma_{j+1}^x \sigma_{j+2}^x$

$$\nu \sigma_j^z = \sigma_j^x \sigma_{j+1}^x \qquad \nu \sigma_j^x = \sigma_1^z \sigma_2^z \cdots \sigma_j^z$$

$$\nu^2 \sigma_j^z = \sigma_{j+1}^z \qquad \nu^2 \sigma_j^x = \sigma_1^x \sigma_{j+1}^x$$

Ising sectors

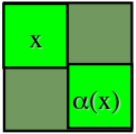
$$M_2 \otimes M_2 \otimes M_2 \otimes M_2 \otimes \cdots \subset \text{Cuntz algebra } \mathcal{O}_2 = C^*(s_+, s_-)$$

$$s_+ s_+^* + s_- s_-^* = 1$$

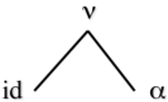
$$\nu(s_+ \pm s_-) = \sqrt{2}(s_+ s_\pm s_\pm^* + s_- s_\mp s_\mp^*)$$

$$\nu^2(x) = s_+ x s_+^* + s_- \alpha x s_-^*$$

$$\alpha : s_+ \leftrightarrow s_-$$



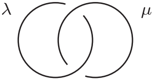
$$\nu^2 = id + \alpha$$

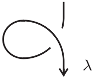


Braided Subfactors

λ endomorphism, N type III_1 factor: $\lambda\mu = \sum_\nu N_{\lambda\mu}^{\nu}\nu$

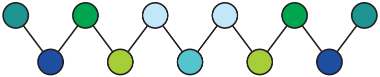
$\lambda\mu = \mathrm{Ad}u(\lambda,\mu)\mu\lambda,$ $u(\lambda,\mu) =$ 

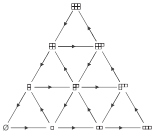
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to [S_{\lambda\mu}]$ 

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to [T_{\lambda\mu}]$ 

Braided Subfactors

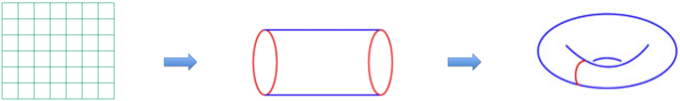
- Loop groups $SU(2), \ldots, SU(N)$ etc Wassermann
  $\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I'} SU(n))'$
  $\lambda = 0 : \quad N = N$, III_1 factor, $\quad \lambda N \subset N$





- Quantum double of finite group, Haagerup subfactor etc
  $\mathcal{X}$ on N : $A \subset N \otimes N^{opp}$, can endo $\Sigma_{\lambda \in \mathcal{X}} \lambda \otimes \lambda^{opp}$ Ocneanu Izumi

Modular Invariants



$$Z \to \quad \mathrm{tr} e^{2\pi i \tau (L_0 - c/24)} e^{-2\pi i \overline{\tau} (\overline{L_0} - c/24)}$$

$$= \quad \sum Z_{\lambda\mu} ch_\lambda(\tau) ch_\mu(\tau)^* \quad = \quad Z(\tau) \quad = \quad Z((a\tau + b)/(c\tau + d))$$

$$ch = \mathrm{trace}\, q^{L_0 - c/24} \qquad q = e^{2\pi i \tau}$$

Understand:

$$Z = [Z_{\lambda\mu}] \in SL(2, \mathbb{Z})'$$

$$Z_{\lambda\mu} \in 0, 1, 2, 3, \ldots \qquad\qquad Z_{00} = 1$$

Ising Model

Dynkin diagram A_3, $\lambda \in \{\bullet, +, -\}$
Fermions $g_a : a \in \mathbb{N} - 1/2$ *or* $\mathbb{N}$
$L_0 = \Sigma_{r\in\mathbb{N}-1/2} r g_r^* g_r \to \chi_\pm$ $L_0 = \Sigma_{n\in\mathbb{N}} n g_n^* g_n \to \chi_\bullet$

$$\chi_+ \pm \chi_- = q^{-1/48}\Pi_{n\in\mathbb{N}}(1 \pm q^{n-1/2})$$

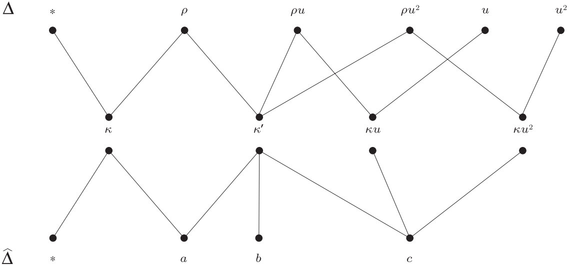
$$\chi_\bullet = q^{1/24}\Pi_{n\in\mathbb{N}}(1 + q^n)$$

$$\chi = \mathrm{trace}\, q^{L_0 - c/24} \quad q = e^{2\pi i\tau}$$

$$\tau \to -1/\tau \qquad S = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & 1-\sqrt{2} \\ 1 & -\sqrt{2} & 0 \end{pmatrix}$$

$$\tau \to \tau + 1 \qquad T = \mathrm{diag}(e^{-\pi i/24}, e^{-\pi i/12}, e^{-\pi i 23/24})$$

Principal graphs of the Haagerup $(5+\sqrt{13})/2$ subfactor

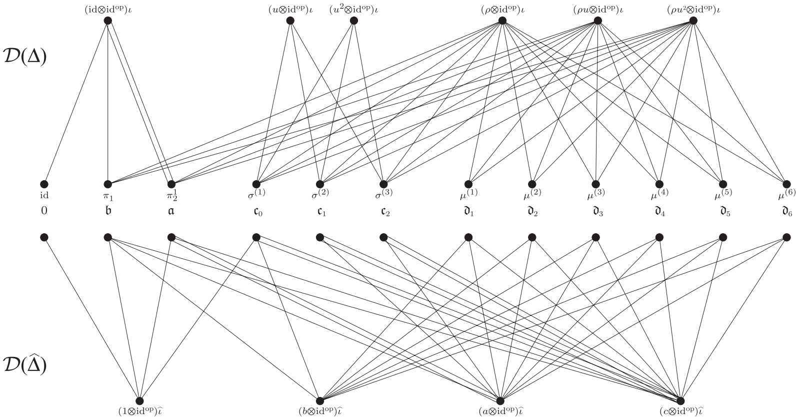


$$\alpha^3 = 1\,, \quad \rho\alpha = \alpha^2\rho\,, \quad \rho^2 = 1 + \rho + \rho\alpha + \rho\alpha^2$$

$$d_\rho^2 = 1 + 3d_\rho; \quad d_\rho = (3+\sqrt{1}3)/2$$

$$d_\rho = [M, \rho M]^{1/2}$$

Group	$\nu^2 + 4$	$\sharp$ *subfactors*	
0	5	1	A_4 subfactor
$\mathbb{Z}_3$	13	1	Haagerup
$\mathbb{Z}_5$	29	1	Izumi
$\mathbb{Z}_3 \times \mathbb{Z}_3$	85	0	Izumi
$\mathbb{Z}_7$	53	1	
$\mathbb{Z}_9$	$85 = 5.17$	2	
$\mathbb{Z}_{11}$	$125 = 5^3$	2	$\mathrm{E} - \mathrm{Gannon}$
$\mathbb{Z}_{13}$	173	1	
$\mathbb{Z}_{15}$	229	1	
$\mathbb{Z}_{17}$	293	1	
$\mathbb{Z}_{19}$	$365 = 5.73$	2	



Dual principal graphs for doubles of Δ and $\widehat{\Delta}$ for Haagerup

$$
\begin{aligned}
|ch_0 + ch_{\mathfrak{b}} + 2ch_{\mathfrak{a}}|^2 &= Z_{\Delta} \\
|ch_0 + ch_{\mathfrak{b}} + ch_{\mathfrak{a}} + ch_{\mathfrak{c}_0}|^2 &= Z_{\widehat{\Delta}}
\end{aligned}
$$

Modular data for Haagerup $\mathcal{D}$Hg

$$S = \frac{1}{3}\begin{pmatrix}
x & 1-x & 1 & 1 & 1 & 1 & y & y & y & y & y & y \\
1-x & x & 1 & 1 & 1 & 1 & -y & -y & -y & -y & -y & -y \\
1 & 1 & 2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & -y & 0 & 0 & 0 & 0 & c(1) & c(2) & c(3) & c(4) & c(5) & c(6) \\
y & -y & 0 & 0 & 0 & 0 & c(2) & c(4) & c(6) & c(5) & c(3) & c(1) \\
y & -y & 0 & 0 & 0 & 0 & c(3) & c(6) & c(4) & c(1) & c(2) & c(5) \\
y & -y & 0 & 0 & 0 & 0 & c(4) & c(5) & c(1) & c(3) & c(6) & c(2) \\
y & -y & 0 & 0 & 0 & 0 & c(5) & c(3) & c(2) & c(6) & c(1) & c(4) \\
y & -y & 0 & 0 & 0 & 0 & c(6) & c(1) & c(5) & c(2) & c(4) & c(3)
\end{pmatrix}$$

$$T = \mathrm{diag}(1, 1, 1, 1, \xi_3, \overline{\xi_3}, \xi_{13}^{6}, \xi_{13}^{-2}, \xi_{13}^{2}, \xi_{13}^{5}, \xi_{13}^{-6}, \xi_{13}^{-5})$$

$$x = (13-3\sqrt{13})/26 \qquad y = 3/\sqrt{13} \qquad c(j) = -2y\cos(2\pi j/13) \qquad \xi = e^{2\pi i/13}$$

$$N_{i,j}^{k} = \sum_{l} \frac{S_{i,l}}{S_{0,l}} S_{j,l} S_{k,l}^{*}\,, \quad S_{i,j} = \overline{T}_{i,i}\overline{T}_{j,j}\,T_{0,0}\sum_{k} T_{k,k} S_{k,0} N_{i,j}^{k}\,.$$

Modular data for $SO(13)_2$

$$S = \frac{1}{3}\begin{pmatrix} y/2 & y/2 & 3/2 & 3/2 & y & y & y & y & y & y \\ y/2 & y/2 & -3/2 & -3/2 & y & y & y & y & y & y \\ 3/2 & -3/2 & 3/2 & -3/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/2 & -3/2 & -3/2 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & y & 0 & 0 & -c(1) & -c(2) & -c(3) & -c(4) & -c(5) & -c(6) \\ y & y & 0 & 0 & -c(2) & -c(4) & -c(6) & -c(5) & -c(3) & -c(1) \\ y & y & 0 & 0 & -c(3) & -c(6) & -c(4) & -c(1) & -c(2) & -c(5) \\ y & y & 0 & 0 & -c(4) & -c(5) & -c(1) & -c(3) & -c(6) & -c(2) \\ y & y & 0 & 0 & -c(5) & -c(3) & -c(2) & -c(6) & -c(1) & -c(4) \\ y & y & 0 & 0 & -c(6) & -c(1) & -c(5) & -c(2) & -c(4) & -c(3) \end{pmatrix}$$

$$T = \mathrm{diag}(-1, -1; -i, i; -\xi_{13}^{6l^2})$$

$$y = 3/\sqrt{13} \qquad c(j) = -2y\cos(2\pi j/13) \qquad \xi = e^{2\pi i/13}$$

Characters for $\mathcal{D}$Hg, $c = 8$, $\gamma = 0, 1$

$$\begin{pmatrix} ch_0(\tau) \\ ch_{\mathfrak{b}}(\tau) \\ ch_{\mathfrak{a}}(\tau) = ch_{\mathfrak{c}_0}(\tau) \\ ch_{\mathfrak{c}_1}(\tau) \\ ch_{\mathfrak{c}_2}(\tau) \\ ch_{\mathfrak{d}_1}(\tau) \\ ch_{\mathfrak{d}_2}(\tau) \\ ch_{\mathfrak{d}_3}(\tau) \\ ch_{\mathfrak{d}_4}(\tau) \\ ch_{\mathfrak{d}_5}(\tau) \\ ch_{\mathfrak{d}_6}(\tau) \end{pmatrix} = \begin{pmatrix} q^{2/3}\left(q^{-1} + (6+13\gamma) + (120+78\gamma)q + (956+351\gamma)q^2 + (6010+1235\gamma)q^3 + \cdots\right) \\ q^{2/3}\left((80-13\gamma) + (1250-78\gamma)q + (10630-351\gamma)q^2 + (65042-1235\gamma)q^3 + \cdots\right) \\ q^{2/3}\left(81 + 1377q + 11583q^2 + 71037q^3 + \cdots\right) \\ 3 + 243q + 2916q^2 + 21870q^3 + \cdots \\ q^{1/3}\left(27 + 594q + 5967q^2 + 39852q^3 + \cdots\right) \\ q^{5/39}\left((7-\gamma) + (292-6\gamma)q + (3204-43\gamma)q^2 + (23010-146\gamma)q^3 + \cdots\right) \\ q^{20/39}\left((42+16\gamma) + (777+121\gamma)q + (7147+547\gamma)q^2 + (45367+2000\gamma)q^3 + \cdots\right) \\ q^{32/39}\left(\gamma q^{-1} + (11\gamma+119) + (73\gamma+1623)q + (300\gamma+12996)q^2 + (76429+1063\gamma)q^3 + \cdots \right. \\ q^{2/39}\left((5-3\gamma) + (229-50\gamma)q + (2738-252\gamma)q^2 + (19942-1032\gamma)q^3 + \cdots\right) \\ q^{8/39}\left((13-5\gamma) + (347-37\gamma)q + (3804-212\gamma)q^2 + (26390-794\gamma)q^3 + \cdots\right) \\ q^{11/39}\left((14+7\gamma) + (441+61\gamma)q + (4445+303\gamma)q^2 + (30329+1167\gamma)q^3 + \cdots\right) \end{pmatrix}$$