The AT property is not preserved by finite extensions

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Plan of talk

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1 Measurable dynamical systems and the Connes-Krieger dictionary

A **measurable dynamical system** is a quadruple $(X, \mathcal{B}, \mu, T)$, consisting of a measure space and an invertible bimeasurable transformation T. Sometimes we replace T with a group Γ of transformations acting on X. For us, the system will usually be *ergodic*: the only invariant sets are null or conull.

In the 1930's, Von Neumann classified factors into Types I_n, $n = 1, 2, \ldots, \infty$, Types II_1 and II_∞, and Type III.

He gave examples of factors by starting with $L^\infty(X, \mu)$ and taking what we call today the crossed product by the action of T. Actually, in all his examples, μ was an infinite product measure on an infinite product of 2-point spaces.

In a mathematical *tour de force*, Alain Connes and Wilfried Krieger and some others, showed that von Neumann's insight was good. Up to isomorphism, every factor arises this way.

In fact, they found a “dictionary” between factors and measurable dynamical systems:

v.n. factors up to isom $\leftrightarrow$ ergod. meas.dyn. system up to orbit equ. $\leftrightarrow$ flows (action of $\mathbb{R}$)up to conjugacy

ITPF1 factors $\leftrightarrow$ orbit equiv to product $\leftrightarrow$ AT flow

Type $I_n \leftrightarrow X$ has n points $\leftrightarrow$ periodic flow
Type II_1, $II_\infty \leftrightarrow \mu \circ T = \mu \leftrightarrow$ recurrent flow
Type III_λ, $0 \leq \lambda \leq 1 \leftrightarrow$ there is no equivalent preserved measure $\leftrightarrow$ dissipative flows.

All types except III_0 are ITPF1. Type III_0 contain both product type and non product type, and are not unique up to o.e..

2 The AT property

It is therefore important to study the AT property. Let me remind you what it is:

A dynamical system $(X, \mathcal{B}, \mu, T)$ is said to be *approximately transitive* (or to have the *AT property*) if any finite collection of positive L^1 functions may be simultaneously approximated by a positive linear combination of iterates of a single L^1 function, that is given $\epsilon > 0, f_1, \ldots, f_n \in L^1_+(X)$, there exist $f \in L^1_+(X)$, n_j : $j = 1, \ldots, m$ and $\alpha_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m \in \mathbb{R}^+$ such that

$$\left\| f_i - \sum_{i,j} \alpha_{i,j} \mathcal{L}_{n_j} f \right\| \leq \epsilon$$

where $\mathcal{L}_n f(x) = \frac{d\mu \circ T^n}{d\mu}(x) f(T^n x)$ and $\| \cdot \|$ denotes the L^1 norm.

We consider exclusively invertible probability measure-preserving transformations and so $\mathcal{L}_n f(x)$ reduces to $f(T^n x)$.

- The above definition is equivalent to the one with n replaced by 2.
- Jane Hawkins proved that if $(X, \mathcal{B}, \mu, \Gamma)$ is AT, and if $\Gamma_0 \trianglelefteq \Gamma$, and if $\rho : X \to X/\Gamma_0 = Y$, then

  $$(Y, \mathcal{B}_1, \mu \circ \rho, \Gamma/\Gamma_0)$$

  is AT
- Suppose that $(X, \mathcal{B}, \mu, T)$ is AT and F is a finite group, with a measurable cocycle

  $$\omega : X \times \Gamma \to F$$

  We can form the skew product $(X, \mathcal{B}, \mu, T) \times_\omega F$, i.e. Γ acts on $X \times F$ by $\gamma(x, f) = (\gamma x, \omega(x, \gamma) f_0)$, where the space has the measure $d\mu \times d\lambda_F$.

  Then $X \times_\omega F$ is a measurable dynamical system.

  If it's ergodic again,

  Giordano, Putnam & Skau **conjectured** that it has the AT property provided that X is AT.

  We will show that this is false.

3 Rank and entropy

The AT property is also related to the **rank** of a dynamical system.

A transformation has *rank one* if there exist a sequence of subsets B_n and integers k_n with the following properties:

1. The sets $T^i(B_n)$ for $0 \leq i < k_n$ are pairwise disjoint;
2. $\mu(\bigcup_{0\leq i<k_n} T^i(B_n)) \to 1$ as $n \to \infty$;
3. the algebras $\mathcal{A}_n$ generated by the sets $T^i(B_n)$ for $0 \leq i < k_n$ have the property that for all $A \in \mathcal{B}$ and all $\epsilon > 0$, there exists an n_0 such that for $n \geq n_0$, there exists a set $B \in \mathcal{A}_n$ such that $\mu(B \triangle A) < \epsilon$.

A system has *funny rank one* (Thouvenot) if there exist a sequence of subsets B_n, and a sequence of sequences $a_0^n, \ldots, a_{k_n-1}^n$ with the following properties:

1. The sets $T^{a_i^n}(B_n)$ are pairwise disjoint;
2. $\mu(\bigcup_{0 \le i < k_n} T^{a_i^n}(B_n)) \to 1$ as $n \to \infty$;
3. the algebras $\mathcal{A}_n$ generated by the sets $T^{a_i^n}(B_n)$ for $0 \le i < k_n$ have the property that for all $A \in \mathcal{B}$ and all $\epsilon > 0$, there exists an n_0 such that for $n \ge n_0$, there exists a set $B \in \mathcal{A}_n$ such that $\mu(B \triangle A) < \epsilon$.

We have the following known or obvious implications:

Rank one $\Rightarrow$ Funny rank one $\Rightarrow$ AT $\Rightarrow$ zero entropy.

The first implication is obvious, and Ferenczi gave a counter-example to show $\not\Leftarrow$

The second implication is fairly easy to see,and Ferenczi conjectures that the *Morse system* is not Funny Rank 1. Our result below would then show $\not\Leftarrow$.

The last arrow was shown by Connes and Woods using operator algebras. We show it using measure theory. We also show that an example of Furstenberg gives $\not\Leftarrow$.

4 The Morse system

Let $\Omega = \{0,1\}^{\mathbb{Z}^+}$ and define

$$\phi(\omega) = \begin{cases} -1 & \text{if } \omega \text{ has an even number of trailing 1s} \\ +1 & \text{otherwise} \end{cases}$$

Ω is a group under the operation of addition with carry. Let $1 = (1, 0, 0, \dots)$.

Let $X = \Omega \times \{\pm 1\}$ and define the Morse transformation on X by $T(\omega, t) = (\omega + 1, t \cdot \phi(\omega))$, where $\cdot$ denotes multiplication in $\{\pm 1\}$. We let μ denote the Haar measure on Ω and letting c be the normalized counting measure on $\{\pm 1\}$, $\mu \times c$ is known to be a uniquely ergodic invariant measure for the Morse transformation.

This defines the Morse system as an explicit two-point extension of the odometer.

- Del Junco showed that it has simple spectrum, but is not of rank one.
- As it is not rank one, it cannot be tiled by translates of a single word: Ferenczi shows more, that the maximum density of the translates of a single word is 2/3.
- Ferenczi conjectured that it is of funny rank one.

Theorem 1 *The Morse system has the AT property.*

5 Measures of "funny" cylinders and zero entropy

Let ν be a shift-invariant measure on $Y = \{1, \ldots, l\}^{\mathbb{Z}}$. A *funny cylinder set* in Y is a set of the form $C_{x,\Lambda} = \{y \in Y : y_i = x_i \text{ for all } i \in \Lambda\}$, where $x \in Y$ and Λ is a finite set. The ϵ-ball about the funny cylinder $C_{x,\Lambda}$ is the set $B_{\epsilon,x,\Lambda} = \{y \in Y : y_i = x_i \text{ for all but at most } \epsilon|\Lambda| \text{ of the } i \in \Lambda\}$.

Theorem 2 *Suppose that the system $(X, \mathcal{B}, \mu, T)$ has the AT property. Then let $\mathcal{P} = \{A_1, \ldots, A_l\}$ be an arbitrary finite measurable partition. Denote by π, the natural map induced by $\mathcal{P}$ from X to $\{1, \ldots, l\}^{\mathbb{Z}}$ and let $\nu = \mu \circ \pi^{-1}$ on Y. Then for every $\delta > 0$ and $\epsilon > 0$, there exist arbitrarily large finite sets $\Lambda \subset \mathbb{Z}$ and points $x \in Y$ such that $\nu(B_{\epsilon,x,\Lambda}) > (1 - \delta)/|\Lambda|$.*

Corollary 5.1 *(David) If* $(X, \mathcal{B}, \mu, T)$ *has positive entropy, then it does not have the AT property.*

Proof If $(X, \mathcal{B}, \mu, T)$ has positive entropy, then it has a Bernoulli factor $Y = \{0, 1\}^{\mathbb{Z}}$ with probability p of 0's and q of 1's with $0 < p < q$. One can check that the measure of any set $B_{\epsilon,x,\Lambda}$ is bounded above by

$$\binom{n}{\lfloor \epsilon n \rfloor} q^{n - \lfloor \epsilon n \rfloor},$$

where $n = |\Lambda|$. It is not hard to show that for sufficiently small ϵ, this quantity is strictly less than $(1 - \epsilon)/n$ for all large n.

Corollary 5.2 *There exists a zero entropy system that does not have the AT property.*

Proof Let α be an irrational number and consider the transformation of $\mathbb{T}^2$ given by $T(x, y) = (x + \alpha, y + 2x + \alpha)$ (mod 1). This transformation, studied by Furstenberg, is uniquely ergodic, preserving Haar measure on the torus. We use a version of the weak law of large numbers to show that it is not approximately transitive.

6 A finite extension which doesn't have the AT property

Finally, I would like to give a counter-example to the conjecture of Giordano, Putnamn and Skau which I mentioned above.

Theorem 3 *There exists a rank 1 system with the AT property and an ergodic two-point extension of it that fails to have the AT property.*

Proof. We modify a construction of Helson and Parry and again apply a weak law of large numbers. Letting p_n denote the nth prime number, let X be the compact group (with the product topology) $\mathbb{Z}_{2^N} \times \prod_{n=2}^{\infty} \mathbb{Z}_{p_n^2}$, where N is to be determined. Let 1 denote the element $(1, 1, 1 \ldots)$ of X_0, let T be the odometer given by $T(x) = x + 1$ and let μ be Haar measure on X.

Next on the nth factor in X for $n \geq 2$, define

$$\phi_n(i) = \begin{cases} -1 & \text{if } i = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Then since ϕ_k is different from 1 only on a set of measure $1/p_k^2$, it follows that for almost every $(x_1, x_2, \ldots) \in X$, $\phi_k(x_k)$ is eventually equal to 1. Hence for almost every $(x_1, x_2, \ldots) \in X$, $\prod_{k=2}^{\infty} \phi_k(x_k)$ exists. We then define a function $\phi_1\colon \mathbb{Z}/(2^N\mathbb{Z}) \to \{\pm 1\}$ and a function $\phi\colon X \to \{\pm 1\}$ by

$$\phi(x_1, x_2, \ldots) = \prod_{k=1}^{\infty} \phi_k(x_k).$$

The key estimates required are due to Helson and Parry on $c(n) = |\int \phi^{(n)}(x)\, d\mu|$, valid independent of the choices of N and ϕ_1.

Since the measure μ is a product measure and $\phi^{(n)}$ is the product of the $\phi_k^{(n)}$, one has

$$c(n) = \prod_{k=1}^{\infty} \left| \int \phi_k^{(n)}\, d\mu_k \right|,$$

where μ_k denotes the Haar measure on the kth factor. Note that all factors in the product are bounded above by 1. Moreover, in any factor such that $p_k^2 \geq 2n$, the quantity $\phi_k^{(n)}$ is equal to -1 on a set of measure n/p_k^2 and is equal to 1 elsewhere. Accordingly, the kth factor in the above product is equal to $1 - 2n/p_k^2 \leq \exp(-2n/p_k^2)$. Hence they derived the estimate

$$c(n) \leq \exp\left(-2n \sum_{p_k^2 \geq 2n} \frac{1}{p_k^2} \right),$$

from which one gets the crude inequality $c(n) \leq \exp(-2Kn^{1/4})$ for some $K > 0$.

It then follows that the $c(n)$ are summable, and with a little work, we can show that $\sum_{n=1}^{\infty} c(n) < 1/4$. Let M be so large that $\sum_{n=M}^{\infty} c(n) < 1/8$, and now choose $N > 6 + 3\log M$.

By the Chinese Remainder Theorem, every point of X has a dense orbit so T is uniquely ergodic. Let μ denote Haar measure on X. Since T is an ergodic rotation of a compact Abelian group, it has discrete spectrum. Hence by a result of del Junco, T is a rank 1 transformation.

We consider the extension $T_\phi \colon X \times \{\pm 1\} \to X \times \{\pm 1\}$ given by $T_\phi(x,t) = (T(x), t \cdot \phi(x))$ and introduce the partition of $X \times \{\pm 1\}$, $\mathcal{P} = \{X \times \{1\}, X \times \{-1\}\}$. As before, T_ϕ preserves the product measure $\mu \times c$. The partition induces a natural map from $X \times \{\pm 1\}$ to $Y = \{\pm 1\}^{\mathbb{Z}}$ as in Theorem 2. Let ν be the induced measure on Y.

We are now in a position to argue as in Corollary 5.2. Let Λ be any finite subset of $\mathbb{Z}$ and let $z \in Y$ be fixed. For $y \in Y$ and $i \in \Lambda$, let $A_i(y) = z_i y_i$. Let T be the sum of the A_i over $i \in \Lambda$. The A_i have expectation 0 (since ν is invariant under flipping the entire sequence). We have

$$\begin{aligned}\mathrm{Cov}(A_i, A_j) &= z_i z_j \int y_i y_j \, d\nu = z_i z_j \int y_0 y_{|i-j|} \, d\nu \\ &= z_i z_j \int \phi^{(|i-j|)}(x) \, d\mu(x),\end{aligned}$$

so that $|\text{Cov}(A_i, A_j)| = |\text{Cov}(A_0, A_{|i-j|})| = c(|i-j|)$. We estimate

$$\begin{aligned}\text{Var}(T) &\leq \sum_{i\in\Lambda}\sum_{j\in\Lambda} c(|i-j|) \\ &\leq \sum_{i\in\Lambda}\sum_{j\in\mathbb{Z}} c(|i-j|) < 3|\Lambda|/2.\end{aligned}$$

Working exactly as before, we eventually see that $\nu(B_{15/16,z,\Lambda}) < 48/49$. This contradicts the conclusion of Theorem 2 showing that T_ϕ fails to have the AT property.

7 Applications to operator algebras

Giordano and Skau conjectured that a subfactor of finite index of an ITPF1 factor is again ITPF1.

The above example shows that this is false.

Consider $(X, \mathcal{B}, \mu, T)$ and construct the flow of constant height 1 over it. Then it is easy to see that the flow is AT if and only if the original system is AT.

Thus in the above example, we obtain a flow which is not AT, which is a 2 point extension of an AT flow.

Now, the Jones tunnel construction (which I don't understand!!) allows us to construct factors A, A_1, where A_1 is a subfactor of A of index two, A is ITPF1 but A_1 is **not** ITPF1.

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