Classification of graph C^*-algebras

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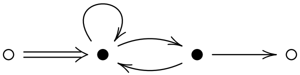
Program

Outline

Graph algebras

Any countable graph $E = (E^0, E^1, r, s)$ defines a C^*-algebra $C^*(E)$ given as a universal C^*-algebra by **projections** $\{p_v : v \in E^0\}$ and **partial isometries** $\{s_e : e \in E^1\}$ subject to the *Cuntz-Krieger relations*:

1. $p_v p_w = 0$ when $v \neq w$
2. $(s_e s_e^*)(s_f s_f^*) = 0$ when $e \neq f$
3. $s_e^* s_e = p_{r(e)}$ and $s_e s_e^* \leq p_{s(e)}$
4. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every v with $0 < |\{e \mid s(e) = v\}| < \infty$.



Shifts of finite type

Any finite graph $E = (E^0, E^1, r, s)$ with no sinks nor sources defines a subshift

$$\mathsf{X}_E = \{(e_i) \in (E^1)^{\mathbb{Z}} \mid r(e_i) = s(e_{i+1})\}$$

This is a compact dynamical system under

$$\sigma((e_i)) = (e_{i+1})$$

We say that two shift spaces X and Y are *flow equivalent* and write $\mathsf{X} \sim_{FE} \mathsf{Y}$ when their suspension flows

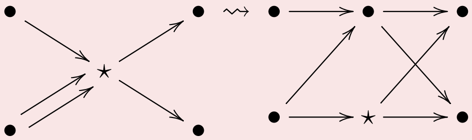
$$S\mathsf{X} = \mathsf{X} \otimes \mathbb{R} / \langle (x, t) \sim (\sigma(x), t+1) \rangle$$

are homeomorphic in a way preserving the direction of the flow lines.

Three moves

Move (I)

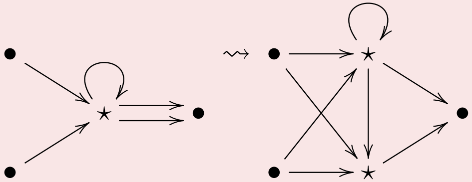
Insplit at $\star$



Three moves

Move **(O)**

Outsplit at $\star$



Three moves

Move **(R)**

Reduce a configuration with a transitional vertex, as

$$\bullet \rightrightarrows \star \longrightarrow \bullet \;\rightsquigarrow\; \bullet \rightrightarrows \bullet$$

Theorem (Parry-Sullivan 75, Franks 84, Matsumoto-Matui 13)

Let A_E, A_F denote adjacency matrices of finite graphs which are strongly connected, have no sinks nor sources, and are not single cycles. The following are equivalent

1. $X_E \sim_{FE} X_F$
2. $\mathbb{Z}^n / \operatorname{im}(I - \mathrm{A}_E) \simeq \mathbb{Z}^m / \operatorname{im}(I - \mathrm{A}_F)$ *and* $\det(I - \mathrm{A}_E) = \det(I - \mathrm{A}_F)$
3. *E can be transformed into F by a finite number of moves* **(I)**, **(O)**, **(R)**
4. $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$ *in a way preserving the canonical MASA*

Theorem (Rørdam 93)

Let A_E, A_F denote adjacency matrices of finite graphs which are strongly connected, have no sinks nor sources, and are not single cycles. The C^*-algebras $C^*(E)$ and $C^*(F)$ are purely infinite and simple, and the following are equivalent

2. $\mathbb{Z}^n / \operatorname{im}(I - \mathsf{A}_E) \simeq \mathbb{Z}^m / \operatorname{im}(I - \mathsf{A}_F)$
4. $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

Outline

Graph algebras

Any countable graph $E = (E^0, E^1, r, s)$ defines a C^*-algebra $C^*(E)$ given as a universal C^*-algebra by **projections** $\{p_v : v \in E^0\}$ and **partial isometries** $\{s_e : e \in E^1\}$ subject to the *Cuntz-Krieger relations*:

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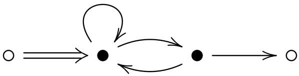
3. $s_e^* s_e = p_{r(e)}$ and $s_e s_e^* \leq p_{s(e)}$
4. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every v with $0 < |\{e \mid s(e) = v\}| < \infty$.

Partitioning the adjacency matrix



$$A_E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \infty & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad A_E^{\bullet} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A_E^{\circ} = \begin{bmatrix} 0 & 0 \\ \infty & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Classification by K-theory

Theorem (Elliott 76, Kirchberg-Phillips 95)

When $C^*(E)$ and $C^*(F)$ have no non-trivial gauge invariant ideals, then

$$K_*(C^*(E)) \simeq K_*(C^*(F)) \Longleftrightarrow C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$$

K-groups are particularly easy to compute for graph algebras. Indeed

$$K_0(C^*(E)) = \mathrm{cok}(I^\bullet - \mathsf{A}_E^\bullet) \qquad K_1(C^*(E)) = \ker(I^\bullet - \mathsf{A}_E^\bullet)$$

where

$$I^\bullet = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \dots & 0 \\ \\ 0 & \dots & 0 \end{bmatrix}$$

Filtrated K-theory

Definition

Let $\mathfrak{A}$ be a C^*-algebra having only finitely many ideals. The collection of all sequences

$$\begin{array}{ccccc}
K_0(\mathfrak{J}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{J}) \\
\uparrow & & & & \downarrow \\
K_1(\mathfrak{K}/\mathfrak{J}) & \longleftarrow & K_1(\mathfrak{K}/\mathfrak{I}) & \longleftarrow & K_1(\mathfrak{J}/\mathfrak{I})
\end{array}$$

with $\mathfrak{I} \lhd \mathfrak{J} \lhd \mathfrak{K} \lhd \mathfrak{A}$ t ideals is called the *filtered K-theory* of $\mathfrak{A}$ and denoted $FK^\gamma(\mathfrak{A})$. Equipping all K_0-groups with order we arrive at the *ordered, filtered K-theory* $FK^+(\mathfrak{A})$.

We write $FK^+(\mathfrak{A}) \simeq FK^+(\mathfrak{B})$ when the lattices ideals of $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic via some map κ, so that there exist K-theory isomorphisms

$$K_*(\mathfrak{J}/\mathfrak{I}) \simeq K_*(\kappa(\mathfrak{J})/\kappa(\mathfrak{I}))$$

which are consistent with all maps in all six-term exact sequences.

Working conjecture (E-Restorff-Ruiz 10)

$FK^+(-)$ is a complete invariant for graph algebras with finitely many ideals.

Tempered ideal space

Definition

We define a **temperature map** $\tau : \mathrm{Prim}(\mathfrak{A}) \to \mathbb{Z}$ by

$$\tau(\mathfrak{I}) = \begin{cases} -1 & \mathfrak{I}/\mathfrak{I}_0 \text{ is } AF \\ \operatorname{rank} K_0(\mathfrak{I}/\mathfrak{I}_0) - \operatorname{rank} K_1(\mathfrak{I}/\mathfrak{I}_0) & \text{otherwise} \end{cases}$$

where $\mathfrak{I}_0$ is a maximal proper ideal of $\mathfrak{I}$ (which may or may not be zero).

The status of the working conjecture depends on the tempered ideal space of the graph C^*-algebra in question. Large body of work by Arklint, Bentmann, E, Katsura, Restorff, Ruiz.

Outline

Theorem (Rørdam 93)

Let A_E, A_F denote adjacency matrices of finite graphs which are strongly connected, have no sinks nor sources, and are not single cycles. The C^*-algebras $C^*(E)$ and $C^*(F)$ are purely infinite and simple, and the following are equivalent

2. $\mathbb{Z}^n / \operatorname{im}(I - \mathsf{A}_E) \simeq \mathbb{Z}^m / \operatorname{im}(I - \mathsf{A}_F)$
4. $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

Move **(C)**

"Cuntz splice" on a vertex supporting two cycles

Theorem (Cuntz-Rørdam 93)

Let A_E, A_F denote adjacency matrices of finite graphs which are strongly connected, have no sinks nor sources, and are not single cycles. The C^*-algebras $C^*(E)$ and $C^*(F)$ are purely infinite and simple, and the following are equivalent

2. $\mathbb{Z}^n/\operatorname{im}(I - \mathsf{A}_E) \simeq \mathbb{Z}^m/\operatorname{im}(I - \mathsf{A}_F)$
3. *E can be transformed into F by a finite number of moves* **(I)**, **(O)**, **(R)**, **(C)**
4. $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

Key result

Theorem (E-Ruiz-Sørensen)

For any pair of graphs with E^0 and F^0 finite, when E and F are related through move **(C)**, *then* $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$.

Moves

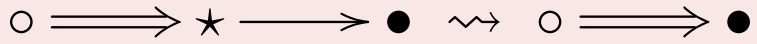
Move **(S)**

Remove a regular source, as

Move **(R)**

Reduce a configuration with a transitional regular vertex, as

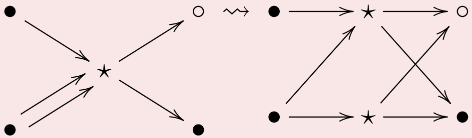
or



Moves

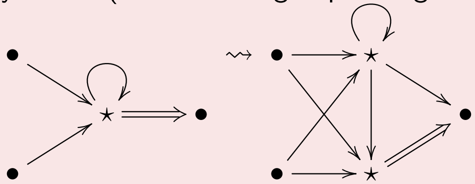
Move **(I)**

Insplit at regular vertex



Move **(O)**

Outsplit at any vertex (at most one group of edges infinite)



Theorem (E-Ruiz-Sørensen)

Let E and F be finite graphs so that $C^*(E)$ and $C^*(F)$ have the same finite primitive ideal space. The following are equivalent

2. $FK^+(C^*(E)) \simeq FK^+(C^*(F))$
3. *E can be transformed into F by a finite number of moves* **(I)**, **(O)**, **(R)**, **(S)**, **(C)**
4. $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

The theorem is true (mut. mut.) for any pair of finite graphs. We conjecture it holds when only the set of vertices is finite. This was established by Sørensen in the simple case.

Outline

Question A

Suppose a C^*-algebra $\mathfrak{A}$ has finitely many ideals with all simple subquotients either AF or purely infinite. Is it possible to tell by K-theory alone when $\mathfrak{A}$ is isomorphic to a graph algebra?

Question B

Suppose

$$0 \longrightarrow C^*(E) \longrightarrow \mathfrak{B} \longrightarrow C^*(F) \longrightarrow 0$$

is a short exact sequence, with $\mathfrak{B}$ having finitely many ideals. Does

$$\begin{array}{ccccc} K_0(C^*(E)) & \longrightarrow & K_0(\mathfrak{B}) & \longrightarrow & K_0(C^*(F)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(F)) & \longleftarrow & K_1(\mathfrak{B}) & \longleftarrow & K_0(C^*(E)) \end{array}$$

contain the answer to when $\mathfrak{B}$ is a graph algebra?